Alpha-Beta Pruning: Algorithm and Analysis

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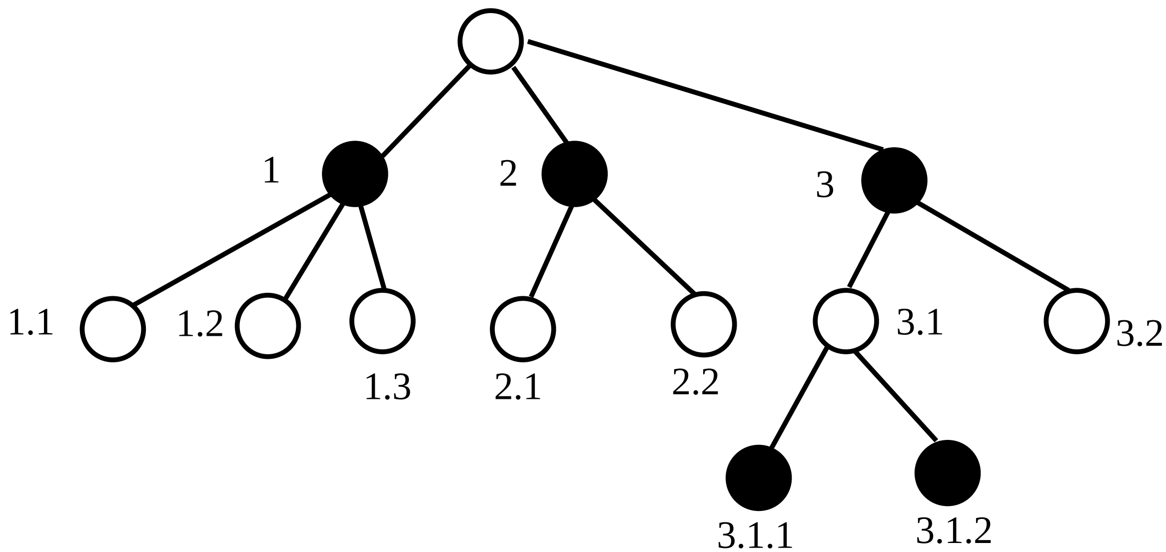
Introduction

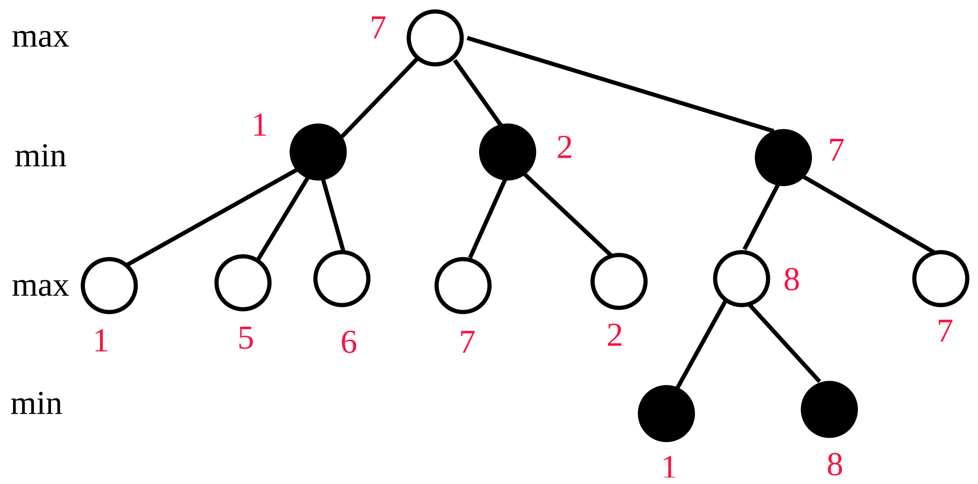
- Alpha-beta pruning is the standard searching procedure used for 2-person perfect-information zero sum games.
- Definitions:
  - A *position* p.
  - The value of a position p, $f(p)$, is a numerical value computed from evaluating p.
    - ▷ *Value is computed from the root player's point of view.*
    - ▷ *Positive values mean in favor of the root player.*
    - ▷ *Negative values mean in favor of the opponent.*
    - ▷ *Since it is a zero sum game, thus from the opponent's point of view, the value can be assigned $-f(p)$.*
  - A terminal position: a position whose *value* can be know.
    - ▷ *A position where win/loss/draw can be concluded.*
    - ▷ *A position where some constraints are met.*
  - A position p has b legal moves $p_1, p_2, \ldots, p_b$.

Tree node numbering



- **From the root, number a node in a search tree by a sequence of integers $a_1.a_2.a_3.a_4 \cdots$**
  - **Meaning from the root, you first take the a_1th branch, then the a_2th branch, and then the a_3th branch, and then the a_4th branch $\cdots$**
  - **The root is specified as an empty sequence.**
  - **The depth of a node is the length of the sequence of integers specifying it.**
- **This is called "Dewey decimal system."**

Mini-max formulation



- **Mini-max formulation:**
  - $$F'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ max\{G'(p_1), \ldots, G'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - $$G'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ min\{F'(p_1), \ldots, F'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - **An indirect recursive formula!**
  - **Equivalent to AND-OR logic.**

Algorithm: Mini-max

- **Algorithm F'(position p) // max node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := -\infty$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $t := G'(p_i)$
    - ▷   ***if*** $t > m$ ***then*** $m := t$ // ***find max value***
  - **end; return m**
- **Algorithm G'(position p) // min node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := \infty$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $t := F'(p_i)$
    - ▷   ***if*** $t < m$ ***then*** $m := t$ // ***find min value***
  - **end; return m**
- **A brute-force method to try all possibilities!**

Mini-max: revised (1/2)

- **Algorithm F'(position p) // max node**
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node

    or depth reaches the cutoff threshold // from iterative deepening

    or time is running up // from timing control

    or some other constraints are met // add knowledge here

    then return $f(p)$// current board value

    else begin

    ▷ $m := -\infty$ // *initial value*

    ▷ ***for*** $i := 1$ ***to*** b ***do*** // *try each child*

    ▷ ***begin***

    ▷ $t := G'(p_i)$

    ▷ ***if*** $t > m$ ***then*** $m := t$ // *find max value*

    ▷ ***end***

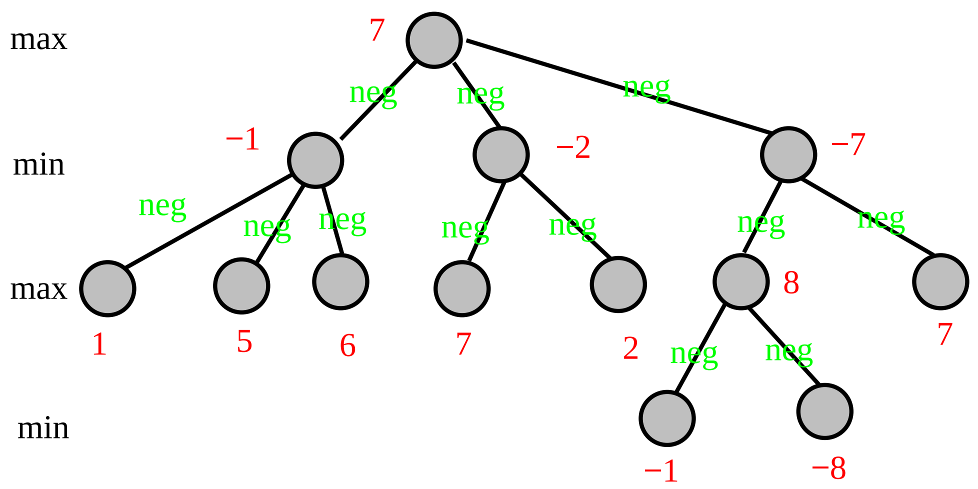
    end
  - return m

Mini-max: revised (2/2)

- **Algorithm G'(position p) // min node**
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or depth reaches the cutoff threshold // from iterative deepening
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
    then return $f(p)$// current board value
    else begin
      - ▷ $m := \infty$ // *initial value*
      - ▷ ***for*** $i := 1$ ***to*** b ***do*** // *try each child*
      - ▷ ***begin***
      - ▷ $t := F'(p_i)$
      - ▷ ***if*** $t < m$ ***then*** $m := t$ // *find min value*
      - ▷ ***end***

    end
  - return m

Nega-max formulation



- **Nega-max formulation:**
Let $F(p)$ be the greatest possible value achievable from position p against the optimal defensive strategy.
  - $$F(p) = \begin{cases} h(p) & \textbf{if } b = 0 \\ max\{-F(p_1), \ldots, -F(p_b)\} & \textbf{if } b > 0 \end{cases}$$
    - ▷ $$h(p) = \begin{cases} f(p) & \textit{if depth of } p \textit{ is 0 or even} \\ -f(p) & \textit{if depth of } p \textit{ is odd} \end{cases}$$

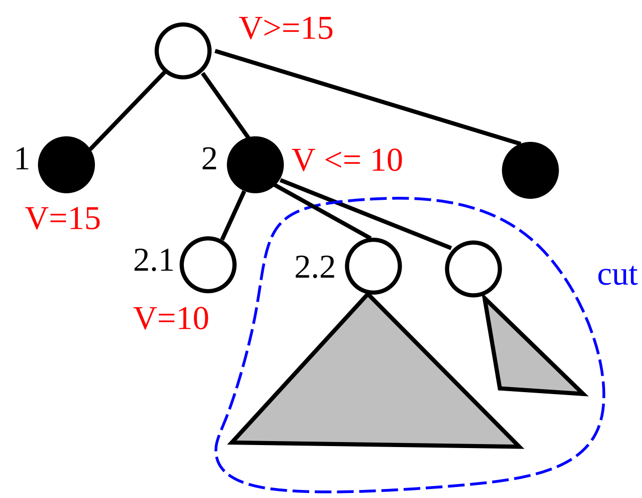
Algorithm: Nega-max

- **Algorithm F(position p)**
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or depth reaches the cutoff threshold // from iterative deepening
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - then return $h(p)$ else
  - begin
    - ▷ $m := -\infty$
    - ▷ *for* $i := 1$ *to* b *do*
    - ▷ *begin*
    - ▷ $t := -F(p_i)$ // *recursive call, the returned value is negated*
    - ▷ *if* $t > m$ *then* $m := t$ // *always find a max value*
    - ▷ *end*
  - end
  - return m
- **Also a brute-force method to try all possibilities, but with a simpler code.**

Intuition for improvements

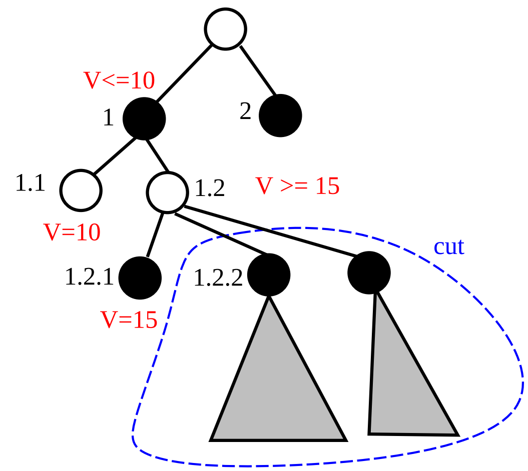
- **Branch-and-bound: using information you have so far to cut or prune branches.**
  - **A branch is cut means we do not need to search it anymore.**
  - **If you know for sure the value of your result is more than x and the current search result for this branch so far can give you no more than x,**
    - ▷ ***then there is no need to search this branch any further.***
- **Two types of approaches**
  - **Exact algorithms: through mathematical proof, it is guaranteed that the branches pruned won't contain the solution.**
    - ▷ ***Alpha-beta pruning: reinvented by several researchers in the 1950's and 1960's.***
    - ▷ ***Scout.***
    - ▷ $\cdots$
  - **Approximated heuristics: with a high probability that the solution won't be contained in the branches pruned.**
    - ▷ ***Obtain a good estimation on the remaining cost.***
    - ▷ ***Cut a branch when it is in a very bad position and there is little hope to gain back the advantage.***

