Two-Player Perfect Information Games: A Brief Survey

Tsan-sheng Hsu

徐讚昇

tshsu@iis.sinica.edu.tw

http://www.iis.sinica.edu.tw/~tshsu

Abstract

- Domain: two-player games.
- Which game characters are predominant when the solution of a game is the main target?
  - It is concluded that decision complexity is more important than state-space complexity.
  - There is a trade-off between knowledge-based methods and brute-force methods.
  - There is a clear correlation between the first-player's initiative and the necessary effort to solve a game.

Definitions (1/4)

- **Domain: 2-person zero-sum games with perfect information.**
  - **Result: win, loss or draw.**
  - **Zero-sum means one player's loss is exactly the other player's gain, and vice versa.**
    - ▷ ***There is no way for both players to win at the same time.***
- **Initiative: the right to move first.**
- **Game-theoretic value of a game: the outcome, i.e., win, loss or draw, when all participants play optimally.**
  - **Classification of games' solutions according to L.V. Allis [Ph.D. thesis 1994] if they are considered solved.**
    - ▷ ***Ultra-weakly solved: the game-theoretic value of the initial position has been determined.***
    - ▷ ***Weakly solved: for the initial position a strategy has been determined to achieve the game-theoretic value against any opponent.***
    - ▷ ***Strongly solved: a strategy has been determined for all legal positions.***
  - **The game-theoretical values of most games are unknown or are only known for some legal positions.**
    - ▷ ***A legal position is one that can be reached from the initial position.***

Definitions (2/4)

- **State-space complexity of a game: the number of the legal positions in a game.**
- **Game-tree (or decision) complexity of a game: the number of the nodes in a solution search tree.**
  - **Actually, it is usually a game graph, not tree.**
  - **A solution search tree is a tree where the game-theoretic value of the root position can be decided.**
  - **Each node in the tree is a legal position. The children of a parent node P are the positions that P can reach in one step.**
  - **Some legal states may not be in a solution search tree.**
    - ▷ *These are unreasonable positions.*
  - **Some children of a node may not be in a solution search tree.**
- **A fair game: the game-theoretic value is draw and both players have roughly an equal probability to make a mistake.**
  - **Examples:**
    - ▷ *Paper-scissor-stone.*
    - ▷ *Roll a dice and the one getting a larger number wins.*
  - **Many popular games are not fair.**
  - **It is difficult to prove a non-trivial game is fair or to design a fair one.**

Definitions (3/4)

- **A convergent game: the size of the state space decreases as the game progresses.**
  - **Start with many pieces on the board and pieces are gradually removed during the course of the game.**
    - ▷ *Example: Checkers.*
  - **It means the number of possible configurations decreases as the game progresses.**
- **A divergent game: the size of the state space increases as the game progresses.**
  - **May start with an empty board, and pieces are gradually added during the course of the game.**
    - ▷ *Example: Connect-5 before the board is almost filled.*
  - **It means the number of possible configurations increases as the game progresses.**

Definitions (4/4)

- **A game may be convergent at one stage and then divergent at other stage.**
  - **Most games are dynamic.**
  - **For the game of Tic-Tac-Toe, assume you have played x plys with x being even.**
    - *Then you have a possible of*

$$\binom{9}{x/2}\binom{9-x/2}{x/2}$$

    different configurations.
  - **This number is not monotone increasing or decreasing.**

Predictions made in 1990

- **Predictions were made in 1990 [Allis et al 1991] for the year 2000 concerning the expected playing strength of computer programs.**

solved	over champion	world champion	grand master	amateur
Connect-four	Checkers $(8 * 8)$	Chess	Go $(9 * 9)$	Go $(19 * 19)$
Qubic	Renju	Draughts $(10 * 10)$	Chinese chess	
Nine Men's Morris	Othello		Bridge	
Go-Moku	Scrabble			
Awari	Backgammon			

  - ▷ *Over champion means definitely over the best human player.*
  - ▷ *World champion means equaling to the best human player.*
  - ▷ *Grand master means beating most human players.*

A double dichotomy of the game space

$\log\log(\text{state-space complexity}) \rightarrow$

category 3 if solvable at all, then by knowledge-based methods	**category 4** currently unsolvable by any method
category 1 solvable by any method	**category 2** if solvable at all, then by brute-force methods

$\log\log(\text{game-tree complexity}) \rightarrow$

Questions to be researched

- **Can perfect knowledge obtained from solved games be translated into rules and strategies which human beings can assimilate?**
- **Are such rules generic, or do they constitute a multitude of ad hoc recipes?**
- **Can methods be transferred between games?**
  - **More specifically, are there generic methods for all category-n games, or is each game in a specific category a law unto itself?**

