Chance Node Searching

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Abstract

- **Searching stochastic games**
- **Alpha-beta based techniques**
  - Star0: exhaustive enumeration without cuts
  - Star0.5: cuts in between choices
  - Star1: cuts inside choices using bounds from a move ordering
  - Star2: use a probing strategy to find a good move ordering
  - Star2.5: using an even better probing strategy
- **MCTS based approaches**
  - Sparse sampling

Stochastic games

- **Stochastic games have nodes whose outcome or move selections cannot be decided completely by players.**
    - **Pure stochastic: no action can be taken by a player before or after a random toss.**
        - ▷ *A dice game.*
    - **A priori chance node: a random toss is made first and then you make a decision based on the toss.**
        - ▷ *EinStein Würfelt Nicht (EWN) [Lorentz et al '12]: you make a random toss to decide what pieces that you can move, and then you make a move.*
    - **A posteriori chance node: you make a decision first and then followed by a random toss.**
        - ▷ *Chinese dark chess [Yen et al '14]: you pick a dark piece to flip, and then the piece is revealed decided by a random toss*

Determinacy (1/2)

- **Determinacy** is the degree that a player can control in playing the game.
  - Usually from a fractional number to 1, where 1 means a deterministic game.
  - May be different as the game progresses.
  - Roughly equal to the chance when you want to do a particular ply, the chance that you can do it.
    - ▷ *Each ply may have a different determinacy.*
    - ▷ *Expected determinacy: average over all possible plys.*

- **In most, but not all games, having the right to move (initiative) is beneficial. Hence having a high determinacy helps.**

Determinacy (2/2)

- **For a ply of moving a piece w in EWN, it is $\frac{y+1}{6}$ where y is the cardinality of the set of removed pieces in the two continuous neighboring segments adjacent to the piece numbered w.**
  - When a piece is removed, the determinacy of some piece(s) increase.
  - Notes:
    - ▷ *When both players have only 1 piece, the game is deterministic.*
    - ▷ *When one player has 1 piece and the other player has more than 1 pieces, the game is "half" stochastic which makes an interesting special case for a stochastic game.*

- **For a flipping ply in CDC, it is $\frac{x}{|D|}$ where x is the number of kinds of pieces you wish to have and $|D|$ is the number of unrevealed pieces.**
  - At first, the determinacy is low.
  - After each flip, $|D|$ decreases by 1.
  - When $|D|$ is 1, $x = 1$, but x can be 1 even $|D| > 1$.
  - When $x = 1$, CDC becomes a deterministic game.

Searching stochastic games

- **Because of a coin toss, the search space is greatly enlarged.**
  - **Example: In the opening phase, Chinese dark chess game tree has a very large branching factor.**
    - ▷ *After using reduction in symmetry, the first ply has $7 * 8$ possible outcomes.*
    - ▷ *The second ply has upto $14 * 31$ possible outcomes which is larger than 19x19 Go.*

- **Maybe need to compute all possible results from the coin toss to decide a good playing strategy.**
  - **The expected value of all possible outcomes is needed which may make it difficult to apply any cuts.**

Search with chance nodes

- **Example: Chinese dark chess (CDC)**
 - Two-player, zero sum
  - **Complete information**
  - **Perfect information**
  - **Stochastic**
  - There is a **chance** node during searching.
    - ▷ *The value of a chance node is a distribution, not a fixed value.*

- **Previous work**
  - Alpha-beta based [Ballard 1983]
  - Monte-Carlo based [Lancoto et al 2013] [Jouandeau and Cazenave '14]

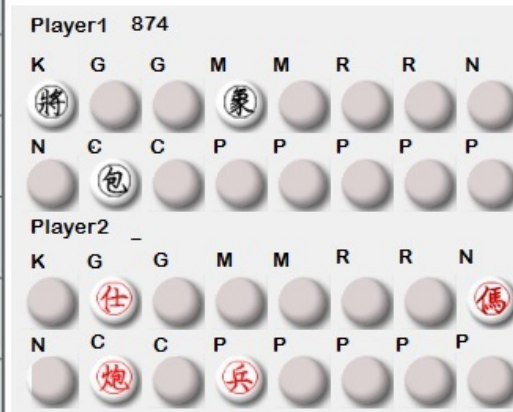# Example (1/4)

- **It's BLACK turn and BLACK has 6 different possible legal moves which includes the four different moving moves made by its elephant and the two flipping moves at a1 or a8.**
  - It is difficult for BLACK to secure a win by moving its elephant along any of the 3 possible directions, namely up, right or left, or by capturing the RED pawn at the left hand side.
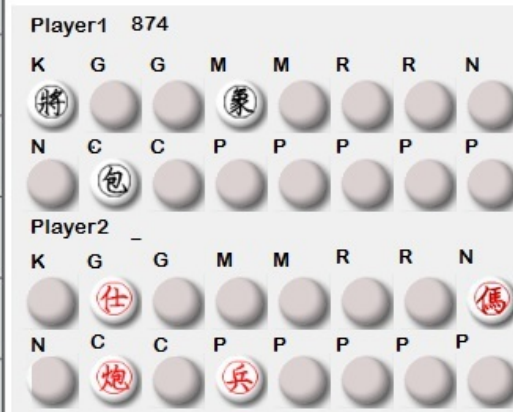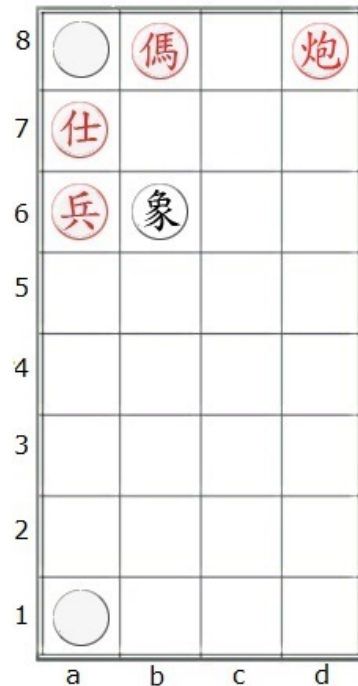
Example (2/4)