Alpha cut-off



- **Alpha cut-off:**
  - **On a max node**
    - ▷ ***Assume you have finished exploring the branch at** 1 **and obtained the best value from it as** $bound$**.***
    - ▷ ***You now search the branch at** 2 **by first searching the branch at** 2.1**.***
    - ▷ ***Assume branch at** 2.1 **returns a value that is** $\leq bound$**.***
    - ▷ ***Then no need to evaluate the branch at** 2.2 **and all later branches of** 2**, if any, at all.***
    - ▷ ***The best possible value for the branch at** 2 **must be** $\leq bound$**.***
    - ▷ ***Hence we should take value returned from the branch at** 1 **as the best possible solution.***

Beta cut-off

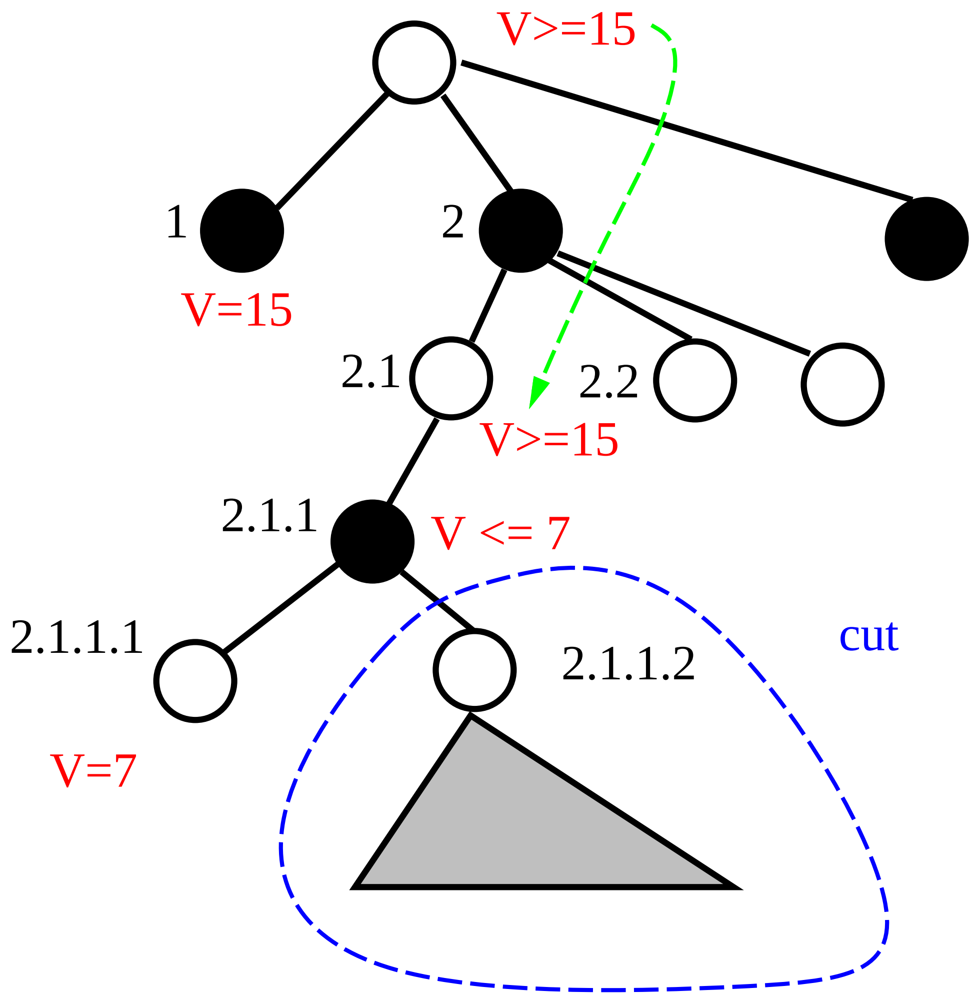


- **Beta cut-off:**
  - **On a min node**
    - ▷ ***Assume you have finished exploring the branch at** 1.1 **and obtained the best value from it as** $bound$.*
    - ▷ ***You now search the branches at** 1.2 **by first exploring the branch at** 1.2.1.*
    - ▷ ***Assume the branch at** 1.2.1 **returns a value that is** $\geq bound$.*
    - ▷ ***Then no need to evaluate the branch at** 1.2.2 **and all later branches of** 1.2, **if any, at all.***
    - ▷ ***The best possible value for the branch at** 1.2 **is** $\geq bound$.*
    - ▷ ***Hence we should take value returned from the branch at** 1.1 **as the best possible solution.***

Deep alpha cut-off

- **For alpha cut-off:**
  - ▷ ***For a min node u, the branch of its ancestor (e.g., elder brother of its parent) produces a lower bound V_l.***
  - ▷ ***The first branch of u produces an upper bound V_u for v.***
  - ▷ ***If $V_l \geq V_u$, then there is no need to evaluate the second branch and all later branches, of u.***
- **Deep alpha cut-off:**
  - ▷ ***Def: For a node u in a tree and a positive integer g, Ancestor(g, u) is the direct ancestor of u by tracing the parent's link g times.***
  - ▷ ***When the lower bound V_l is produced at and propagated from u's great grand parent, i.e., Ancestor(3,u), or any Ancestor($2i+1$,u), $i \geq 1$.***
  - ▷ ***When an upper bound V_u is returned from the a branch of u and $V_l \geq V_u$, then there is no need to evaluate all later branches of u.***
- **We can find similar properties for deep beta cut-off.**

Illustration — Deep alpha cut-off



Ideas for refinements

- **During searching, maintain two values $alpha$ and $beta$ so that**
  - $alpha$ **is the current lower bound of the possible returned value;**
  - $beta$ **is the current upper bound of the possible returned value.**
- **If during searching, we know for sure $alpha > beta$, then there is no need to search any more in this branch.**
  - **The returned value cannot be in this branch.**
  - **Backtrack until it is the case $alpha \leq beta$.**
- **The two values $alpha$ and $beta$ are called the ranges of the current search window.**
  - **These values are dynamic.**
  - **Initially, $alpha$ is $-\infty$ and $beta$ is ∞.**

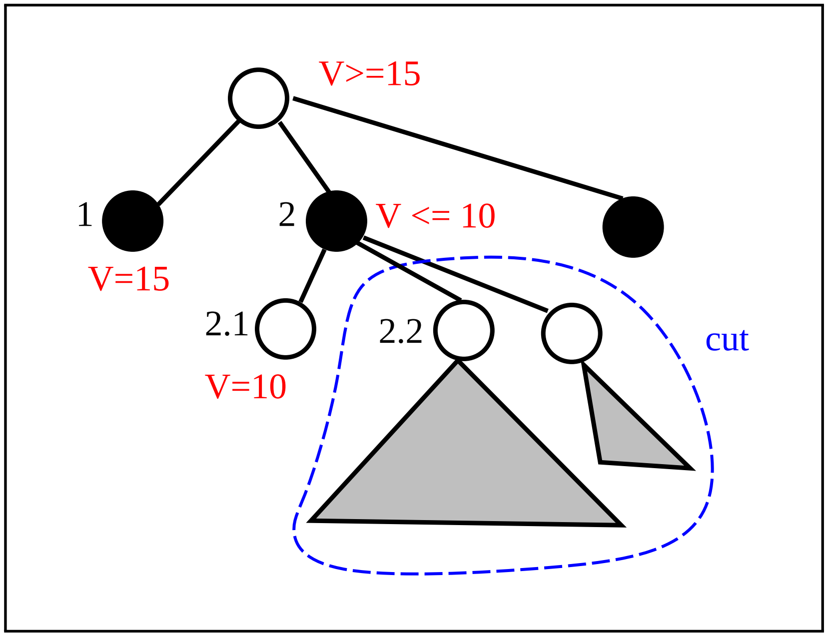
Alpha-beta pruning algorithm: Mini-Max

- **Algorithm $F2'$(position p, value $alpha$, value $beta$) // max node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := alpha$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $t := G2'(p_i, m, beta)$
    - ▷   ***if*** $t > m$ ***then*** $m := t$
    - ▷   ***if*** $m \geq beta$ ***then return***(m) ***// beta cut off***
  - **end; return m**
- **Algorithm $G2'$(position p, value $alpha$, value $beta$) // min node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := beta$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $t := F2'(p_i, alpha, m)$
    - ▷   ***if*** $t < m$ ***then*** $m := t$
    - ▷   ***if*** $m \leq alpha$ ***then return***(m) ***// alpha cut off***
  - **end; return m**

Example

Initial call: $F2'$(root,$-\infty$,∞)

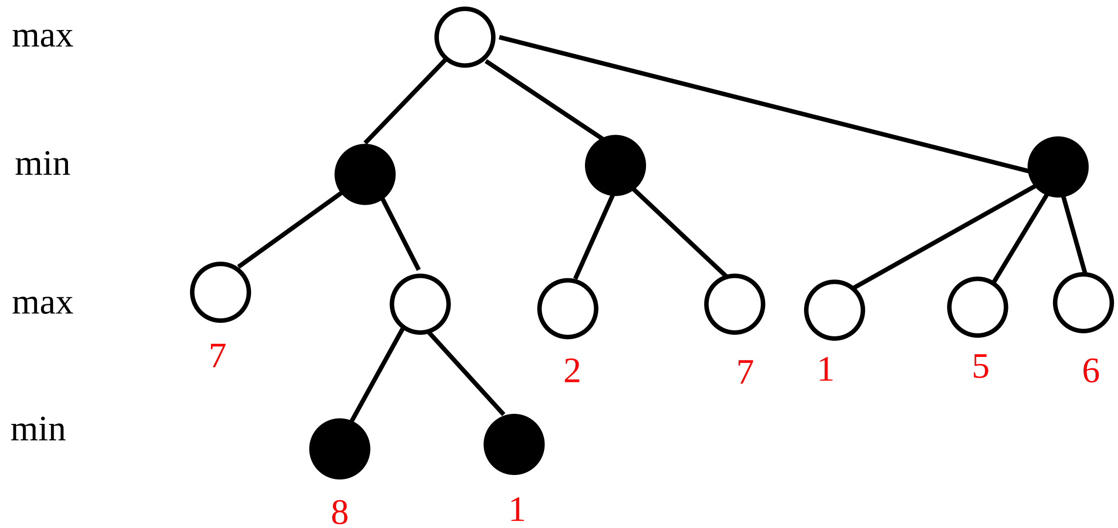
- $m = -\infty$
- **call** $G2'$**(node 1,**$-\infty$**,**∞**)**
  - ▷ ***it is a terminal node***
  - ▷ ***return value*** 15
- $t = 15$**;**
  - ▷ ***since*** $t > m$***,*** m ***is now*** 15
- **call** $G2'$**(node 2,**15**,**∞**)**
  - ▷ ***call*** $F2'$***(node 2.1,***15***,***∞***)***
  - ▷ ***it is a terminal node; return*** 10
  - ▷ $t = 10$***; since*** $t < \infty$***,*** m ***is now*** 10
  - ▷ $alpha$ ***is*** 15***,*** m ***is*** 10***, so we have an alpha cut off***
  - ▷ ***no need to call*** $F2'$***(node 2.2,15,10)***
  - ▷ $\cdots$