Convergent games

- **Since most games are dynamic, here we consider games whose ending phases are convergent.**
  - **Can be solved by the method of endgame databases if we can enumerate and store all possible positions at a certain stage.**
- **Problems solved:**
  - **Nine Men's Morris: in the year 1995, a total of 7,673,759,269 states.**
    - ▷ ***The game theoretic value is draw.***
  - **Mancala games**
    - ▷ ***Awari: in the year 2002.***
    - ▷ ***Kalah: in the year 2000 upto, but not equal, Kalah(6,6)***
  - **Checkers**
    - ▷ ***By combining endgame databases, middle-game databases and verification of opening-game analysis.***
    - ▷ ***Solved the so called 100-year position in 1994.***
    - ▷ ***The game is proved to be a draw in 2007.***
  - **Chess endgames**
  - **Chinese chess endgames**

Divergent games

- Since most games are dynamic, here we consider games whose INITIAL phases are divergent.
- Connection games
  - Connect-four $(6 * 7)$
  - Qubic $(4 * 4 * 4)$
  - Go-Moku $(15 * 15)$
  - Renju
  - k-in-a-row games
  - Hex $(10 * 10$ or $11 * 11)$
- Polynmino games: place pieces inside a board without overlapping and alternatively until one cannot place more.
  - Pentominoes
  - Domineering
- Othello
- Chess
- Chinese chess
- Shogi
- Go

Connection games (1/2)

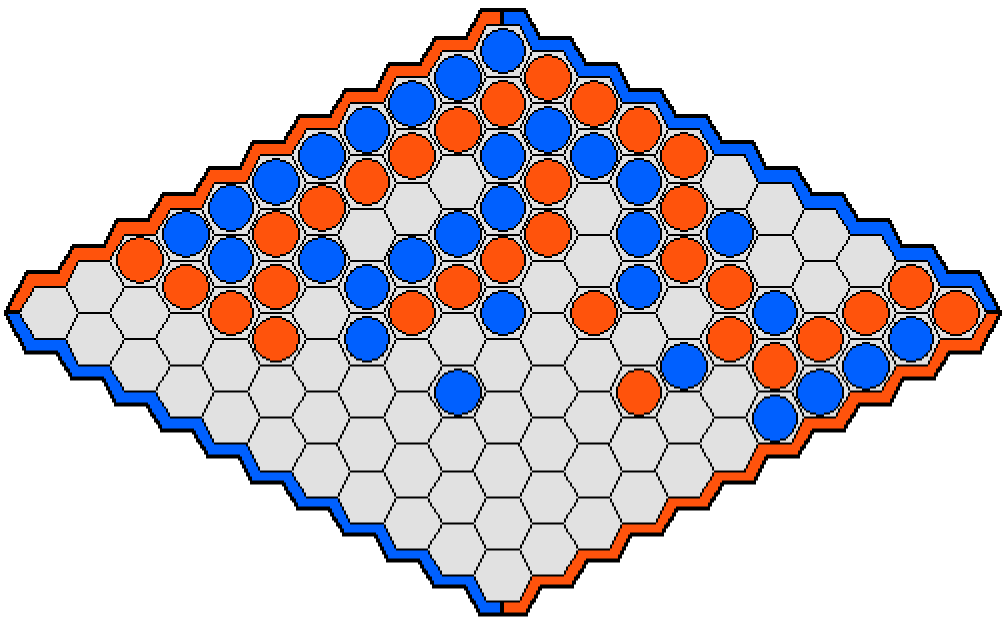
- **Connect-four** ($6 * 7$)
  - **Solved by J. Allen in 1989 using a brute-force depth first search with alpha-beta pruning, a transposition table, and killer-move heuristics.**
  - **Also solved by L.V. Allis in 1988 using a knowledge-based approach by combining 9 strategic rules that identify potential threats of the opponent.**
    - ▷ *Threats are something like forced moved or moves you have little choices.*
    - ▷ *Threats are moves with predictable counter-moves.*
  - **It is first-player win.**
  - **Weakly solved on a SUN-4 workstation using 300+ hours.**
- **Qubic** ($4 * 4 * 4$)
  - **A three-dimensional version of Tic-Tac-Toe.**
  - **Connect-four played on a** $4 * 4 * 4$ **game board.**
  - **Solved in 1980 by O. Patashnik by combining the usual depth-first search with expert knowledge for ordering the moves.**
    - ▷ *It is first-player win for the 2-player version.*

Connection games (2/2)