- **If BLACK flips a1, then there are 2 possible cases.**
  - **If a1 is BLACK cannon, then it is difficult for RED to win.**
    - ▷ *RED guard is in danger.*
  - **If a1 is BLACK king, then it is difficult for BLACK to lose.**
    - ▷ *BLACK king can go up through the right.*
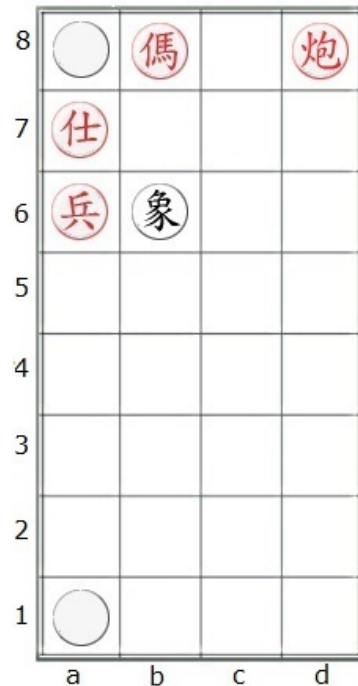
Example (3/4)

- **If BLACK flips a8, then there are 2 possible cases.**
  - **If a8 is BLACK cannon, then it is easy for RED to win.**
    - ▷ *RED cannon captures it immediately.*
  - **If a8 is BLACK king, then it is also easy for RED to win.**
    - ▷ *RED cannon captures it immediately.*
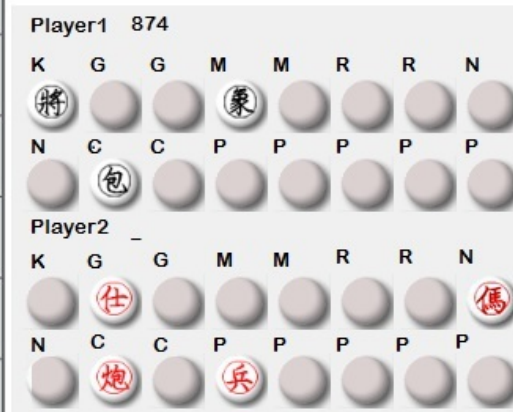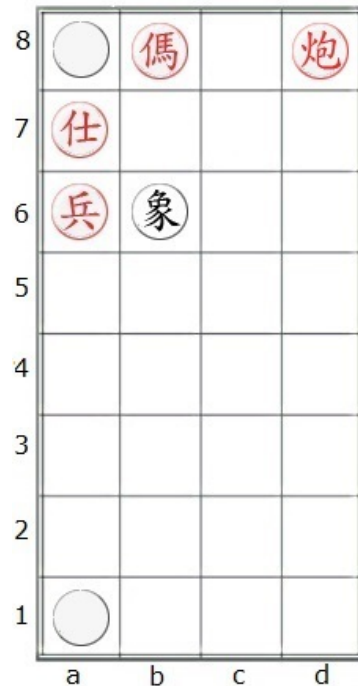
Example (4/4)

- **Conclusion:**
  - **It is bad for BLACK to move its elephant.**
  - **It is vary bad for BLACK to flip a8.**
  - **It is better for BLACK to flip a1.**
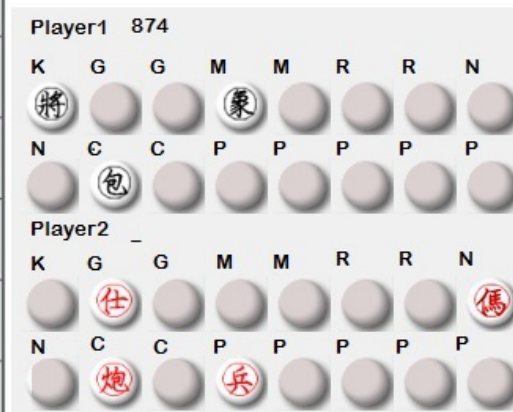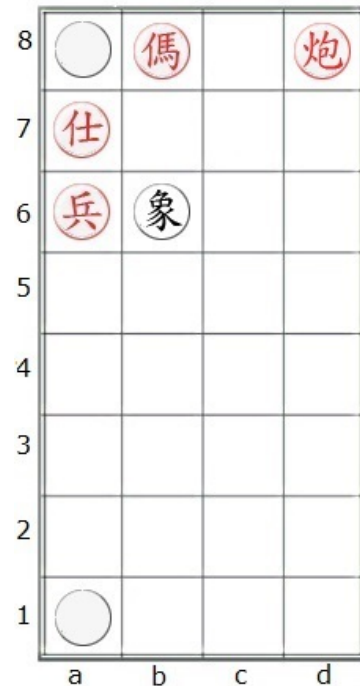
Example: illustration

- **Conclusion:**
  - **It is vary bad for BLACK to flip a8.**
  - **It is bad for BLACK to move its elephant.**
  - **It is better for BLACK to flip a1.**
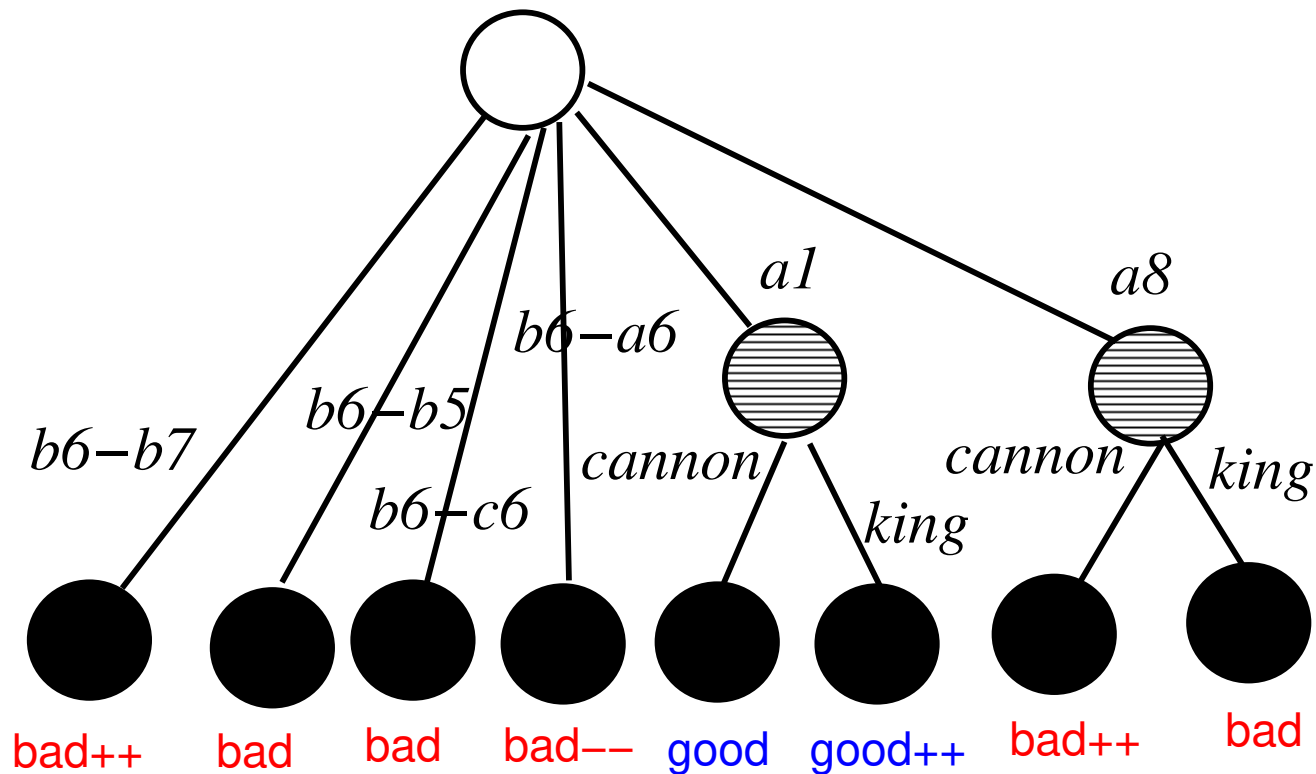
Basic ideas for searching chance nodes