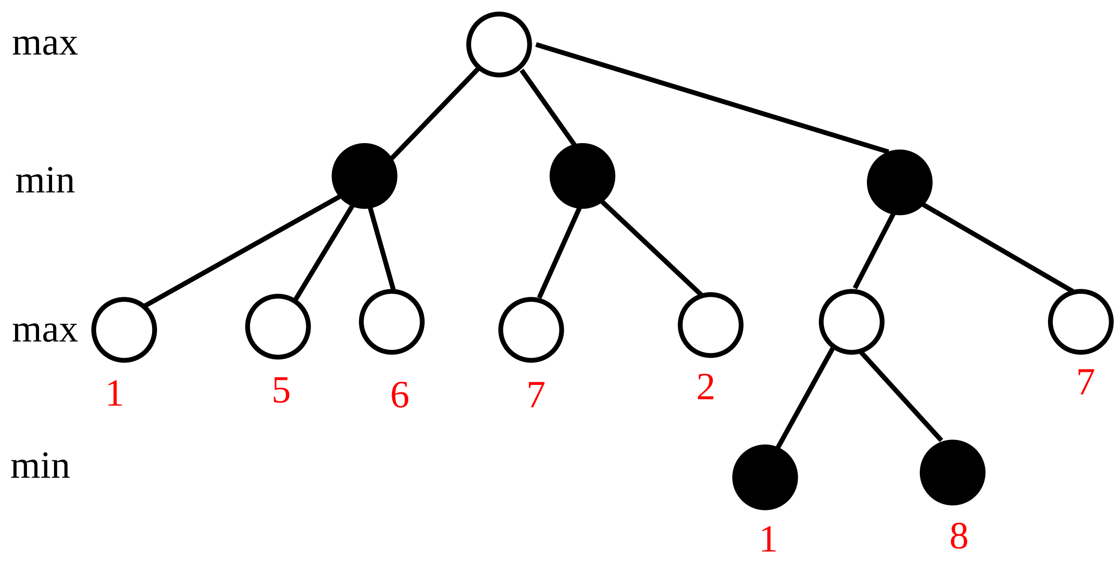


Alpha-beta pruning algorithm: Nega-max

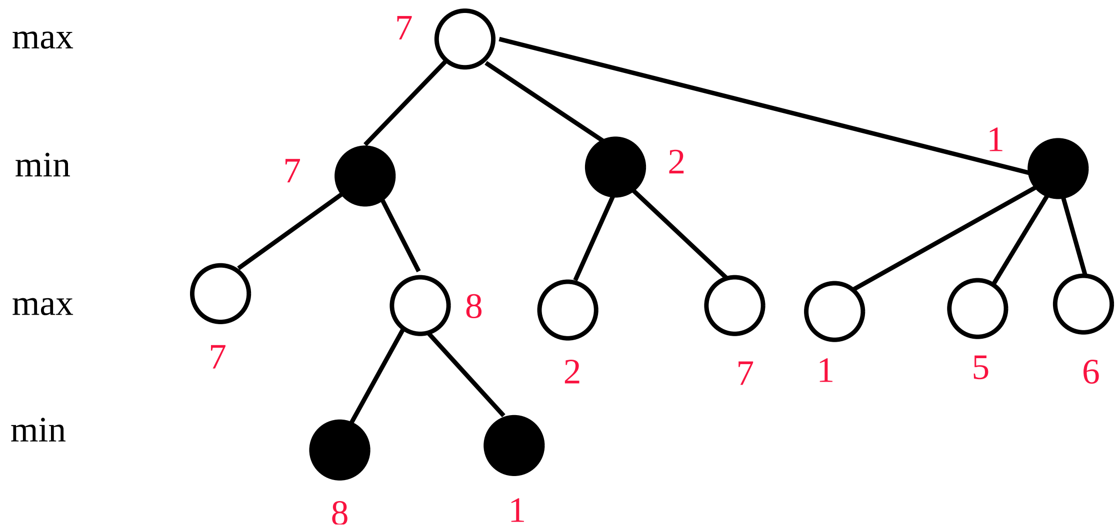
- **Algorithm** $F2$(**position** p, **value** $alpha$, **value** $beta$)
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or depth reaches the cutoff threshold // from iterative deepening
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - then return $h(p)$ else
  - begin
    - ▷ $m := alpha$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ $t := -F2(p_i, -beta, -m)$
    - ▷ ***if*** $t > m$ ***then*** $m := t$
    - ▷ ***if*** $m \geq beta$ ***then return***(m) // cut off
    - ▷ ***end***
  - end
  - return m

Examples (1/4)





Examples (2/4)


max
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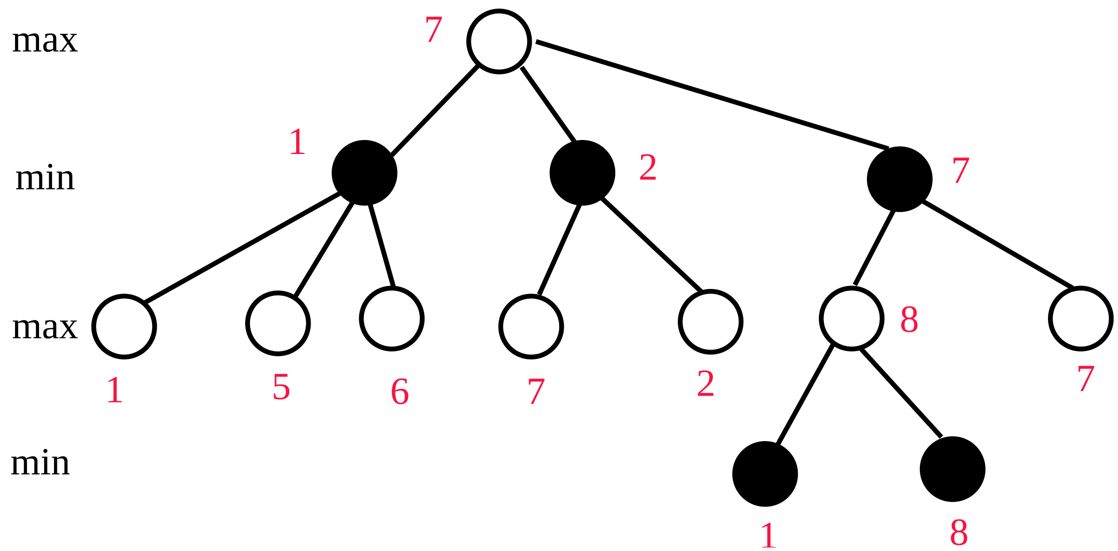
1


max
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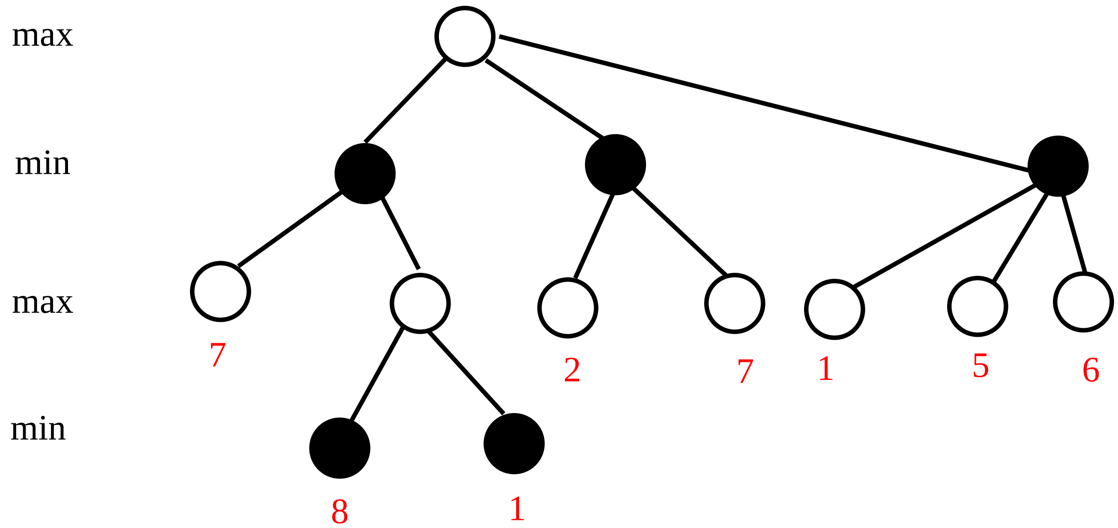
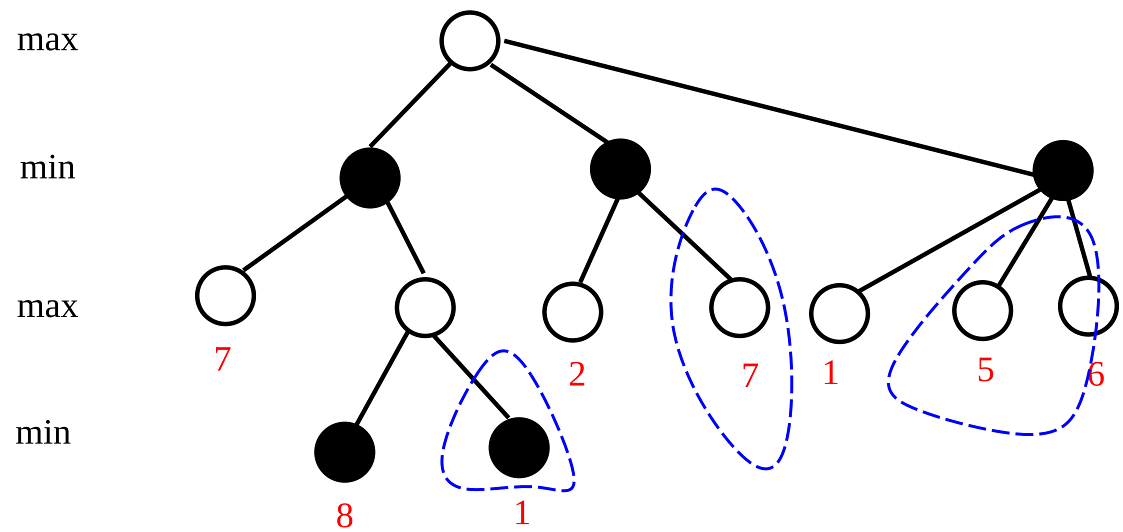
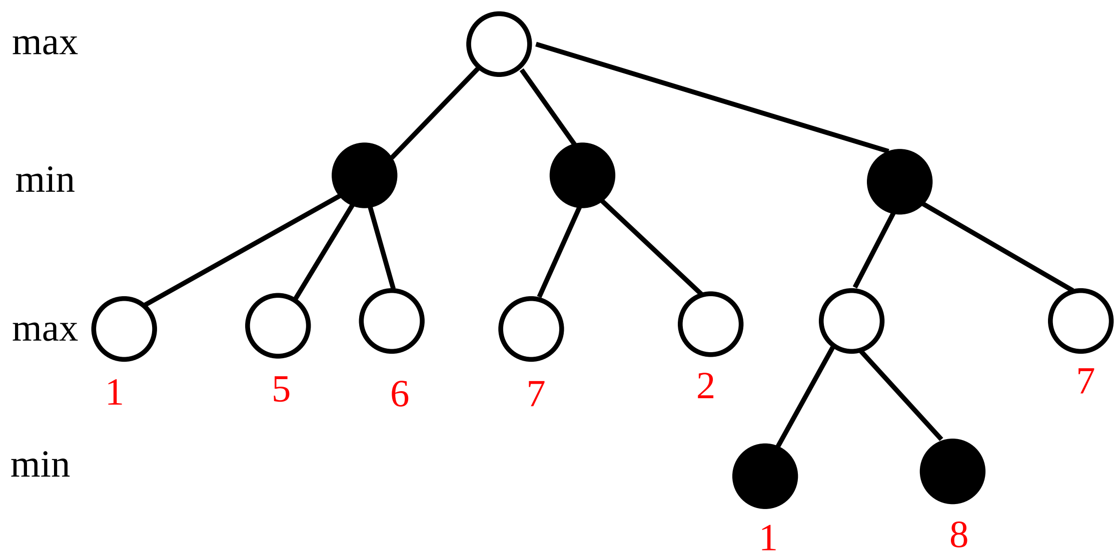
Examples (3/4)



Examples (3/4)



Examples (4/4)



Lessons from the previous examples

- **It looks like for the same tree, different move orderings give very different cut branches.**
- **It looks like if a node can evaluate a child with the best possible outcome earlier, then it can decide to cut earlier.**
  - **For a min node, this means to evaluate the child branch that gives the lowest value first.**
  - **For a max node, this means to evaluate the child branch that gives the highest value first.**
- **Q: In the best possible scenario, how many nodes are cut?**

Analysis of a possible best case

- **Definitions:**
  - **A path in a search tree is a sequence of numbers indicating the branches selected in each level using the Dewey decimal system.**
  - **A position is denoted as a path $a_1.a_2.\cdots.a_\ell$ from the root.**
  - **A position $a_1.a_2.\cdots.a_\ell$ is critical if**
    - ▷ $a_i = 1$ ***for all even values of*** i ***or***
    - ▷ $a_i = 1$ ***for all odd values of*** i***.***
  - **Note: as a special case, the root is critical.**
  - **Examples:**
    - ▷ ***2.1.4.1.2, 1.3.1.5.1.2, 1.1.1.2.1.1.1.3 and 1.1 are critical***
    - ▷ ***1.2.1.1.2 is not critical***