- **Go-Moku ($15 * 15$)**
  - **First-player win.**
  - **Weakly solved by L.V. Allis in 1995 using a combination of threat-space search and database construction.**
- **Renju**
  - **Does not allow the first player to play certain moves.**
  - **An *asymmetric* game.**
  - **Weakly solved by Wágner and Viráag in 2000 by combining search and knowledge.**
    - ▷ ***Took advantage of an iterative-deepening search based on threat sequences up to 17 plies.***
    - ▷ ***It is still first-player win.***
- **k-in-a-row games**
  - **mnk-Game: a game playing on a board of m rows and n columns with the goal of obtaining a straight line of length k.**
  - **Variations: first ply picks only one stone, the rest picks two stones in a ply.**
    - ▷ ***Connect 6.***
    - ▷ ***Try to balance the advantage of the initiative!***

Hex ($10 * 10$ or $11 * 11$)

- **Properties:**
  - **It is a finite game.**
  - **It is not possible for both players to win at the same time.**
  - **Exactly one of the players can win.**



Red won

Courtesy of ICGA web site

Proof on exactly one player win (1/2)

- **It is easy to known there cannot be two winners.**
  - ▷ ***When the first player wins, allow the second player to play one more time. If the second player also wins, then the game is tie.***
- **A topological argument.**
  - **A vertical chain can only be cut by a horizontal chain and vice versa because each cell is connected with 6 adjacent cells.**
    - ▷ ***Note if a cell has 4 neighbors as in the case of Go, then it is possible to cut off a vertical chain by cells that are not horizontally connected and vice versa.***
- **Other arguments such as one using graph theory exist.**

Proof on exactly one player win (2/2)

- **We then prove there is at least one winner.**
  - **Assume there is no winner.**
- **W.l.o.g. let R be the set of red cells that can be reached by chains originated from the rightmost column.**
  - **R must contain a cell of the leftmost column; otherwise we have a contradiction.**
- **Let $N(R)$ be the blue cells that can be reached by R originated from the rightmost column.**
  - **$N(R)$ must contain a cell in the top row.**
    - ▷ ***Otherwise, R contains all cells in the first row, which is a contradiction.***
  - **$N(R)$ must contain a cell in the bottom row.**
    - ▷ ***Otherwise, R contains all cells in the bottom row, which is a contradiction.***
  - **$N(R)$ must be connected.**
    - ▷ ***Otherwise, R can advance further.***
  - **Hence $N(R)$ is a blue winning chain.**

Illustration of the ideas (1/3)