- **Assume a chance node x has a score probability distribution function $Pr(*)$ with the range of possible outcomes from 1 to N where N is a positive integer.**
  - **For each possible outcome i, we need to compute $score(i)$.**
  - **The expected value $E = \sum_{i=1}^{N} score(i) * Pr(x = i)$.**
  - **The minimum value is $m = \min_{i=1}^{N}\{score(i) \mid Pr(x = i) > 0\}$.**
  - **The maximum value is $M = \max_{i=1}^{N}\{score(i) \mid Pr(x = i) > 0\}$.**
- **Example: open game in Chinese dark chess.**
  - **For the first ply, $N = 14 * 32$.**
    - ▷ *Using symmetry, we can reduce it to 7*8.*
  - **We now consider the chance node of flipping the piece at the cell a1.**
    - ▷ *$N = 14$.*
    - ▷ *Assume $x = 1$ means a BLACK King is revealed and $x = 8$ means a RED King is revealed.*
    - ▷ *Then $score(1) = score(8)$ since the first player owns the revealed king no matter its color is.*
    - ▷ *$Pr(x = 1) = Pr(x = 8) = 1/14$.*

Illustration

max

expected value

chance

min
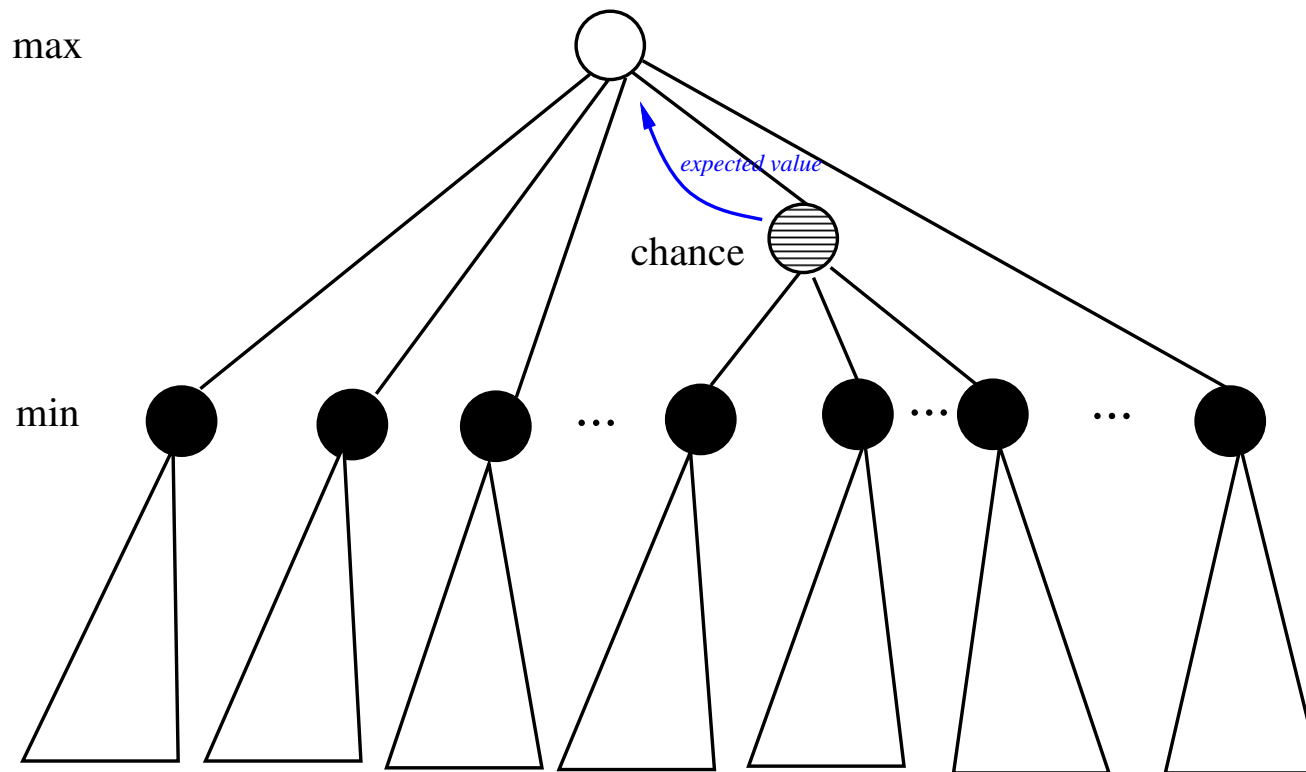
The probability distribution

- **General case**
  - **Assume a chance node x has c choices $k_1, \ldots, k_c$.**
  - **The ith choice happens with the probability Pr_i.**
    - ▷ $\sum_{i=1}^{c} Pr_i = 1$

- **Special cases**
  - **Special case 1, called uniform (EQU): $Pr_i = 1/c$.**
    - ▷ *All choices happen with an equal chance.*
    - ▷ *Example: EinStein Würfelt Nicht (EWN).*
  - **Special case 2, called GCD: $Pr_i = w_i/D$ where each w_i is an integer and D is also an integer.**
    - ▷ $D = \sum_{i=1}^{c} w_i$ *as in Chinese dark chess.*

- **The above two special cases usually happen in game playing and can be used to do some optimization in arithmetic calculations.**