Perfect-ordering tree

- **A perfect-ordering tree:**

$$F(a_1.\cdots.a_\ell) = \begin{cases} h(a_1.\cdots.a_\ell) & \textbf{if } a_1.\cdots.a_\ell \textbf{ is a terminal} \\ -F(a_1.\cdots.a_\ell.1) & \textbf{otherwise} \end{cases}$$

  - **The first successor of every non-terminal position gives the best possible value.**

Theorem 1

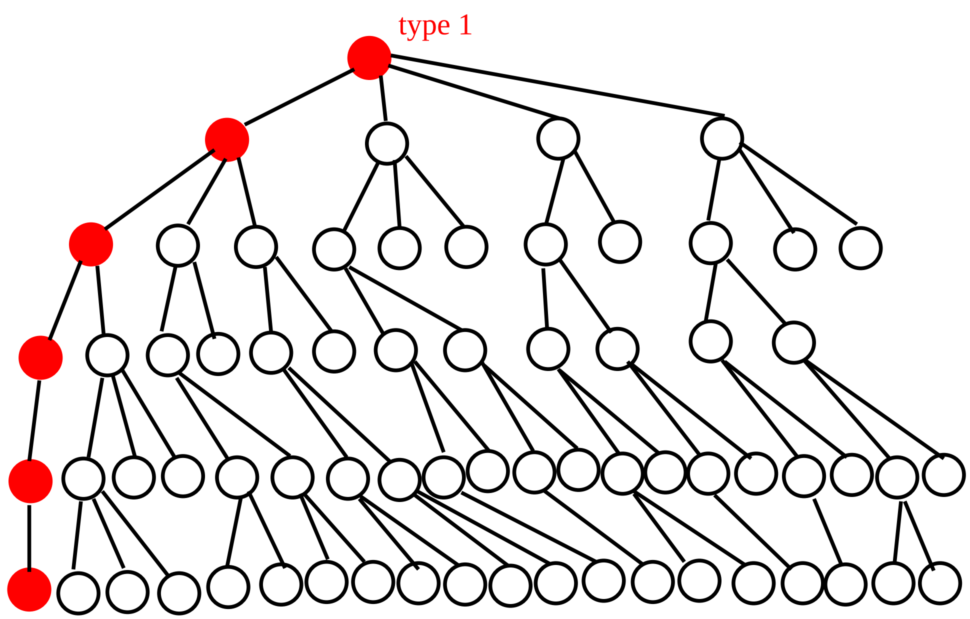
- **Theorem 1: $F2$ examines precisely the critical positions of a perfect-ordering tree.**
- **Proof sketch:**
  - **Classify the critical positions, a.k.a. nodes.**
    - ▷ *You must evaluate the first branch from the root to the bottom.*
    - ▷ *Alpha cut off happens at odd-depth nodes as soon as the first branch of this node is evaluated.*
    - ▷ *Beta cut off happens at even-depth nodes as soon as the first branch of this node is evaluated.*
  - **For each type of nodes, try to associate them with the types of pruning occurred.**

Types of nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index, if exists, such that $a_j \neq 1$ and ℓ is the last index.**
  - **Def: let $IS1(a_i)$ be a boolean function so that it is 0 if it is not the value 1 and it is 1 if it is.**
    - ▷ ***We call this $IS1$ parity of a number.***
  - **If j exists and $\ell > j$, then**
    - ▷ *$a_{j+1} = 1$ **because this position is critical and thus the** $IS1$ **parities of** a_j **and** a_{j+1} **are different.***
  - **Since this position is critical, if $a_j \neq 1$, then $a_h = 1$ for any h such that $h - j$ is odd.**
- **We now classify critical nodes into 3 types.**

Type 1 nodes

- **type 1: the root, or a node with all the a_i are 1;**
  - **This means j does not exist.**
  - **Nodes on the leftmost branch.**
  - **The leftmost child of a type 1 node except the root.**



Type 2 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 2: $\ell - j$ is zero or even;**
  - **type 2.1: $\ell - j = 0$.**
    - ▷ ***It is in the form of*** $\underline{1.1.1.\cdots.1.1.1}.a_\ell$ ***and*** $a_\ell \neq 1$.
    - ▷ ***The non-leftmost children of a type 1 node.***
  - **type 2.2: $\ell - j > 0$ and is even.**
    - ▷ ***It is in the form of*** $\underline{1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1}.a_\ell$.
    - ▷ ***Note, we will show*** $1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1$ ***is a type 3 node later.***
    - ▷ ***All of the children of a type 3 node.***

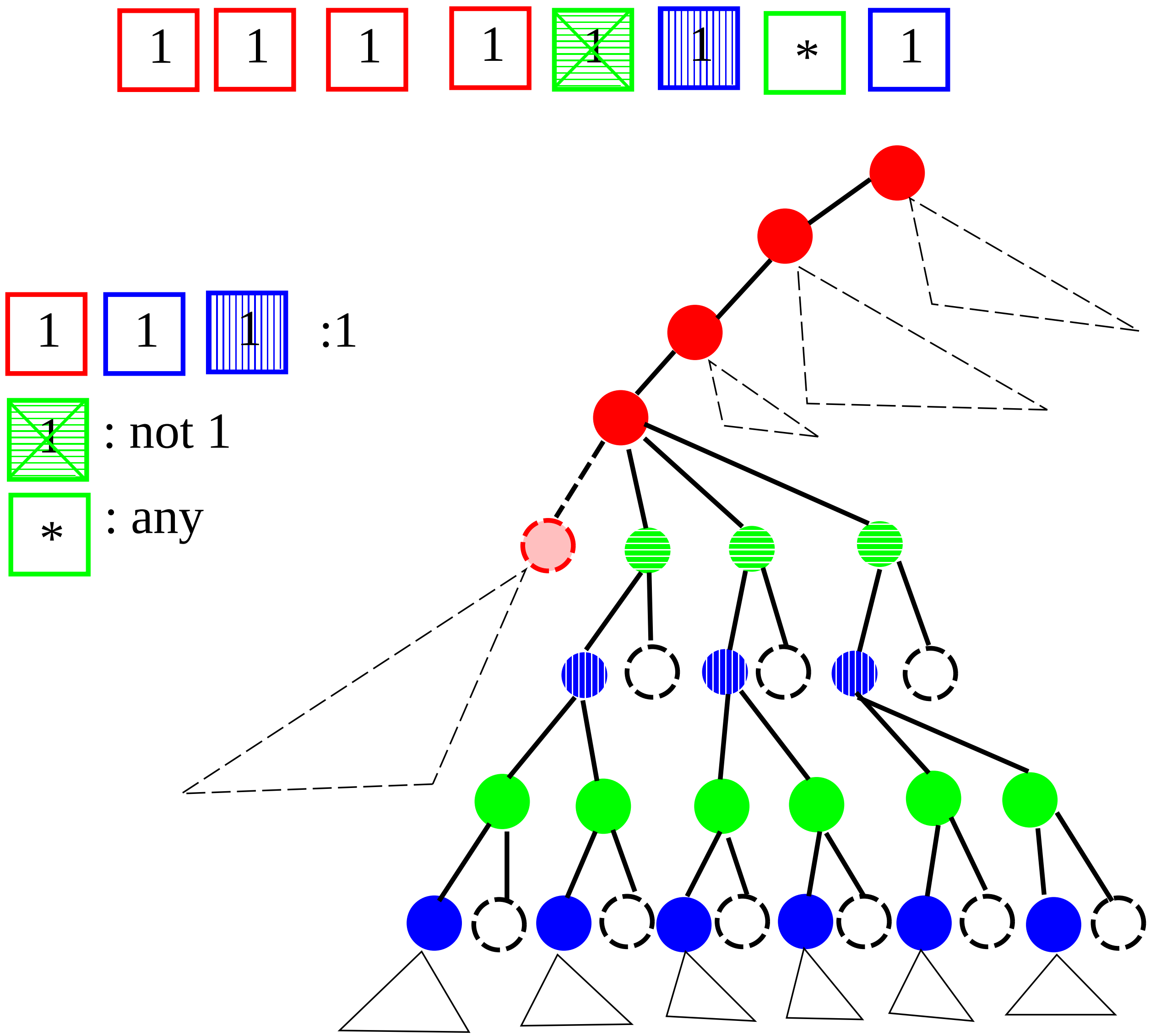
Type 3 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 3: $\ell - j$ is odd;**
  - **type 3.1: $\ell = j + 1$.**
    - ▷ ***It is of the form*** $\underline{1.1.\cdots.1.a_j}.1$
    - ▷ ***The leftmost child of a type 2.1 node.***
  - **type 3.2: $\ell > j + 1$.**
    - ▷ ***It is of the form*** $\underline{1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}}.1$
    - ▷ ***The leftmost child of a type 2.2 node.***

Comments

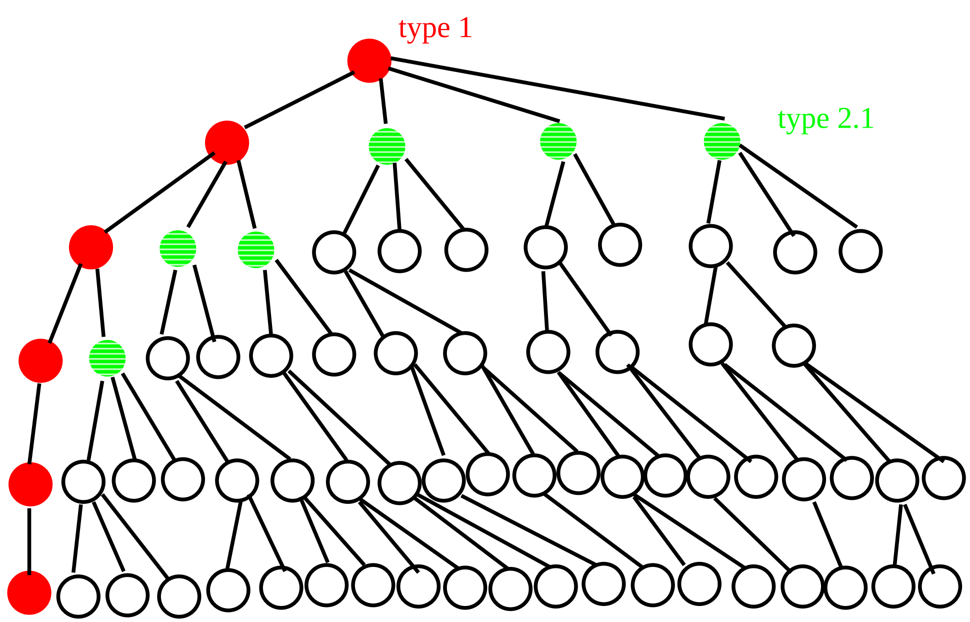
- **Nodes of the same have common properties.**
- **These properties can be used in solving other problems.**
  - **Efficient parallel processing.**
- **Main techniques used: you cannot have two consecutive non-1 numbers in the ID of a critical node.**

Illustration — critical nodes



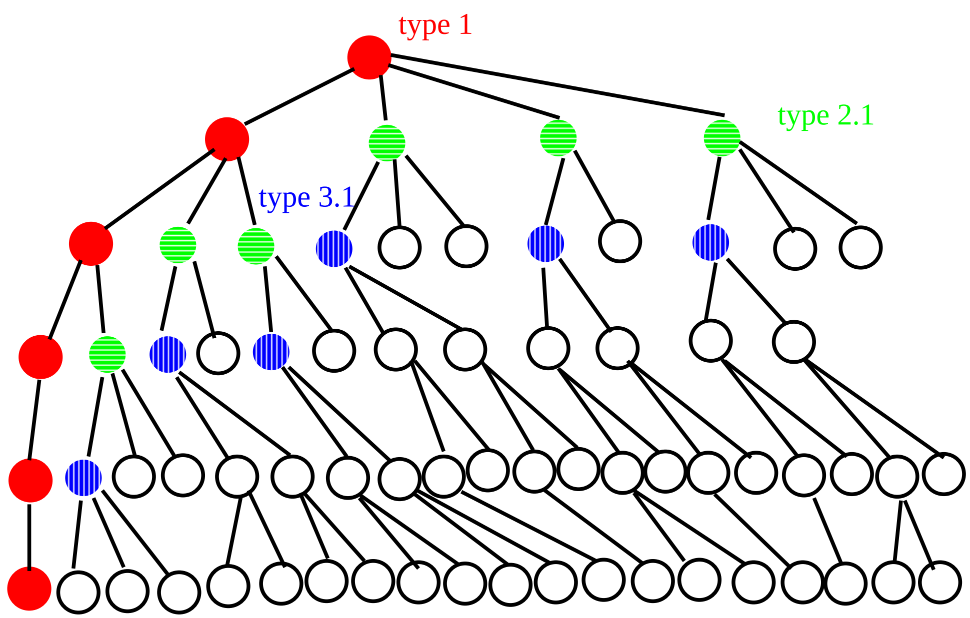
Type 2.1 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 2: $\ell - j$ is zero or even;**
  - **type 2.1: $\ell - j = 0$.**
    - ▷ ***Then $\ell = j$.***
    - ▷ ***It is in the form of $\underline{1.1.1.\cdots.1.1.1}.a_\ell$ and $a_\ell \neq 1$.***
    - ▷ ***The non-leftmost children of a type 1 node.***