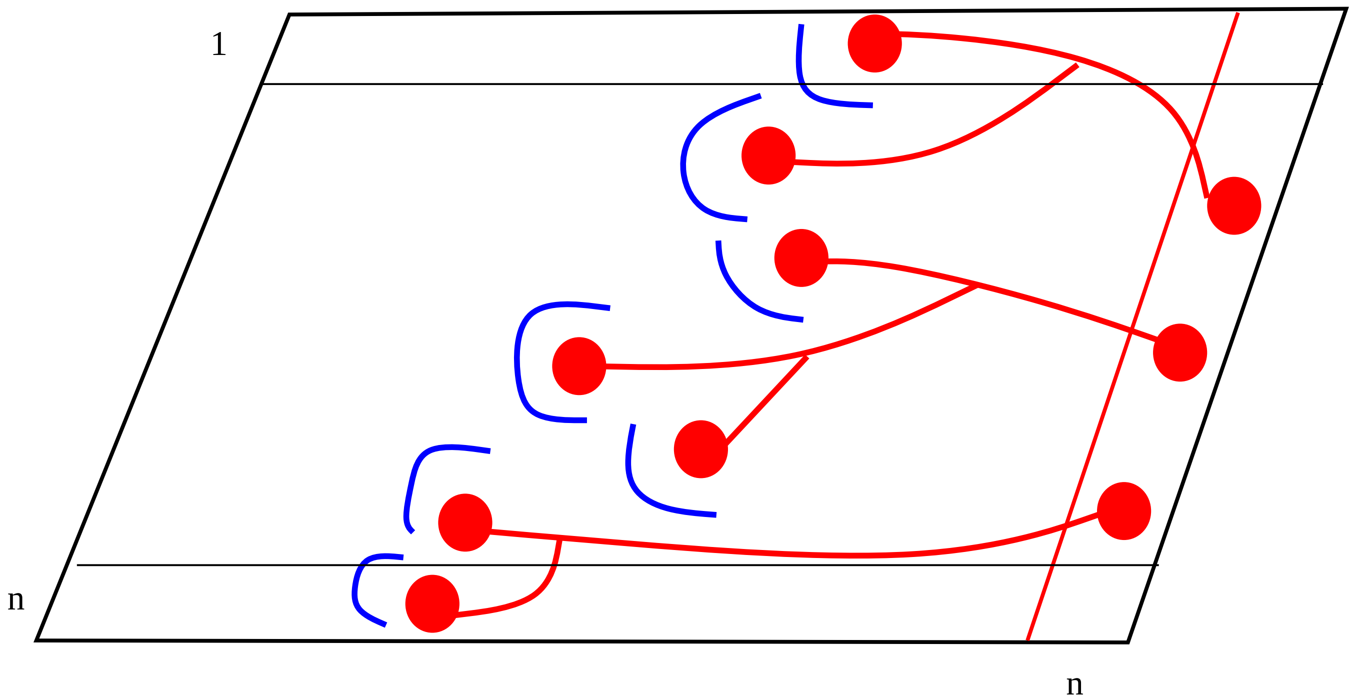


Illustration of the ideas (2/3)

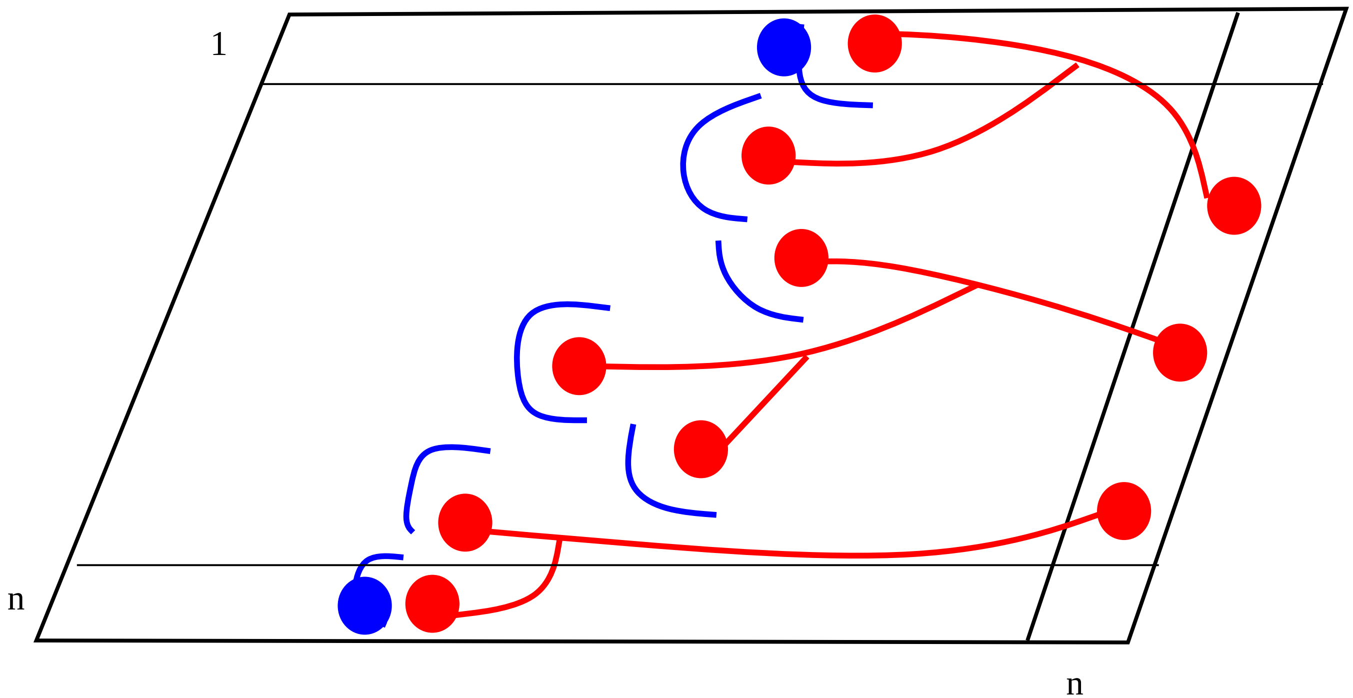
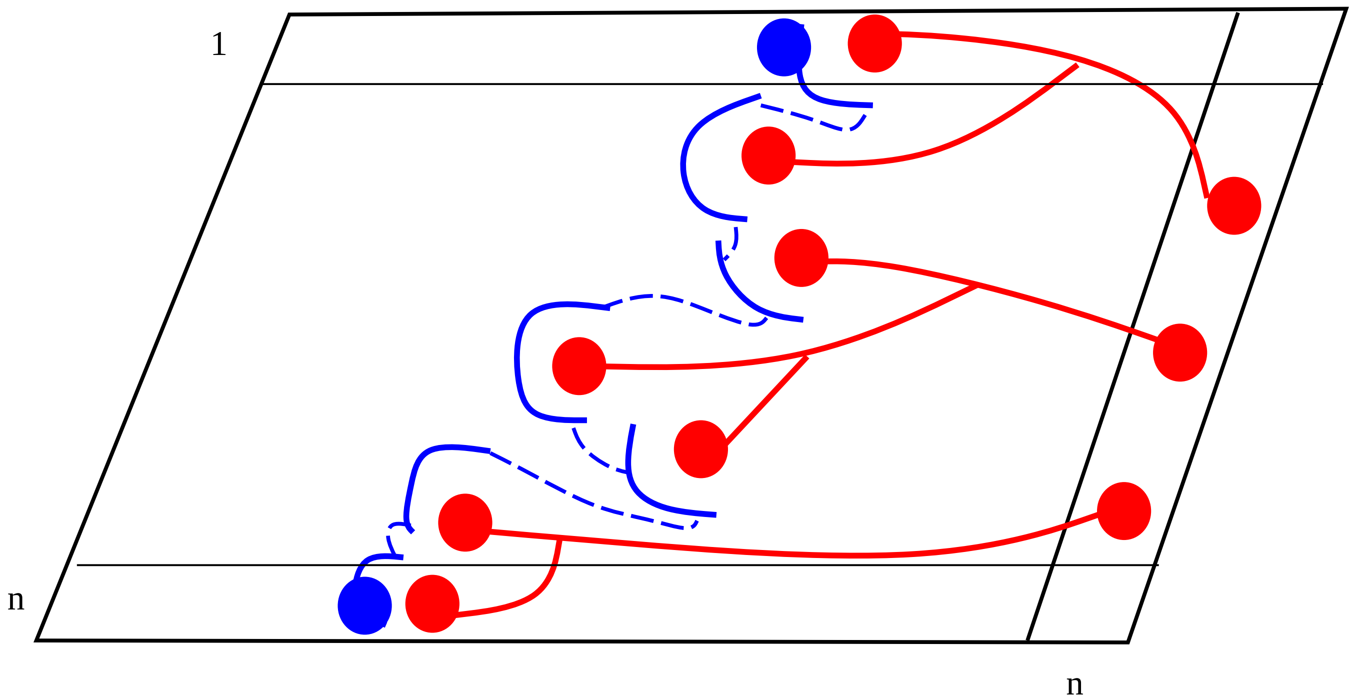