Comments about EWN (1/4)

- $\sum_{i=1}^{c} Pr_i$ **is always** 1**.**
- **In EWN when there are only two pieces left, it appears that the above claim is not true.**
  - **Example I: 1 and 6 with both probabilities being selected may look like** $\frac{5}{6}$**.**
    - ▷ *Assume the winning rates in example I are 0.75 and 0.23 for 1 and 6 being picked respectively.*
  - **Example II: 1 and 2 may look like the probability of 1 being selected is** $\frac{1}{6}$**, but is** $\frac{5}{6}$ **for 2 to be picked.**
    - ▷ *Assume the winning rates in example II are also 0.75 and 0.23 for 1 and 2 being picked respectively.*
- **Example I is favored over example II not because the sum of probabilities is larger, but because the average determinacy is larger!!!**

Illustration for EWN

max

expected value

chance

choice = 1   2   3   4   5   6

pick piece &
direction

min

...   .
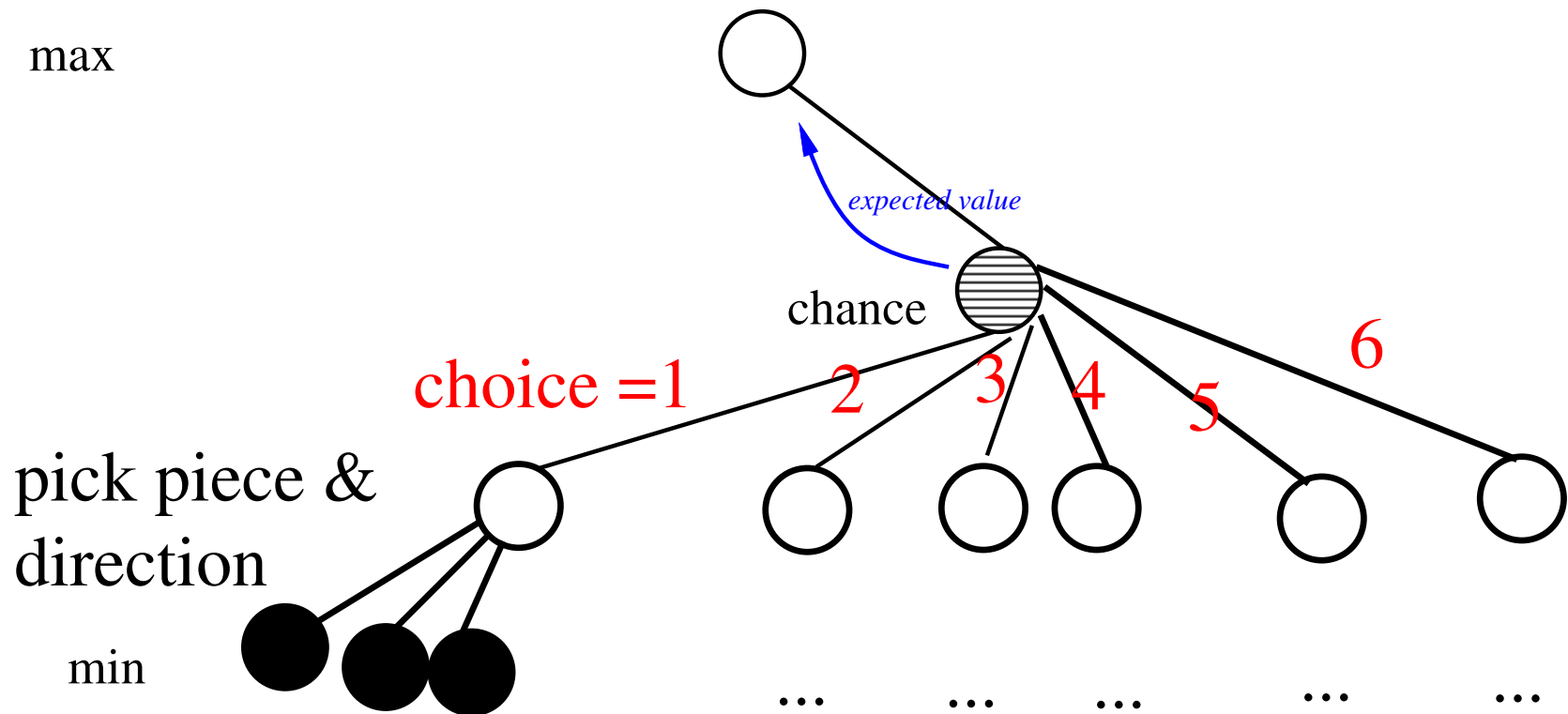
Comments about EWN (2/4)

- **EWN always has SIX choices: uniform case.**
- **Example I:**
  - **For choices 1 to 5, we can choose to move piece 1.**
  - **For choices 2 to 6, we can choose to move piece 6.**
  - **It appears that for choices 2 to 5, we have an equal chance of choosing either piece 1 or 6.**
    - ▷ *However, due to the difference in winning rates, we always choose piece 1.*
  - **This means 1 is chosen with a probability of $\frac{5}{6}$ and 6 is picked with a probability of $\frac{1}{6}$.**
    - ▷ *Hence the expected winning rate is $5 * \frac{1}{6} * 0.75 + 1 * \frac{1}{6} * 0.23 = 0.6633$*
- **Example II:**
  - **For choice 1, we can choose to move piece 1.**
  - **For choices 2 to 6, we can choose to move piece 2.**
  - **This means 1 is chosen with a probability of $\frac{1}{6}$ and 2 is picked with a probability of $\frac{5}{6}$.**
    - ▷ *Hence the expected winning rate is $1 * \frac{1}{6} * 0.75 + 5 * \frac{1}{6} * 0.23 = 0.3167$*

Comments about EWN (3/4)

- **In example I above**
  - Only **ONE** piece can be picked when dice $= 1$ or 6.
  - If piece i is not being captured, then choice i can only pick that piece.
  - When dice is between 2 and 5, if the corresponding piece is being removed, then it has at most **TWO** pieces to choose from.