Type 3.1 nodes

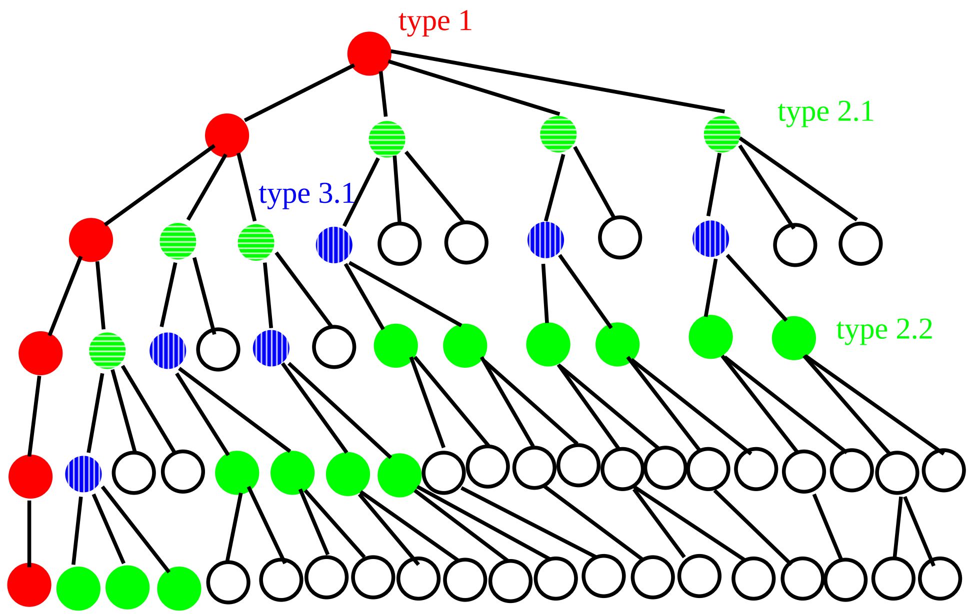
- **Classification of critical positions** $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ **where** j **is the least index such that** $a_j \neq 1$ **and** ℓ **is the last index.**
- **type 3:** $\ell - j$ **is odd;**
  - **type 3.1:** $\ell = j + 1$.
    - ▷ ***It is of the form*** $\underline{1.1.\cdots.1.a_j}.1$ ***and*** $a_\ell \neq 1$.
    - ▷ ***The leftmost child of a type 2.1 node.***



Type 2.2 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 2: $\ell - j$ is zero or even;**
  - **type 2.2: $\ell - j > 0$ and is even.**
    - ▷ ***The $IS1$ parties of a_j and a_{j+1} are different.***
      $\Longrightarrow$ Since $a_j \neq 1$, $a_{j+1} = 1$.
    - ▷ ***$(\ell - 1) - j$ is odd:***
      $\Longrightarrow$ The $IS1$ parties of $a_{\ell-1}$ and a_j are different.
      $\Longrightarrow$ Since $a_j \neq 1$, $a_{\ell-1} = 1$.
    - ▷ ***It is in the form of $\underline{1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1}.a_\ell$.***
    - ▷ ***Note, we will show $1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1$ is a type 3 node later.***
    - ▷ ***All of the children of a type 3 node.***

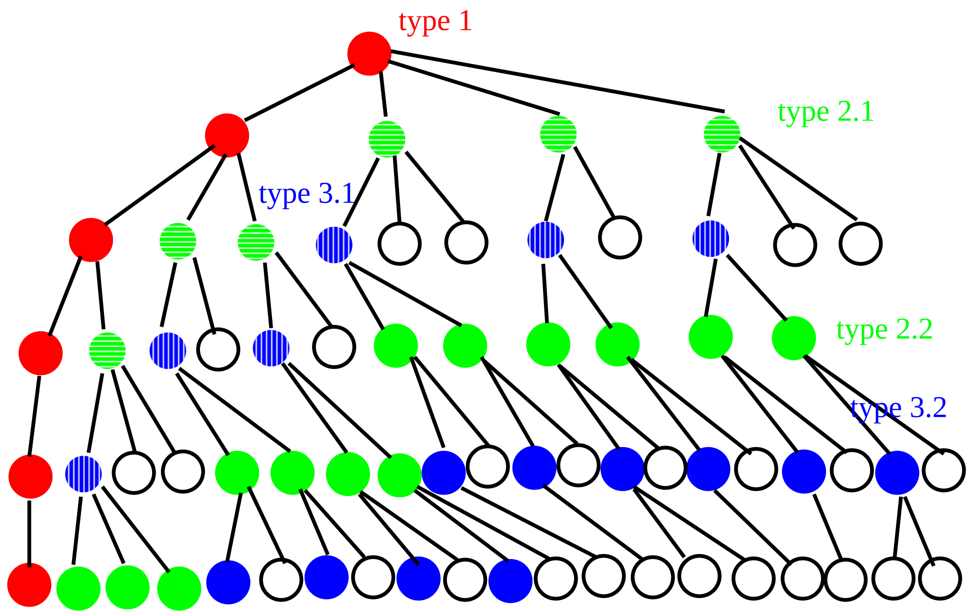
Type 2.2 nodes



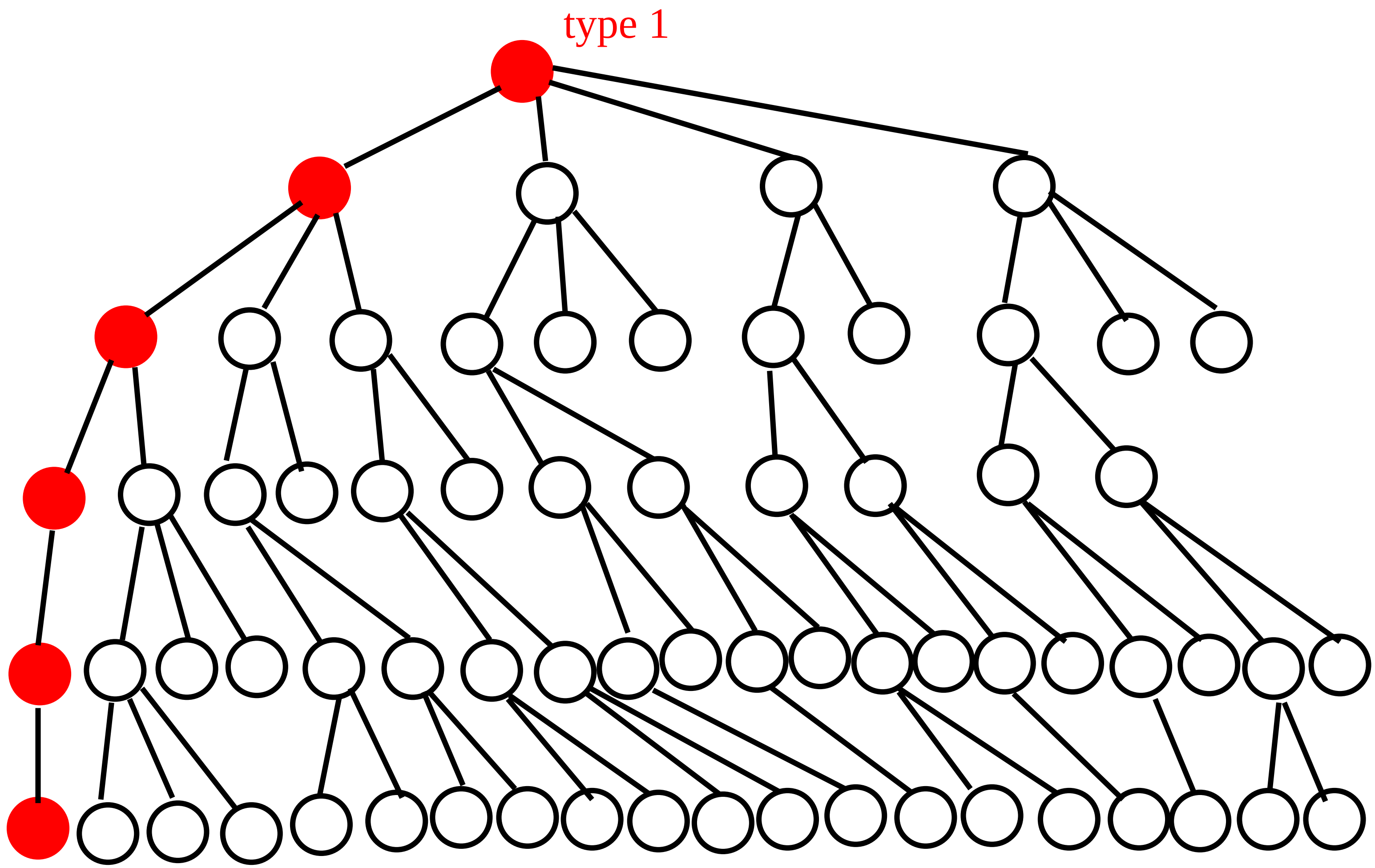
Type 3.2 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 3: $\ell - j$ is odd;**
  - $a_j \neq 1$ **and** $\ell - j$ **is odd**
    - ▷ ***Since this position is critical, the $IS1$ parities of a_j and a_ℓ are different.***
      $\Longrightarrow a_\ell = 1$
      $\Longrightarrow a_{j+1} = 1$
  - **It is in the form of**
    - ▷ $1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}.1.$
  - **The leftmost child of a type 2 node.**
  - **type 3.1: $\ell = j + 1$.**
    - ▷ ***It is of the form*** $\underline{1.1.\cdots.1.a_j}.1$
    - ▷ ***The leftmost child of a type 2.1 node.***
  - **type 3.2: $\ell > j + 1$.**
    - ▷ ***It is of the form*** $\underline{1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}}.1$
    - ▷ ***The leftmost child of a type 2.2 node.***