Illustration of the ideas (3/3)



Strategy-stealing argument (1/5)

- **The** ***unrestricted form of Hex is a first-player win game. using the "strategy-stealing" argument made by John Nash in 1949.***
  - **If there is a winning strategy for the second player, the first player can still win by making an arbitrary first move and using the second-player strategy from then on.**
    - ▷ *The first player ignores the arbitrary first move by assuming that move does not exist.*
    - ▷ *Hence the second move made by the second player becomes the first move.*
    - ▷ *The third move made by the first player becomes the second move.*
  - **If using the second-player strategy requires playing the chosen first move or any move played before, then make another arbitrary move.**
    - ▷ *An arbitrary extra move can never be a disadvantage in Hex.*
  - **We have obtained a contradiction, and thus the second player cannot win from the initial empty board.**
  - **Since we have proved there is no draw, and there is always a winner, and both players cannot win at the same time, the first player must have a winning strategy from the initial empty board.**

Strategy-stealing argument (2/5)

- **Assume the second player P_2 has a winning function $f(B)$ that tells the next ply towards winning when seeing the board B.**
  - **Assume the initial board position is B_0 which is an empty board.**
  - **$f(B)$ has a value only for the case B is a legal position for the second player.**
    - ***$f(B)$ returns the x-y coordinates of a location and the color of the piece to play.***
  - **$rev(m)$: flip the color and coordinate of a ply m to play.**
    - ***Let $m = (x_m, y_m)$ be the location to play.***
    - ***Let c be the color of the piece to play.***
    - ***Let $\bar{c}$ be the color flipped.***
    - ***Return the location (y_m, x_m) and the color $\bar{c}$.***

Strategy-stealing argument (3/5)