- **Average determinacy**
  - **Example I:**
    - ▷ *Assume three legal plys for each piece.*
    - ▷ *For each ply, you have a determinacy of $\frac{5}{6}$.*
    - ▷ *Average determinacy is also $\frac{5}{6}$.*

  - **Example II:**
    - ▷ *Assume three legal plys for each piece.*
    - ▷ *For a ply moving piece 1, you have a determinacy of $\frac{1}{6}$.*
    - ▷ *For a ply moving piece 2, you have a determinacy of $\frac{5}{6}$.*
    - ▷ *Average determinacy is thus $\frac{3}{6}$.*

Comments about EWN (4/4)

- **Other possible important affecting factors considering only the material values without positional information:**
  - **Variance of determinacy.**
  - **Number of pieces left.**
- **Using transposition tables will help a lot in searching when some pieces are removed!!!**
  - **Example: when the pieces left are 1 and 6, then dice = 2 to 5 makes no difference in searching.**
    - ▷ *You first search the subtree T_1 for dice =1, and then the subtree T_6 for dice = 6.*
    - ▷ *Assume $exp\_best(T_i)$ is the returned expected best value for T_i.*
    - ▷ *Then the overall expected best value is $\frac{1}{6}(exp\_best(T_1) + exp\_best(T_6) + 4 \cdot k)$ where $k = \max\{exp\_best(T_1), exp\_best(T_6)\}$.*

Algorithm: Chance_Search with Star0 (MAX)

- **Algorithm $F3.0'$(position p, value $alpha$, value $beta$, integer $depth$)**
    - **// max node**
    - **determine the successor positions $p_1, \ldots, p_b$**
    - **if $b = 0$ // a terminal node**
      or $depth = 0$ // remaining depth to search
      or time is running up // from timing control
      or some other constraints are met // add knowledge here
    - **then return $f(p)$ else begin**
        - $\triangleright$ $m := -\infty$
        - $\triangleright$ *for $i := 1$ to b do*
        - $\triangleright$ *begin*
        - $\triangleright$ *if p_i is to play a chance node x*
          then $t := Star0\_F3.0'(p_i, x, max\{alpha, m\}, beta, depth - 1)$
        - $\triangleright$ *else $t := G3.0'(p_i, max\{alpha, m\}, beta, depth - 1)$*
        - $\triangleright$ *if $t > m$ then $m := t$*
        - $\triangleright$ *if m is max or $m \geq beta$ then return(m) // beta cut off*
        - $\triangleright$ *end*
    - **end;**
    - **return m**

Algorithm: Chance_Search with Star0 (MIN)

- **Algorithm** $G3.0'$**(position** p, **value** $alpha$, **value** $beta$, **integer** $depth$**)**
  - **// min node**
  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$ **// a terminal node**
    or $depth = 0$ **// remaining depth to search**
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return** $f(p)$ **else begin**
    - $\triangleright$ $m := \infty$
    - $\triangleright$ **for** $i := 1$ **to** b **do**
    - $\triangleright$ **begin**
    - $\triangleright$ **if** p_i **is to play a chance node** x
      then $t := Star0\_G3.0'(p_i, x, alpha, min\{beta, m\}, depth - 1)$
    - $\triangleright$ **else** $t := F3.0'(p_i, alpha, min\{beta, m\}, depth - 1)$
    - $\triangleright$ **if** $t < m$ **then** $m := t$
    - $\triangleright$ **if** m **is min or** $m \le alpha$ **then return**(m) **// alpha cut off**
    - $\triangleright$ **end**
  - **end;**
  - **return** m

Algorithm: $Star0$, uniform case (MAX)

- **version when all choices have equal probabilities**
- **max node**
- **Algorithm $Star0\_EQU\_F3.0'$(position p, node x, value $alpha$, value $beta$, integer $depth$)**
  - // **a max chance node x with c equal probability choices k_1, ..., k_c**
  - // **exhaustive search all possibilities and return the expected value**
  - **determine the possible values of the chance node x to be $k_1, \ldots, k_c$**
  - $vsum = 0$; // **initial sum of expected value**
  - **for $i = 1$ to c do**
  - **begin**
    - ▷ *let p_i be the position of assigning k_i to x in p;*
    - ▷ $vsum \mathrel{+}= G3.0'(p_i, -\infty, +\infty, depth)$;
  - **end**
- **return $vsum/c$; // return the expected score**

Algorithm: $Star0$, uniform case (MIN)

- **version when all choices have equal probabilities**
- **min node**
- **Algorithm $Star0\_EQU\_G3.0'$(position p, node x, value $alpha$, value $beta$, integer $depth$)**
  - **// a min chance node x with c equal probability choices k_1, ..., k_c**
  - **// exhaustive search all possibilities and return the expected value**
  - **determine the possible values of the chance node x to be $k_1, \ldots, k_c$**
  - $vsum = 0$; **// initial sum of expected value**
  - **for $i = 1$ to c do**
  - **begin**
    - ▷ **let p_i be the position of assigning k_i to x in p;**
    - ▷ $vsum \mathrel{+}= F3.0'(p_i, -\infty, +\infty, depth)$;
  - **end**
- **return $vsum/c$; // return the expected score**

Star0: note

- $depth$ **stays the same in Star0 search since we are unwrapping a chance node.**

- **The search window from normal alpha-beta pruning cannot be applied in a chance node searching since we are looking at the average of the outcome.**
  - **It is okay for one choice to have a very large or small value because it may be evened out by values from other choices.**
  - **Thus the search window is reset to $(-\infty, \infty)$.**

Algorithm: $Star0$, general case (MAX)

- **Algorithm** $Star0\_F3.0'$**(position** p, **node** x, **value** $alpha$, **value** $beta$,**integer** $depth$**)**
    - // **a max chance node** x **with** c **choices** k_1, ..., k_c
    - // **the** i**th choice happens with the probability Pr**$_i$
    - // **exhaustive search all possibilities and return the expected value**
    - **determine the possible values of the chance node** x **to be** $k_1, \ldots, k_c$
    - $vexp = 0$; // **initial sum of expected value**
    - **for** $i = 1$ **to** c **do**
    - **begin**
        - ▷ *let* p_i *be the position of assigning* k_i *to* x *in* p;
        - ▷ $vexp$ += **Pr**$_i * G3.0'(p_i, -\infty, +\infty, depth)$;
    - **end**
- **return** $vexp$; // **return the expected score**

Algorithm: $Star0$, general case (MIN)

- **Algorithm $Star0\_G3.0'$(position p, node x, value $alpha$, value $beta$, integer $depth$)**
  - // a min chance node x with c choices k_1, ..., k_c
  - // the ith choice happens with the probability $\Pr_i$
  - // exhaustive search all possibilities and return the expected value
  - determine the possible values of the chance node x to be $k_1, \ldots, k_c$
  - $vexp = 0$; // initial sum of expected value
  - for $i = 1$ to c do
  - begin
    - ▷ let p_i be the position of assigning k_i to x in p;
    - ▷ $vexp \mathrel{+}= \Pr_i * F3.0'(p_i, -\infty, +\infty, depth)$;
  - end
- **return $vexp$; // return the expected score**