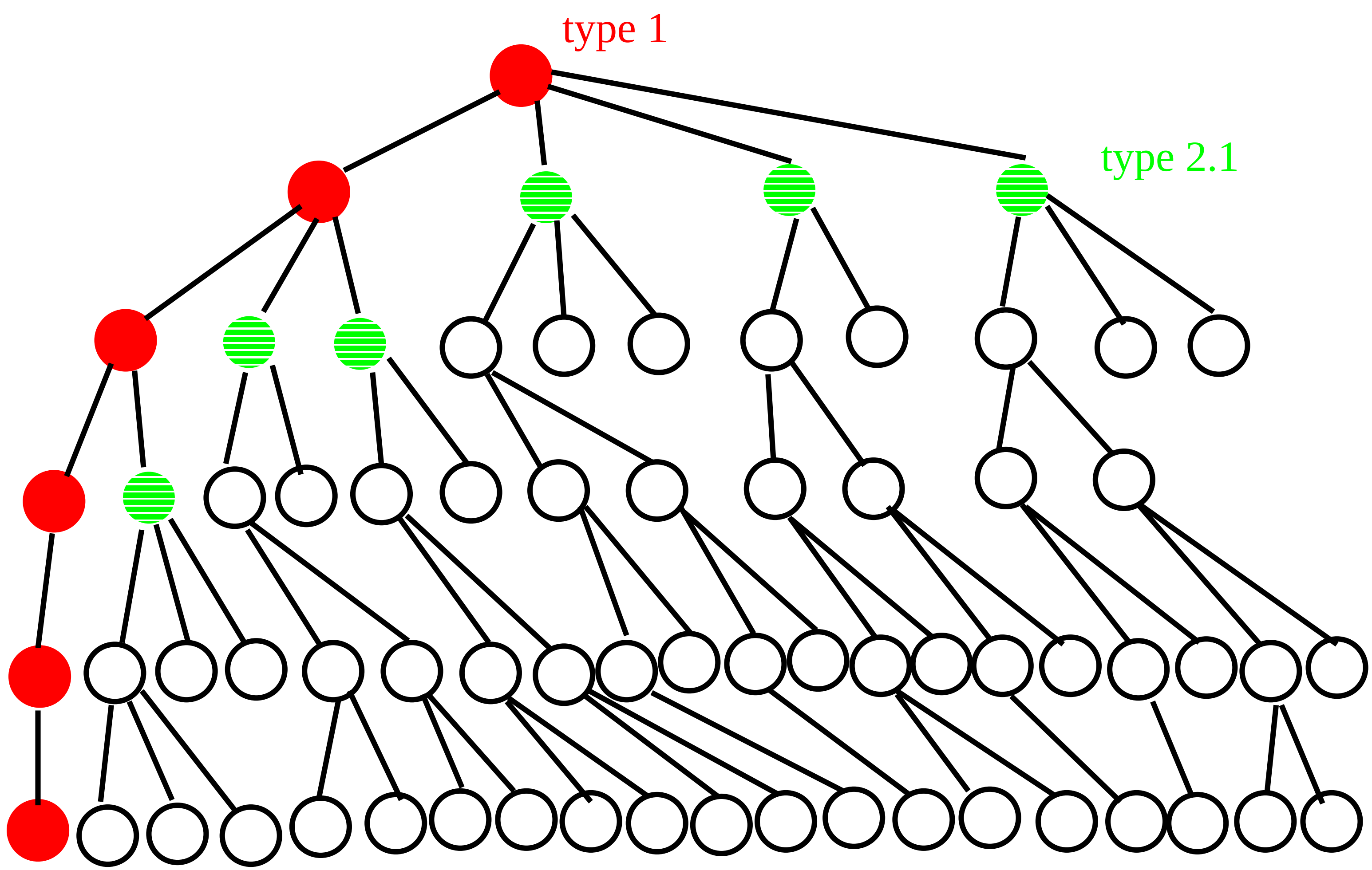
Type 3.2 nodes



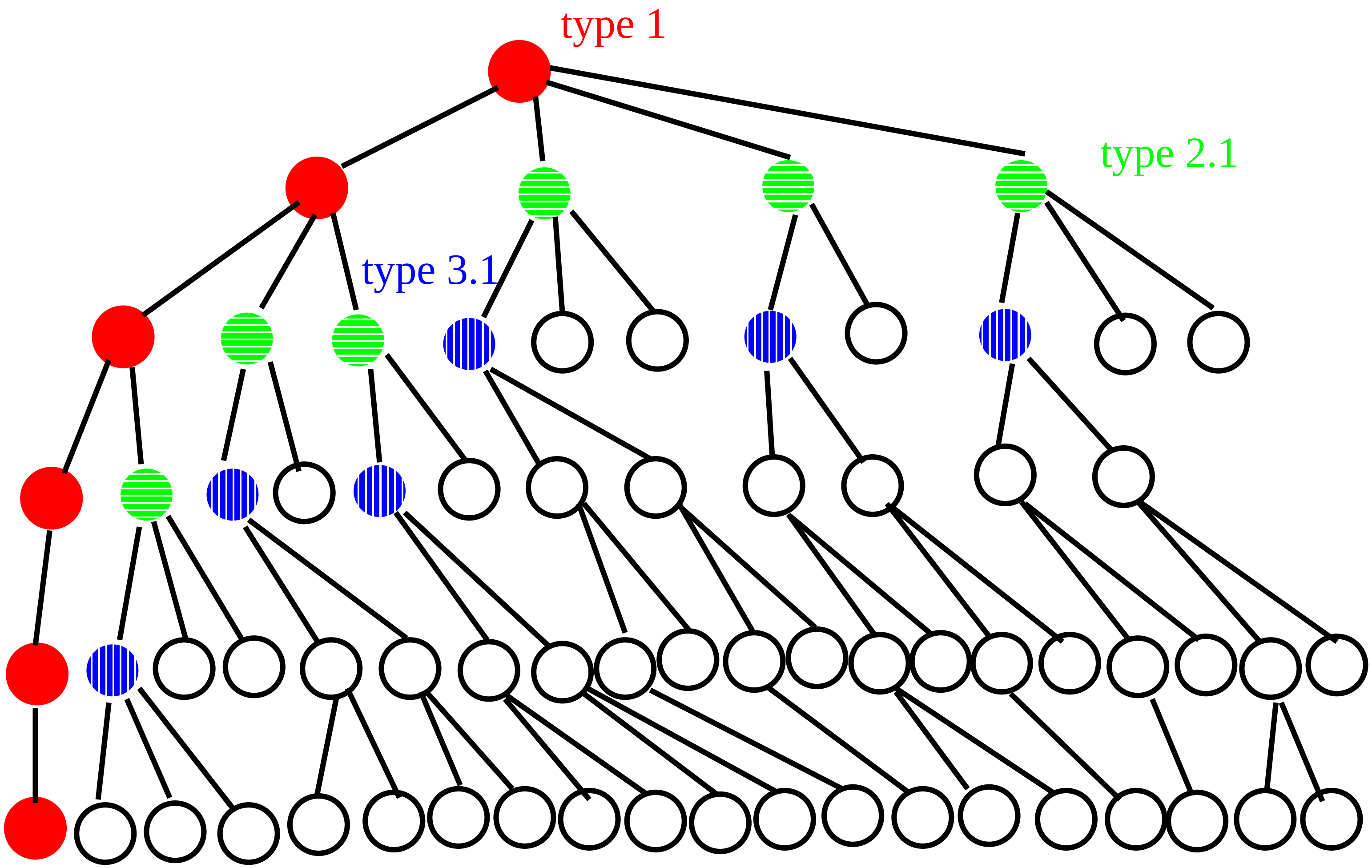
Illustration



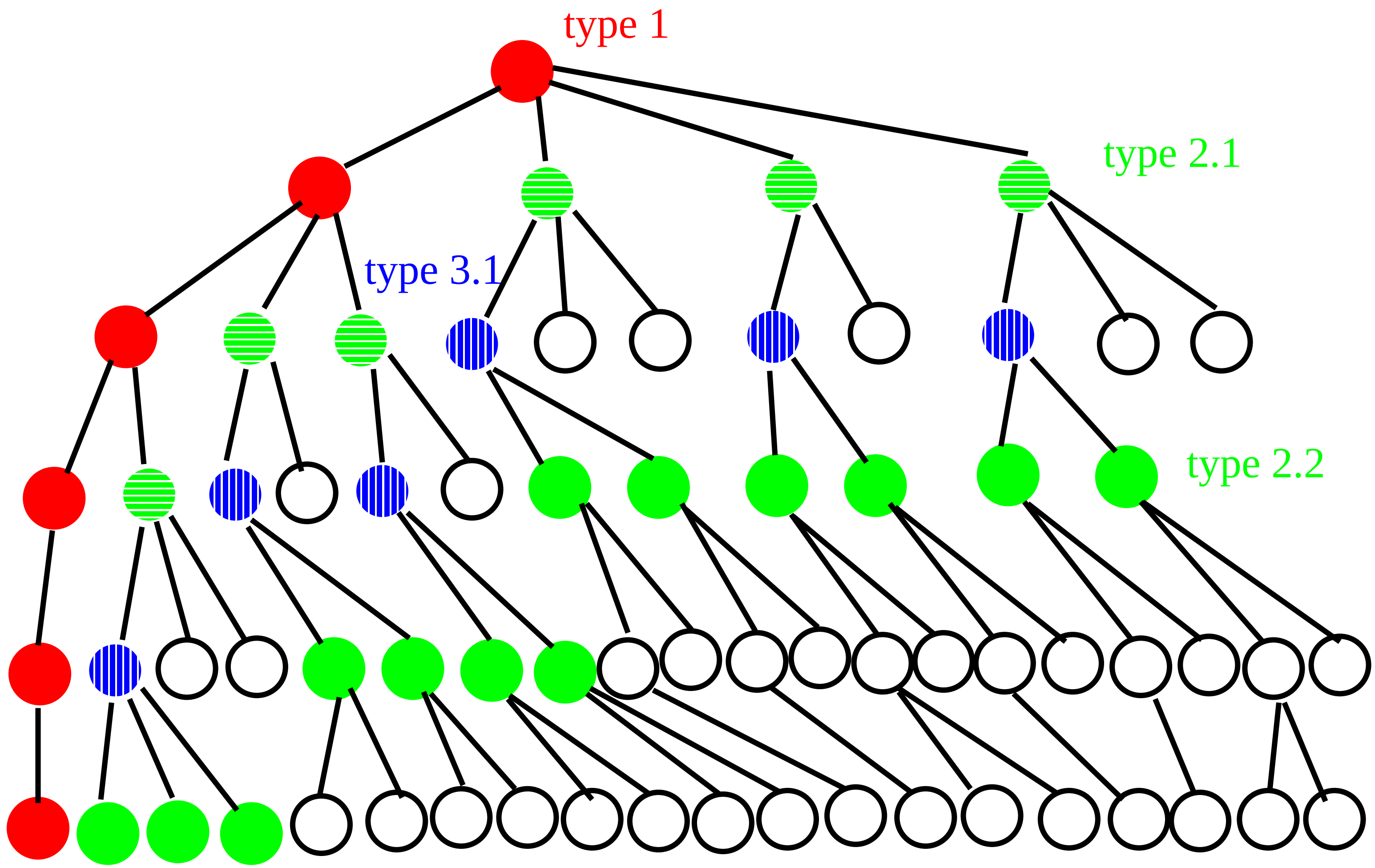
Illustration



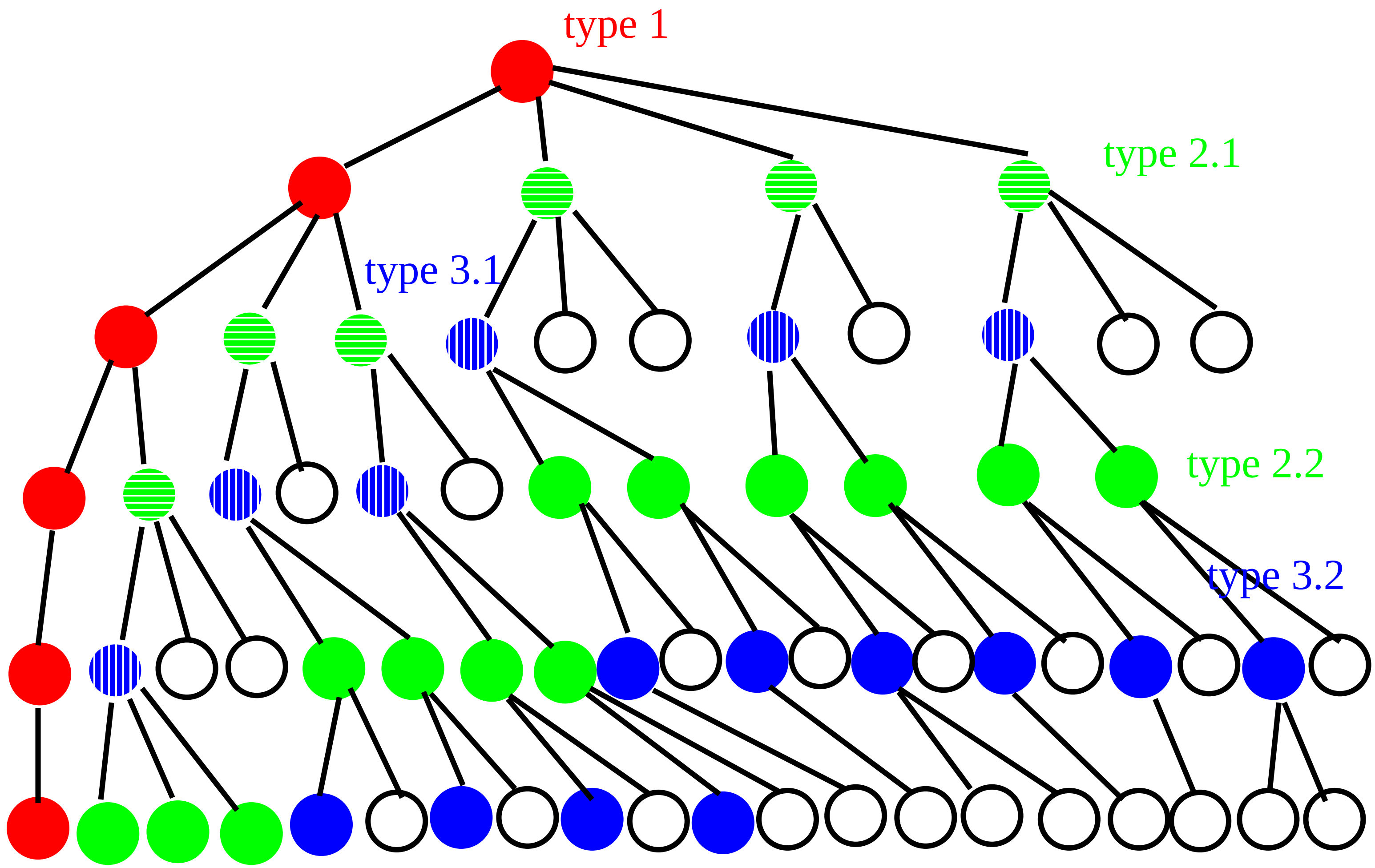
Illustration



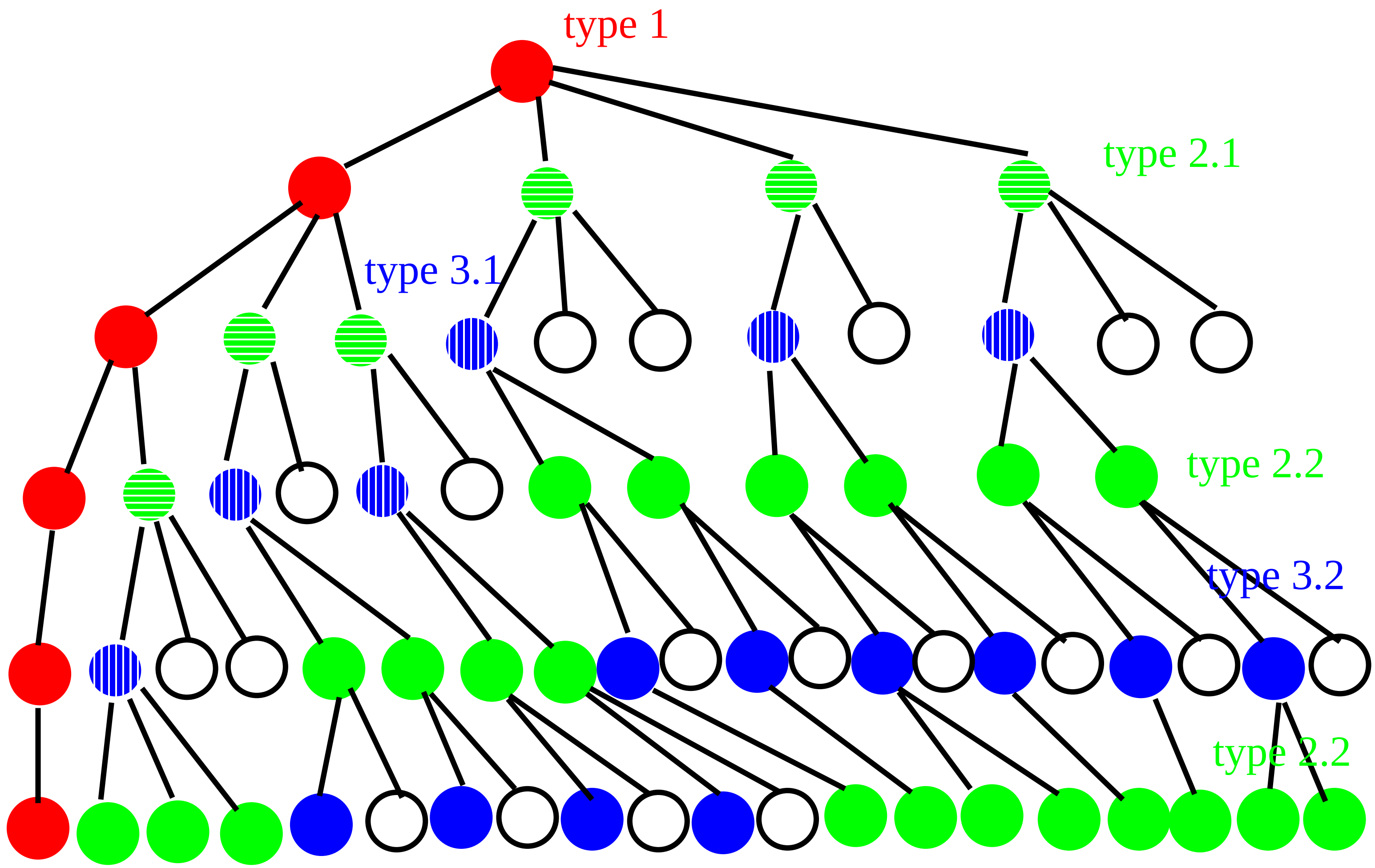
Illustration



Illustration



Illustration



Proof sketch for Theorem 1

- **Properties (invariants)**
  - **A type 1 position p is examined by calling $F2(p, -\infty, \infty)$**
    - ▷ ***p's first successor p_1 is of type 1***
    - ▷ $F(p) = -F(p_1) \neq \pm\infty$
    - ▷ ***p's other successors $p_2, \ldots, p_b$ are of type 2***
    - ▷ ***p_i, $i > 1$, are examined by calling $F2(p_i, -\infty, F(p_1))$***
  - **A type 2 position p is examined by calling $F2(p, -\infty, beta)$ where $-\infty < beta \leq F(p)$**
    - ▷ ***p's first successor p_1 is of type 3***
    - ▷ $F(p) = -F(p_1)$
    - ▷ ***p's other successors $p_2, \ldots, p_b$ are not examined***
  - **A type 3 position p is examined by calling $F2(p, alpha, \infty)$ where $\infty > alpha \geq F(p)$**
    - ▷ ***p's successors $p_1, \ldots, p_b$ are of type 2***
    - ▷ ***they are examined by calling $F2(p_1, -\infty, -alpha)$, $F2(p_2, -\infty, -\max\{m_1, alpha\}), \ldots,$ $F2(p_i, -\infty, -\max\{m_{i-1}, alpha\})$ where $m_i = F2(p_i, -\infty, -\max\{m_{i-1}, alpha\})$***
- **Using an inductive argument to prove all and also only critical positions are examined.**

Analysis: best case

- **Corollary 1: Assume each position has exactly b successors**
  - **The number of positions examined by the alpha-beta procedure on level i is exactly**
  $$b^{\lceil i/2 \rceil} + b^{\lfloor i/2 \rfloor} - 1.$$
- **Proof:**
  - **There are $b^{\lfloor i/2 \rfloor}$ sequences of the form $a_1.\cdots.a_i$ with $1 \le a_i \le b$ for all i such that $a_i = 1$ for all odd values of i.**