- **The steps taken by the first player P_1 to also win:**
  - P_1 **makes an arbitrary first ply** m_1**. Call it** m'**.**
  - P_2 **uses** $f(B_0 + m_1)$ **to make the second ply** m_2**.**
  - P_1 **makes the third ply** $m_3 = rev(f(B_0 + rev(m_2)))$**.**
    - ▷ ***If*** $m_3 = m'$***, then make another arbitrary ply and let it be the new*** m'***.***
  - P_2 **uses** $f(B_0 + m_1 + m_2 + m_3)$ **to make the forth ply** m_4**.**
  - P_1 **makes the fifth ply** $m_5 = rev(f(B_0 + rev(m_2) + m_3 + rev(m_4)))$**.**
    - ▷ ***If*** $m_5 = m'$***, then make another arbitrary ply and let it be the new*** m'***.***
  - P_2 **uses** $f(B_0 + m_1 + m_2 + m_3 + m_4 + m_5)$ **to make the 6th ply** m_6**.**
  - $\cdots$

Strategy-stealing argument (4/5)

- **Hence we know it is not possible for the second player to win.**
- **We also know these.**
  - **There is exactly one winner when the board is completely filled.**
  - **The game is finite.**
  - **Hence we can enumerate the whole solution search tree.**
    - ▷ *In this solution search tree, there is a way for one player to win all of the times no matter the opponent reacts.*
- **Since the second player cannot win, the first player must have a winning strategy.**

Strategy-stealing argument (5/5)

- This is not a constructive proof.
- It only shows the first player has a winning strategy from the initial empty board, not from an arbitrary position.
- The strategy-stealing argument cannot be used for every game.
  - An arbitrary extra move can never be a disadvantage in Hex.
  - This may not be true for other games.
- The argument works for any game when
  - there is a way for the first player not to lose at the first ply,
  - it is symmetric,
  - it is history independent,
  - it always has exactly one winner, and
    - ▷ *namely, it cannot have a draw by having no winners or 2 winners,*
  - an arbitrary extra move can never be a disadvantage.
    - ▷ *Note: it requires that a player is always possible to place an arbitrary move which may not be true for some games.*

Properties of Hex

- **Variations of Hex**
  - **The one-move-equalization rule: one player plays an opening move and the other player then has to decide which color to play for the reminder of the game.**
    - ▷ ***The revised version is a second-player win game (ultra-weakly).***
- **Hex exhibits considerable mathematical structure.**
  - **Hex in its general form has been proved to be PSPACE-complete by Even and Tarjan in 1976 by converting it to a Shannon switching game.**
  - **The state-space and decision complexities are comparable to those of Go on an equally-sized board.**
- **Solutions**
  - **(Weakly or strongly) solved on a $6 * 6$ board in 1994.**
  - **Maybe possible to solve the $7 * 7$ case.**
    - ▷ ***The $7 * 7$ case was solved in 2001. [Yang et. al. 2001]***
  - **Not likely to solve the $8*8$ version without fundamental breakthroughs.**
    - ▷ ***The $8 * 8$ case was solved in 2009. [Henderson et. al. 2009]***

More divergent games (1/3)

- **Polynmino games: placing 2-D pieces of a connected subset of a square grid to construct a special form.**
  - **Pentominoes**
  - **Domineering**
  - **Games on smaller boards have been solved.**
- **Othello**
  - **M. Buro's LOGISTELLO beat the resigning World Champion by 6-0 in 1997.**
  - **Weakly solved on a $6 * 6$ board by J. Feinstein in 1993.**
    - ▷ ***Second player win.***
- **Chess**
  - **DEEP BLUE beat the human World Champion in 1997!**

More divergent games (2/3)

- **Chinese chess**
  - **Still in progress.**
  - **Professional 7-dan since 2007.**
- **Shogi**
  - **Still in progress.**
  - **Claimed to be professional 2-dan in 2007.**
  - **Defeat a Lady professional player in 2010.**
  - **Defeat a 68-year old 1993 Meijin during 2011 and 2012.**

More divergent games (3/3)

- **Go**
  - **5 by 5 Go was solved in 2002.**
    - ▷ *First player wins and takes all cells using 22 plys.*
  - **Recent success and breakthrough using Monte Carlo UCT based methods between 2004 and 2012.**
  - **Lack major theoretical or practical break through since 2012.**
  - **Amateur 1 – 4 kyu in 2008.**
    - ▷ *Beat a professional 8-dan by having an 8-stone advantage.*
    - ▷ *Beaten by a professional 9-dan by giving a 7-stone advantage.*
  - **Amateur 1 dan in 2010.**
  - **Amateur 3 dan in 2011.**
  - **The program Zen beat a 9-dan professional master at March 17, 2012.**
    - ▷ *First game: Five stone handicap and won by 11 points.*
    - ▷ *Second game: four stones handicap and won by 20 points.*
  - **Solved (19 by 19): AlphaGo beat a human top player by a margin of 4:1 at March 2016.**