Algorithm: $Star0$, GCD case (MAX)

- **Algorithm** $Star0\_GCD\_F3.0'$(**position** p, **node** x, **value** $alpha$, **value** $beta$, **integer** $depth$)
  - // a max chance node x with c choices k_1, ..., k_c
  - // whose occurrence probability are w_1/D, ..., w_c/D
  - // and each w_i is an integer
  - // exhaustive search all possibilities and return the expected value
  - determine the possible values of the chance node x to be $k_1, \ldots, k_c$
  - $vsum = 0$; // initial sum of weight values
  - for $i = 1$ to c do
  - begin
    - ▷ let p_i be the position of assigning k_i to x in p;
    - ▷ $vsum \mathrel{+}= w_i * G3.0'(p_i, -\infty, +\infty, depth)$;
  - end
- **return** $vsum/D$; // return the expected score

Algorithm: $Star0$, GCD case (MIN)

- **Algorithm** $Star0\_GCD\_G3.0'$(**position** p, **node** x, **value** $alpha$, **value** $beta$, **integer** $depth$)
  - // a min chance node x with c choices k_1, ..., k_c
  - // whose occurrence probability are w_1/D, ..., w_c/D
  - // and each w_i is an integer
  - // exhaustive search all possibilities and return the expected value
  - determine the possible values of the chance node x to be $k_1, \ldots, k_c$
  - $vsum = 0$; // initial sum of weight values
  - **for** $i = 1$ **to** c **do**
  - **begin**
    - ▷ let p_i be the position of assigning k_i to x in p;
    - ▷ $vsum \mathrel{+}= w_i * F3.0'(p_i, -\infty, +\infty, depth)$;
  - **end**
- **return** $vsum/D$; // return the expected score

Ideas for improvements

- **During a chance search, an exhaustive search method is used without any pruning.**
- **Ideas for further improvements**
  - When some of the choices turn out very bad or good results, we know information about lower/upper bounds of the final value.
  - When you are in advantage, search for a bad choice first.
    - ▷ *If the worst choice cannot is not too bad, then you can take this chance.*
  - When you are in disadvantage, search for a good choice first.
    - ▷ *If the best choice cannot is not good enough, then there is no need to take this chance.*
- **Examples: the average of two drawings of a dice is similar to a position with 2 choices with scores in [1..6].**
  - If the first drawing is 5, then bounds of the average becomes
    - ▷ *lower bound is 3*
    - ▷ *upper bound is 5.5.*
  - If the first drawing is 1, then bounds of the average becomes
    - ▷ *lower bound is 1*
    - ▷ *upper bound is 3.5.*

Bounds in a chance node

- **Assume the various possibilities of a chance node is evaluated one by one in the order that at the end of phase i, the ith choice is evaluated.**
  - **Assume $v_{min} \leq score(i) \leq v_{max}$.**
- **What are the lower and upper bounds, namely m_i and M_i, of the expected value of the chance node immediately after the end of phase i?**
  - $i = 0$.
    - $\triangleright$ $m_0 = v_{min}$
    - $\triangleright$ $M_0 = v_{max}$
  - $i = 1$, **we first compute** $score(1)$, **and then know**
    - $\triangleright$ $m_1 \geq score(1) * Pr(x = 1) + v_{min} * (1 - Pr(x = 1))$, **and**
    - $\triangleright$ $M_1 \leq score(1) * Pr(x = 1) + v_{max} * (1 - Pr(x = 1))$.
  - $\ldots$
  - $i = i^*$, **we have computed** $score(1), \ldots, score(i^*)$, **and then know**
    - $\triangleright$ $m_{i*} \geq \sum_{i=1}^{i^*} score(i) * Pr(x = i) + v_{min} * (1 - \sum_{i=1}^{i^*} Pr(x = i))$, **and**
    - $\triangleright$ $M_{i*} \leq \sum_{i=1}^{i^*} score(i) * Pr(x = i) + v_{max} * (1 - \sum_{i=1}^{i^*} Pr(x = i))$.

Star0.5: uniform case (1/3)

- **For simplicity, let's assume** $Pr(x = i) = \frac{1}{c}$, **that is, the uniform case.**
- **For all** i, **and the evaluated value of the** ith **choice is** v_i.
- **Assume the search window entering a chance node with** $N = c$ **choices is** $(alpha, beta)$.
- **The value of a chance node after the first** i **choices are explored can be expressed as**
  - **an expected value** $E_i = vsum_i / c$ **obtained so far;**
    - ▷ $vsum_i = \sum_{j=1}^{i} v_j$
    - ▷ *This value is returned **only** when all choices are explored.*
      - ⇒ *The expected value of an un-explored child shouldn't be* $\frac{v_{min} + v_{max}}{2}$.
  - **a range of possible values** $[m_i, M_i]$.
    - ▷ $m_i = (\sum_{j=1}^{i} v_j + v_{min} \cdot (c - i))/c$
    - ▷ $M_i = (\sum_{j=1}^{i} v_j + v_{max} \cdot (c - i))/c$
  - **Invariants:**
    - ▷ $E_i \in [m_i, M_i]$
    - ▷ $E_c = m_c = M_c$

- **Let m_i and M_i be the current lower and upper bounds, respectively, of the expected value of this chance node immediately after the evaluation of the ith node.**
  - $m_i = (\sum_{j=1}^{i-1} v_j + v_i + v_{min} \cdot (c - i))/c$
  - $M_i = (\sum_{j=1}^{i-1} v_j + v_i + v_{max} \cdot (c - i))/c$

- **How to incrementally update m_i and M_i:**
  - $m_0 = v_{min}$
  - $M_0 = v_{max}$
  -
$$m_i = m_{i-1} + (v_i - v_{min})/c \tag{1}$$

  -
$$M_i = M_{i-1} + (v_i - v_{max})/c \tag{2}$$

Star0.5: uniform case (3/3)

- **Let m_i and M_i be the current lower and upper bounds, respectively, of the expected value of this chance node immediately after the evaluation of the ith node.**
  - $m_i = (\sum_{j=1}^{i-1} v_j + v_i + v_{min} \cdot (c - i))/c$
  - $M_i = (\sum_{j=1}^{i-1} v_j + v_i + v_{max} \cdot (c - i))/c$
- **The current search window is $(alpha, beta)$.**
  - **No more searching is needed when**
    - $\triangleright$ $m_i \geq beta$, *chance node cut off I;*
      $\Rightarrow$ *The lower bound found so far is good enough.*
      $\Rightarrow$ *Similar to a beta cut off.*
      $\Rightarrow$ *The returned value is m_i.*
    - $\triangleright$ $M_i \leq alpha$, *chance node cut off II.*
      $\Rightarrow$ *The upper bound found so far is bad enough.*
      $\Rightarrow$ *Similar to an alpha cut off.*
      $\Rightarrow$ *The returned value is M_i.*