  - **There are $b^{\lceil i/2 \rceil}$ sequences of the form $a_1.\cdots.a_i$ with $1 \le a_i \le b$ for all i such that $a_i = 1$ for all even values of i.**
  - **We subtract 1 for the sequence $1.1.\cdots.1.1$ which are counted twice.**
- **Total number of nodes visited is**

$$\sum_{i=0}^{\ell} b^{\lceil i/2 \rceil} + b^{\lfloor i/2 \rfloor} - 1.$$

Analysis: average case

- **Assumptions: Let a random game tree be generated in such a way that**
  - **each position on level j has probability q_j of being nonterminal**
  - **has an average of b_j successors**
- **Properties of the above random game tree**
  - **Expected number of positions on level ℓ is $b_0 \cdot b_1 \cdots b_{\ell-1}$**
  - **Expected number of positions on level ℓ examined by an alpha-beta procedure assumed the random game tree is perfectly ordered is**

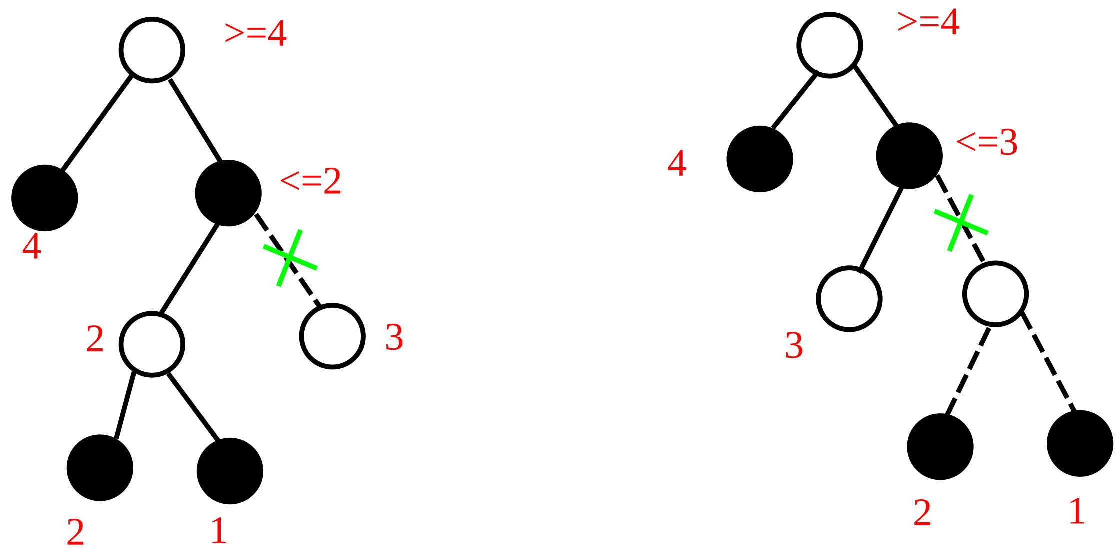
$$b_0 q_1 b_2 q_3 \cdots b_{\ell-2} q_{\ell-1} + q_0 b_1 q_2 b_3 \cdots q_{\ell-2} b_{\ell-1} - q_0 q_1 \cdots q_{\ell-1} \text{ if } \ell \text{ is even;}$$

$$b_0 q_1 b_2 q_3 \cdots q_{\ell-2} b_{\ell-1} + q_0 b_1 q_2 b_3 \cdots b_{\ell-2} q_{\ell-1} - q_0 q_1 \cdots q_{\ell-1} \text{ if } \ell \text{ is odd}$$

- **Proof sketch:**
  - **If x is the expected number of positions of a certain type on level j, then xb_j is the expected number of successors of these positions, and xq_j is the expected number of "numbered 1" successors.**
  - **The above numbers equal to those of Corollary 1 when $q_j = 1$ and $b_j = b$ for $0 \le j < \ell$.**

Perfect ordering is not always best

- **Intuitively, we may "think" alpha-beta pruning would be most effective when a game tree is perfectly ordered.**
  - **That is, when the first successor of every position is the best possible move.**
  - **This is not always the case!**



- **Truly optimum order of game trees traversal is not obvious.**

When is a branch pruned?