Table of complexity

Game	$\log_{10}$(state-space)	$\log_{10}$(game-tree size)
Nine Men's Morris	10	50
Pentominoes	12	18
Awari	12	32
Kalak(6,4)	13	18
Connect-four	14	21
Domineering $(8 * 8)$	15	27
Dakon-6	15	33
Checkers	21	31
Othello	28	58
Qubic	30	34
Draughts	30	54
Chess	46	123
Chinese chess	48	150
Hex $(11 * 11)$	57	98
Shogi	71	226
Renju $(15 * 15)$	105	70
Go-Moku $(15 * 15)$	105	70
Go $(19 * 19)$	172	360

State-space versus game-tree size

- **In 1994, the boundary of solvability by complete enumeration was set at 10^{11}.**
  - **The current estimation is about 10^{13} (since the year 2007).**
- **It is often possible to use heuristics in searching a game tree to cut the number of nodes visited tremendously when the structure of the game is well studied.**
  - **Example: Connect-Four.**

Methods developed for solving games

- **Brute-force methods**
  - **Retrograde analysis**
  - **Enhanced transposition-table methods**
- **Knowledge-based methods**
  - **Threat-space search and λ-search**
  - **Proof-number search**
  - **Depth-first proof-number search**
  - **Pattern search**
    - ▷ *To search for **threat patterns**, which are collections of cells in a position.*
    - ▷ *A threat pattern can be thought of as representing the **relevant area** on the board, an area that human players commonly identify when analyzing a position.*
- **Recent advancements:**
  - **Monte Carlo UCT based game tree simulation.**
    - ▷ *Monte Carlo method has a root from statistic.*
    - ▷ *Biased sampling.*
    - ▷ *Using methods from machine learning.*
    - ▷ *Combining domain knowledge with statistics.*
  - **A majority vote algorithm.**

Brute-force versus knowledge-based methods

- **Games with both a relative low state-space complexity and a low game-tree complexity have been solved by both methods.**
  - Category 1
  - Connect-four and Qubic
- **Games with a relative low state-space complexity have mainly been solved with brute-force methods.**
  - Category 2
  - Namely by constructing endgame databases
  - Nine Men's Morris
- **Games with a relative low game-tree-complexities have mainly been solved with knowledge-based methods.**
  - Category 3
  - Namely, by intelligent (heuristic) searching
  - Sometimes, with the helps of endgame databases
  - Go-Moku, Renju, and k-in-a-row games

Advantage of the initiative

- **Theorem (or argument) made by Singmaster in 1981: The first player has advantages.**
  - **Two kinds of positions**
    - ▷ *P-positions: the previous player can force a win.*
    - ▷ *N-positions: the next player can force a win.*
  - **Arguments**
    - ▷ *For the first player to have a forced win, just one of the moves must lead to a P-position.*
    - ▷ *For the second player to have a forced win, all of the moves must lead to N-positions.*
    - ▷ *It is easier to the first player to have a forced win assuming all positions are randomly distributed.*
    - ▷ *Can be easily extended to games with draws.*
- **Remarks:**
  - **One small boards, the second player is able to draw or even to win for certain games.**
  - **Cannot be applied to the infinite board.**

How to make use of the initiative

- **A potential universal strategy for winning a game:**
  - **Try to obtain a small advantage by using the initiative.**
    - *The opponent must react adequately on the moves played by the other player.*
  - **To reinforce the initiative the player searches for threats, and even a sequence of threats using an evaluation function E.**
  - **Force the opponent to always play the moves you expected.**
- **Threat-space search**
  - **Search for threats only!**

Offsetting the initiative

- **An example of a game with a huge initiative:**
  - **A connection $mn1$-game.**
    - *一子棋 was mentioned in 張系國著名小說"棋王"(1978年出版).*
  - **A connection $mn2$-game.**
  - **A connection $mn3$-game.**
  - **For a connection mni-game, you can have a feeling that the advantage given to the first player through initiative is gradually lessened when i gets larger.**
- **Need to offset the initiative.**
  - **The offsetting rule must be simple.**
  - **The revised game must be as fair as possible.**
    - *It is difficult to prove a game is fair.*
    - *Example: Paper-scissor-stone is fair.*
  - **The revised game needs be fun to play with.**
  - **The revised game cannot be too much different from the original game.**
- **Knowing how to properly offsetting the initiative may uncover some fundamental properties of the game such as its level of difficulty.**

Examples (1/2)

- **Enforce rules so that the first player cannot win by selective patterns.**
  - **Renju.**
    - ▷ ***Still first-player win.***
  - **Go ($19 * 19$).**
    - ▷ ***The first player must win by more than 7 stones.***
    - ▷ ***Komi*** $= 7.5$ ***in 2011.***
    - ▷ ***The value of Komi changes with the time and maybe different because of using different set of rules.***
- **The one-move-equalization rule: one player plays an opening move and the other player then has to decide which color to play for the reminder of the game.**
  - ▷ ***Hex.***
  - ▷ ***Second-player win.***