Example for Star0.5

- **Assumption:**
  - **The range of the scores of Chinese dark chess is $[-10, 10]$ inclusive, $alpha = -10$ and $beta = 10$.**
  - $N = 7$.
  - $Pr(x = i) = 1/N = 1/7$.
- $i = 0$
  - $m_0 = -10$.
  - $M_0 = 10$.
- $i = 1$
  - **if $score(1) = -2$, then**
    - $\triangleright$ $m_1 = -2 * 1/7 + -10 * 6/7 = -62/7 \simeq -8.86$.
    - $\triangleright$ $M_1 = -2 * 1/7 + 10 * 6/7 = 58/7 \simeq 8.26$.
  - **if $score(1) = 3$, then**
    - $\triangleright$ $m_1 = 3 * 1/7 + -10 * 6/7 = -57/7 \simeq -8.14$.
    - $\triangleright$ $M_1 = 3 * 1/7 + 10 * 6/7 = 63/7 = 9$.

Star0.5: uniform case (MAX)

- **Algorithm $Star0.5\_EQU\_F3.0'$(position p, node x, value $alpha$, value $beta$, integer $depth$)**
  - // a max chance node x with c equal probability choices k_1, ..., k_c
  - determine the possible values of the chance node x to be $k_1, \ldots, k_c$
  - $m_0 = v_{min}$, $M_0 = v_{max}$ // initial lower and upper bounds
  - $vsum = 0$; // initial sum of expected values
  - for $i = 1$ to c do
  - begin
    - $\triangleright$ *let p_i be the position of assigning k_i to x in p;*
    - $\triangleright$ *$t := G3.0'(p_i, v_{min}, v_{max}, depth)$*
    - $\triangleright$ *$m_i = m_{i-1} + (t - v_{min})/c$, $M_i = M_{i-1} + (t - v_{max})/c$; // update the bounds*
    - $\triangleright$ *if $m_i \geq beta$ then return m_i; // failed high, chance node cut off I*
    - $\triangleright$ *if $M_i \leq alpha$ then return M_i; // failed low, chance node cut off II*
    - $\triangleright$ *$vsum \mathrel{+}= t$;*
  - end
- **return $vsum/c$;**

Star0.5: uniform case (MIN)

- **Algorithm** $Star0.5\_EQU\_G3.0'$**(position** p**, node** x**, value** $alpha$**, value** $beta$**, integer** $depth$**)**
    - *// **a min chance node** x **with** c **equal probability choices** k_1, ..., k_c*
    - **determine the possible values of the chance node** x **to be** $k_1, \ldots, k_c$
    - $m_0 = v_{min}$, $M_0 = v_{max}$ **// initial lower and upper bounds**
    - $vsum = 0$; **// initial sum of expected values**
    - **for** $i = 1$ **to** c **do**
    - **begin**
        - ▷ *let p_i be the position of assigning k_i to x in p;*
        - ▷ *$t := F3.0'(p_i, v_{min}, v_{max}, depth)$*
        - ▷ *$m_i = m_{i-1} + (t - v_{min})/c$, $M_i = M_{i-1} + (t - v_{max})/c$; // update the bound*
        - ▷ *if $m_i \geq beta$ then return m_i; // failed high, chance node cut off I*
        - ▷ *if $M_i \leq alpha$ then return M_i; // failed low, chance node cut off II*
        - ▷ *$vsum \mathrel{+}= t$;*
    - **end**
- **return** $vsum/c$**;**

Illustration: Star0.5

max

chance   *(alpha,beta)*

(Vmin,Vmax)

min   1 $i-1$ ... i c ...

$v[1]$ $v[i-1]$

?   ?   ?

$[m[i-1],M[i-1]]$
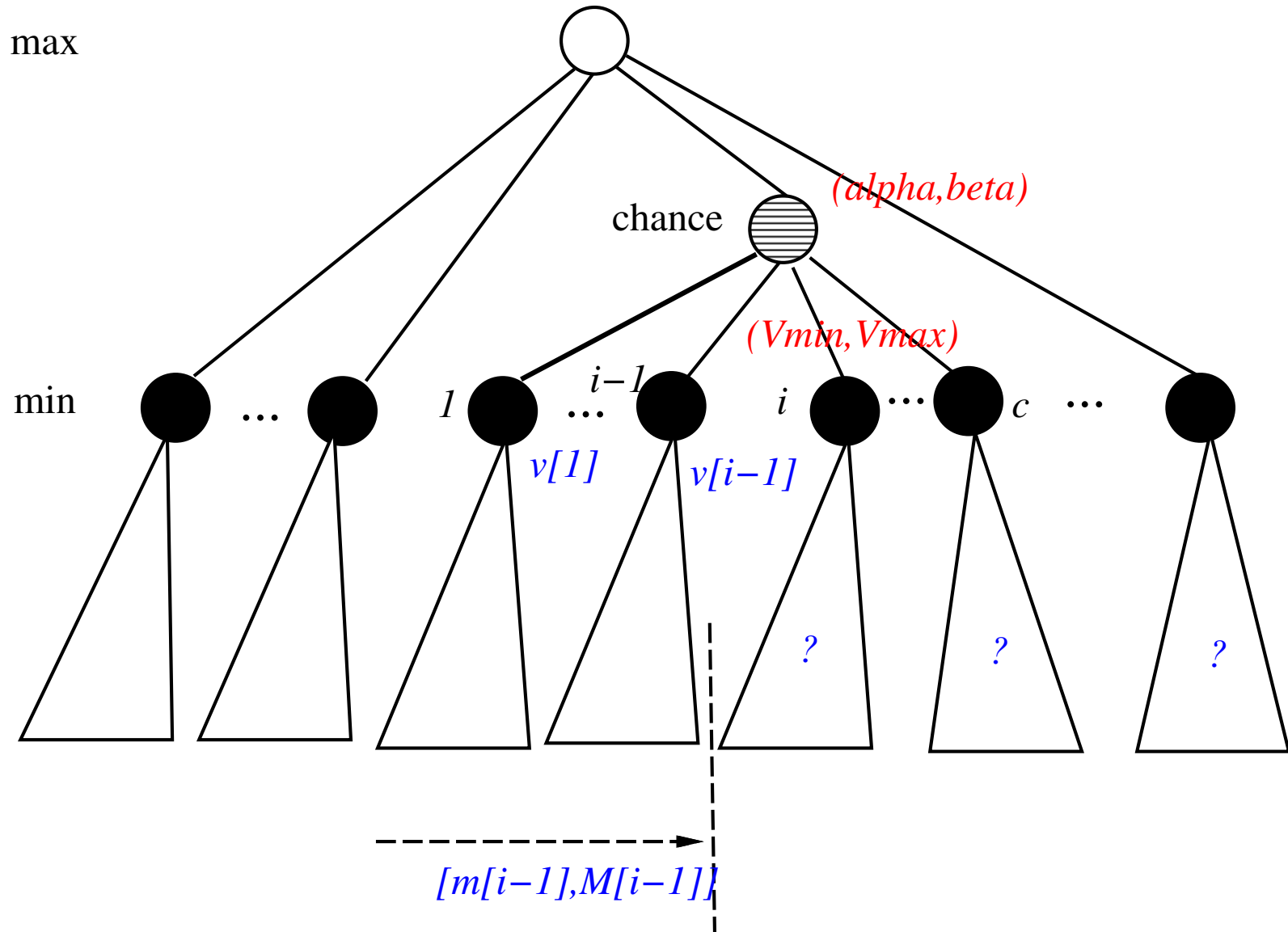
Ideas for further improvements (1/2)