- **Assume a node r has two children u and v with u being visited before v using some move ordering.**
  - **Further assume u produced a new bound $bound$.**
- **Assume node v has a child w.**
  - **If the value new returned from w can cause a range conflict with $bound$, then branches of v later than w are cut.**
- **This means as long as the "relative" ordering of u and v are good enough, then we can have some cut-off.**
  - **There is no need for r to have the best move ordering.**

Theorem 2

- **Theorem 2: Alpha-beta pruning is optimum in the following sense:**
  - **Given any game tree and any algorithm which computes the value of the root position, there is a way to permute the tree**
    - *by reordering successor positions if necessary;*
  - **so that every terminal position examined by the alpha-beta method under this permutation is examined by the given algorithm.**
  - **Furthermore if the value of the root is not ∞ or $-\infty$, the alpha-beta procedure examines precisely the positions which are critical under this permutation.**

Variations of alpha-beta search

- **Initially, to search a tree with the root r by calling $F2(r,-\infty,+\infty)$.**
  - **What does it mean to search a tree with the root r by calling $F2(r,alpha,beta)$?**
    - ▷ ***To search the tree rooted at r requiring that the returned value to be within $alpha$ and $beta$.***
- **In an alpha-beta search with a pre-assigned window $[alpha, beta]$:**
  - **Failed-high means it returns a value that is larger than or equal to its upper bound $beta$.**
  - **Failed-low means it returns a value that is smaller than or equal to its lower bound $alpha$.**
- **Variations:**
  - **Brute force Nega-Max version: F**
    - ▷ ***Always finds the correct answer according to the Nega-Max formula.***
  - **Fail hard alpha-beta cut (Nega-Max) version: $F2$**
  - **Fail soft alpha-beta cut (Nega-Max) version: $F3$**

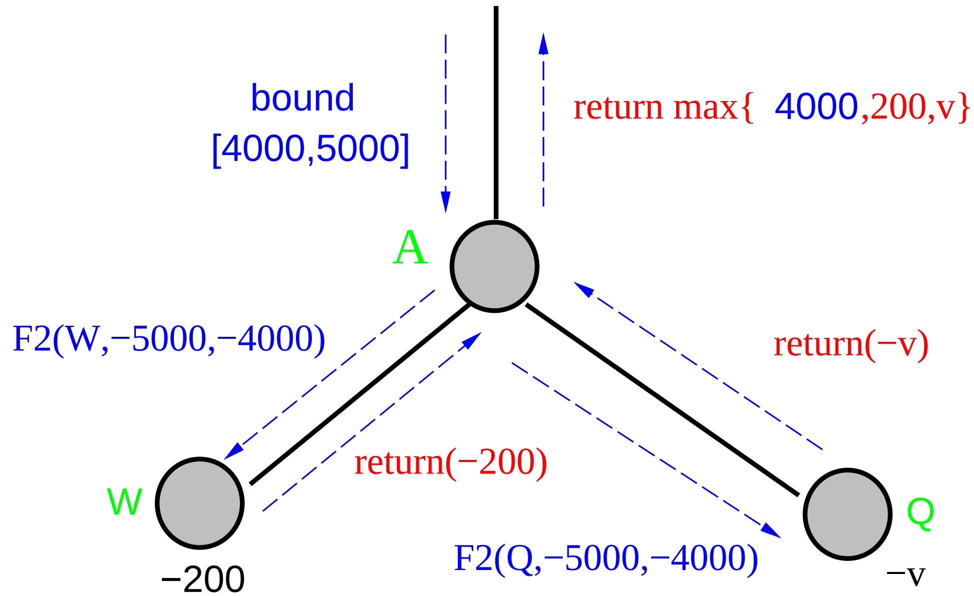
Fail hard version

- **Original version.**
- **Algorithm** $F2$**(position** p**, value** $alpha$**, value** $beta$**)**
  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$ **// a terminal node**
    or depth reaches the cutoff threshold // from iterative deepening
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return** $h(p)$ **else**
  - **begin**
    - ▷ $m := alpha$ ***// hard initial value***
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ $t := -F2(p_i, -beta, -m)$
    - ▷ ***if*** $t > m$ ***then*** $m := t$ ***// the returned value is "used"***
    - ▷ ***if*** $m \geq beta$ ***then return(***m***) // cut off***
    - ▷ ***end***
  - **end**
  - **return** m

Properties and comments

- **Properties:**
  - $alpha < beta$
  - $F2(p, alpha, beta) = alpha$ **if** $F(p) \leq alpha$
  - $F2(p, alpha, beta) = F(p)$ **if** $alpha < F(p) < beta$
  - $F2(p, alpha, beta) = beta$ **if** $F(p) \geq beta$
  - $F2(p, -\infty, +\infty) = F(p)$
- **Comments:**
  - $F2(p, alpha, beta)$**: find the best possible value according to a nega-max formula for the position** p **with the constraints that**
    - ▷ ***If*** $F(p)$ ***is less than the lower bound*** $alpha$***, then*** $F2(p, alpha, beta)$ ***returns with a value*** $alpha$ ***from a terminal position whose value is*** $\leq alpha$***.***
    - ▷ ***If*** $F(p)$ ***is more than the upper bound*** $beta$***, then*** $F2(p, alpha, beta)$ ***returns with value*** $beta$ ***from a terminal terminal position whose value is*** $\geq beta$***.***
  - **The meanings of** $alpha$ **and** $beta$ **during searching:**
    - ▷ ***For a max node: the current best value is at least*** $alpha$***.***
    - ▷ ***For a min node: the current best value is at most*** $beta$***.***
  - $F2$ **always finds a value that is within** $alpha$ **and** $beta$**.**
    - ▷ ***The bounds are hard, i.e., cannot be violated.***

Fail hard version: Example



- **As long as the value of the leaf node W is less than the current $alpha$ value, the returned value of A will be at least the returned value of W.**

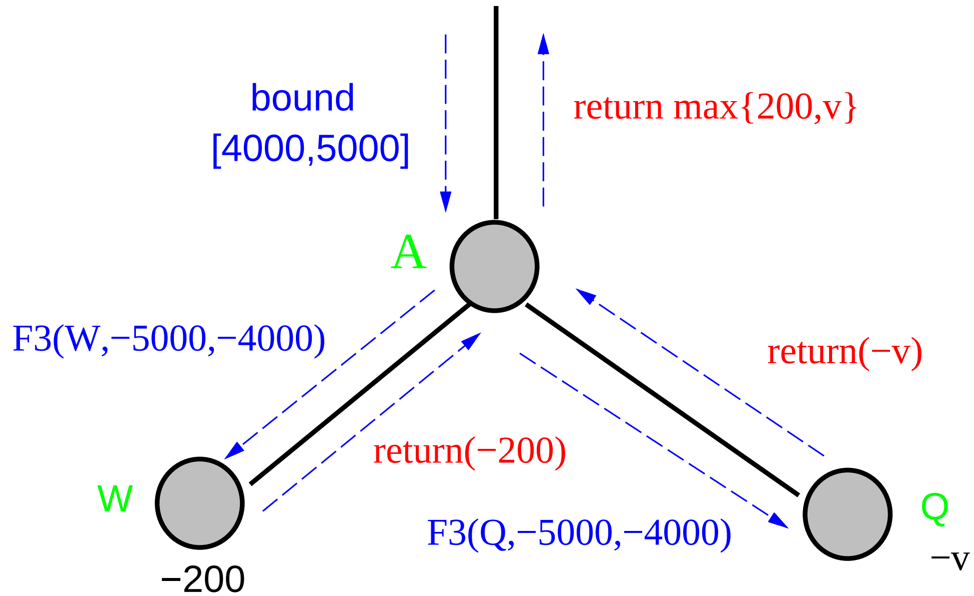
Fail soft version

- **Algorithm** $F3$**(position** p**, value** $alpha$**, value** $beta$**)**
  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$ **// a terminal node**
    or depth reaches the cutoff threshold // from iterative deepening
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return** $h(p)$ **else**
  - **begin**
    - ▷ $m := -\infty$ *// soft initial value*
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ $t := -F3(p_i, -beta, -\max\{m, alpha\})$
    - ▷ ***if*** $t > m$ ***then*** $m := t$ *// the returned value is "used"*
    - ▷ ***if*** $m \geq beta$ ***then return***(m) *// cut off*
    - ▷ ***end***
  - **end**
  - **return** m

Properties and comments

- **Properties:**
  - $alpha < beta$
  - $F3(p, alpha, beta) \leq alpha$ **if** $F(p) \leq F3(p, alpha, beta) \leq alpha$
  - $F3(p, alpha, beta) = F(p)$ **if** $alpha < F(p) < beta$
  - $F3(p, alpha, beta) \geq beta$ **if** $F(p) \geq F3(p, alpha, beta) \geq beta$
  - $F3(p, -\infty, +\infty) = F(p)$
- **$F3$ finds a "better" value when the value is out of the search window.**
  - **Better means a tighter bound.**
    - ▷ ***The bounds are soft, i.e., can be violated.***
  - **When it fails high, $F3$ normally returns a value that is higher than that of $F2$.**
    - ▷ ***Never higher than that of F!***
  - **When it fails low, $F3$ normally returns a value that is lower than that of $F2$.**
    - ▷ ***Never lower than that of F!***

Fail soft version: Example

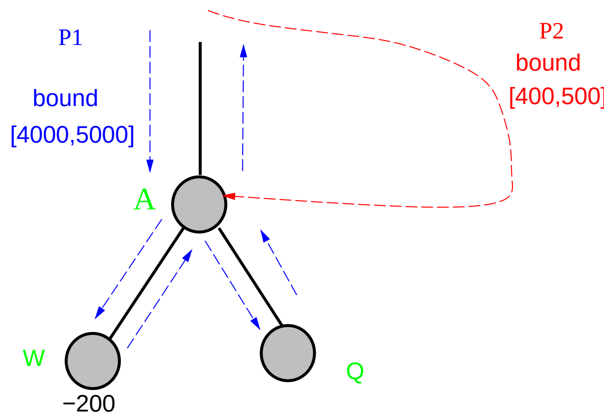


- **Let the value of the leaf node W be u.**
- **If $u < alpha$, then the branch at W will have a returned value of at least u.**

Comparisons between $F2$ and $F3$

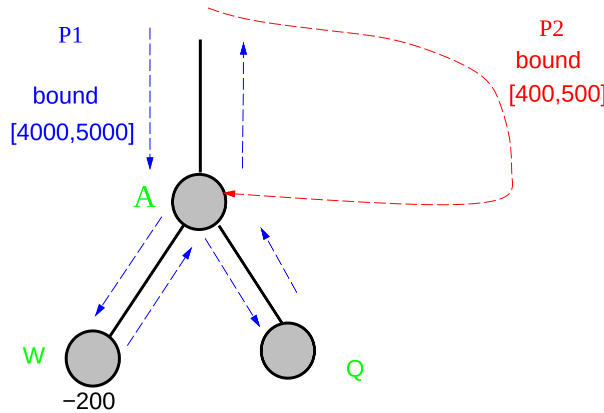
- Both versions find the corrected value v if v is within the window $[alpha, beta]$.
- Both versions scan the same set of nodes during searching.
  - ▷ *If the returned value of a subtree is decided by a cut, then $F2$ and $F3$ return the same value.*
- $F3$ provides more information when the true value is out of the pre-assigned search window.
  - Can provide a feeling on how bad or good the game tree is.
  - Use this "better" value to guide searching later on.
- $F3$ saves about 7% of time than that of $F2$ when a transposition table is used to save and re-use searched results [Fishburn 1983].
  - A transposition table is a data structure to record the results of previous searched results.
  - The entries of a transposition table can be efficiently accessed, i.e., read and write, during searching.
  - Need an efficient addressing scheme, e.g., hash, to translate between a position and its address.

$F2$ and $F3$: Example (1/2)



- **Assume the node A can be reached from the starting position using path P_1 and path P_2.**
  - **If W is visited first along P_1 with a bound of $[4000, 5000]$, and returns a value of 200, then**
    - ▷ *the returned value of W, 200, is stored into the transposition table.*
  - **If A is visited again along P_2 with a bound of $[400, 500]$, then a better value of previously stored value of W helps to decide whether the subtree rooted at W needs to be searched again.**

$F2$ and $F3$: Example (2/2)



- **Fail soft version has a chance to record a better value to be used later when this position is revisited.**
  - **If A is visited again along P_2 with a bound of $[400, 500]$, then**
    - ▷ ***it does not need to be searched again, since the previous stored value of W is -200.***
  - **However, if the value of W is 450, then it needs to be searched again.**
- **The fail hard version does not store the returned value of W after its first visit since this value is less than $alpha$.**

Questions

- **What move ordering is good?**
  - **It may not be good to search the best possible move first.**
  - **It maybe better to cut off a branch with more nodes first.**
- **How about the case when the tree is not uniform?**
- **What is the effect of using iterative-deepening alpha-beta cut off?**
- **How about the case for searching a game graph instead of a game tree?**
  - **Can some nodes be visited more than once?**

References and further readings

* D. E. Knuth and R. W. Moore. An analysis of alpha-beta pruning. *Artificial Intelligence*, 6:293–326, 1975.
* John P. Fishburn. Another optimization of alpha-beta search. *SIGART Bull.*, (84):37–38, 1983.
- J. Pearl. The solution for the branching factor of the alpha-beta pruning algorithm and its optimality. *Communications of ACM*, 25(8):559–564, 1982.