Examples (2/2)

- **The first move plays one stone, the rest plays two stones each.**
  - ▷ *Connect 6.*
  - ▷ *Intuitively, in each turn the initiative goes to different players alternatively.*
  - ▷ *Still not able to prove the game is fair (in 2016).*
- **The first player uses less resource.**
  - **For example: using less time.**
    - ▷ *Chinese chess.*
  - **A resource-auctioning scheme.**
- **Unclear about how to redesign a game to make it fair.**

Conclusions

- **The knowledge-based methods mostly inform us on the structure of the game, while exhaustive enumeration rarely does.**
- **Many ad-hoc recipes are produced currently.**
  - **The database can be used as a corrector or verifier of strategies formulated by human experts.**
- **It may be hopeful to use data mining techniques to obtain cross-game methods.**
  - **Currently not very successful.**

Comments

- **Can combine knowledge-based method with exhaustive enumeration.**
  - **For converging games, build endgame databases when the remaining state spaces is manageable.**
    - *Example: build endgames with at most 5 pieces in Chess and stop searching when the number of pieces on the board is less than 6.*
  - **For diverging games, pre-compute all possible opening moves and solve them one by one in sequence or in parallel.**
- **This is different from the usage of pattern databases in solving one-player games.**
  - **Patterns are used to guide the search in solving one-player games.**
  - **Endgame databases are used here to stop the search earlier. The idea has a flavor like that of bi-directional search.**

1990's Predictions — 2000's Status

- **Predictions were made in 1990 [Allis et al 1991] for the year 2000 concerning the expected playing strength of computer programs.**

solved	over champion	world champion	grand master	amateur
Connect-four	Checkers $(8 * 8)$	Chess	Go $(9 * 9)$	Go $(19 * 19)$
Qubic	Renju	Draughts $(10 * 10)$	Chinese chess	
Nine Men's Morris	Othello		Bridge	
Go-Moku	Scrabble			
Awari	Backgammon			

- **color code**
  - Green: Performs much better than expected
  - Red: right on the target.
  - Black: have some progress towards the target.
  - Blue: not so.

Predictions for 2010

- **Predictions were made at the year 2000 for the year 2010 concerning the expected playing strength of computer programs.**

solved	over champion	world champion	grand master	amateur
Awari	Chess	Go $(9 * 9)$	Bridge	Go $(19 * 19)$
Othello	Draughts $(10 * 10)$	Chinese chess	Shogi	
Checkers $(8 * 8)$	Scrabble	Hex		
	Backgammon	Amazons		
	Lines of Action			

Predictions for 2010 – Status

- **My personal opinion about the status of Prediction-2010 at October, 2010, right after the Computer Olympiad held in Kanazawa, Japan.**

solved	over champion	world champion	grand master	amateur
Awari	Chess	Go $(9 * 9)$	Bridge	Go $(19 * 19)$
Othello	Draughts $(10 * 10)$	Chinese chess	Shogi	
Checkers $(8 * 8)$	Scrabble	Hex		
	Backgammon	Amazons		
	Lines of Action			

- **color code**
  - **Red: right on the target.**
  - **Black: have some progress towards the target.**
  - **Blue: not so.**

References and further readings (1/2)

- **L.V. Allis, H.J. van den Herik, and I.S. Herschberg. Which games will survive? In: D.N.L. Levy, D.F. Beal (Eds.), Heuristic Programming in Artificial Intelligence 2: The Second Computer Olympiad, Ellis Horwood, Chichester, 1991, pp. 232-243.**

* **H. J. van den Herik, J. W. H. M. Uiterwijk, and J. van Rijswijck. Games solved: Now and in the future.** *Artificial Intelligence*, **134:277–311, 2002.**

- **Jonathan Schaeffer. The games computers (and people) play.** *Advances in Computers*, **52:190–268, 2000.**
- **L. V. Allis, M. van der Meulen, and H. J. van den Herik. Proof-number search.** *Artificial Intelligence*, **66(1):91–124, 1994.**

References and further readings (2/2)

- J. Yang, S. Liao, and M. Pawlak. A decomposition method for finding solution in game Hex 7x7. In *Proceedings of International Conference on Application nd Development of Computer games in the 21st century*, pages 93–112, November 2001.
- P. Henderson, B. Arneson, and R. B. Hayward. Solving 8x8 Hex. In *Proceedings of IJCAI*, pages 505–510, 2009.