- **The above two cut offs comes from each time a choice is completely searched.**
  - **When $m_i \geq beta$, chance node cut off I,**
    - ▷ *which means $(\sum_{j=1}^{i-1} v_j + v_i + v_{min} \cdot (c-i))/c \geq beta$.*
  - **When $M_i \leq alpha$, chance node cut off II,**
    - ▷ *which means $(\sum_{j=1}^{i-1} v_j + v_i + v_{max} \cdot (c-i))/c \leq alpha$.*

- **Further cut off can be obtained during searching a choice.**
  - **Assume after searching the first $i-1$ choices, no chance node cut off happens.**
  - **Before searching the ith choice, we know that if v_i is large enough, then it will raise the lower bound of the chance node which may trigger a chance node cut off I.**
  - **How large should v_i be for this to happen?**
    - ▷ *chance node cut off I:*
      $(\sum_{j=1}^{i-1} v_j + v_i + v_{min} \cdot (c-i))/c \geq beta$
    - ▷ $\Rightarrow v_i \geq B_{i-1} = c \cdot beta - (\sum_{j=1}^{i-1} v_j - v_{min} * (c-i))$
    - ▷ B_{i-1} *is the threshold for cut off I to happen.*

Ideas for further improvements (2/2)

- **Similarly,**
  - **Assume after searching the first $i-1$ choices, no chance node cut off happens.**
  - **Before searching the ith choice, we know that if v_i is small enough, then it will lower the upper bound of the chance node which may trigger a chance node cut off II.**
  - **How small should v_i be for this to happen?**
    - ▷ *chance node cut off II:*
      $(\sum_{j=1}^{i-1} v_j + v_i + v_{max} \cdot (c-i))/c \leq alpha$
    - ▷ $\Rightarrow v_i \leq A_{i-1} = c \cdot alpha - (\sum_{j=1}^{i-1} v_j - v_{max} * (c-i))$
    - ▷ A_{i-1} *is the threshold for cut off II to happen.*

Example: Star1

- **Example: the average of 2 drawings of a dice is similar to a position with 2 choices with scores in [1..6].**
  - $[m_0, M_0] = [v_{min}, v_{max}] = [1, 6]$
  - **Assume** $(alpha, beta) = (3.25, 3.95)$
- **The first drawing $v_1 = 3$. Then bounds of the average:**
  - **lower bound is 2; upper bound is 4.5.**
  - $[m_1, M_1] = [2, 4.5]$
- **Before the second drawing, the search will**
  - **failed-low if $\frac{v_2+3}{2} \leq alpha = 3.25$ which means the search fails low if $v_2 \leq 3.5$.**
  - **failed-high if $\frac{v_2+3}{2} \geq beta = 3.95$ which means the search fails high if $v_2 \geq 4.9$.**
- **Hence we can set the search window for the second search to be $(3.5, 4.9)$ instead of $[1, 6]$.**
  - ▷ *We only need to do a test on whether v_2 is 4 or not.*

Formulas for the uniform case: Star1

- **Set the window for searching the ith choice to be (A_{i-1}, B_{i-1}) which means no further search is needed if the result is not within this window.**
  - (A_{i-1}, B_{i-1}) **is the window for searching the ith choice instead of using $(alpha, beta)$.**
- **How to incrementally update A_i and B_i?**
  -

$$A_0 = c \cdot (alpha - v_{max}) + v_{max} \tag{3}$$

  -

$$B_0 = c \cdot (beta - v_{min}) + v_{min} \tag{4}$$

  -

$$A_i = A_{i-1} + v_{max} - v_i \tag{5}$$

  -

$$B_i = B_{i-1} + v_{min} - v_i \tag{6}$$

- **Comment:**
  - **May want to use zero-window search to test first.**

Algorithm: Chance_Search with Star1 (MAX)

- **Algorithm $F3.1'$(position p, value $alpha$, value $beta$, integer $depth$)**
    - // **max node**
    - **determine the successor positions** $p_1, \ldots, p_b$;
    - **if** $b = 0$ // **a terminal node**
      or $depth = 0$ // **remaining depth to search**
      or time is running up // **from timing control**
      or some other constraints are met // **add knowledge here**
    - **then return** $f(p)$; **else begin**
        - ▷ $m := -\infty$;
        - ▷ **for** $i := 1$ **to** b **do**
        - ▷ **begin**
        - ▷   **if** p_i **is to play a chance node** x
          then $t := Star1\_F3.1'(p_i, x, \max\{alpha, m\}, beta, depth - 1)$;
        - ▷   **else** $t := G3.1'(p_i, \max\{alpha, m\}, beta, depth - 1)$;
        - ▷   **if** $t > m$ **then** $m := t$;
        - ▷   **if** $m \geq beta$ **then** $return(m)$; // **beta cut off**
        - ▷ **end**;
    - **end**;
    - **return** m;

Algorithm: Chance_Search with Star1 (MIN)

- **Algorithm** $G3.1'$(**position** p, **value** $alpha$, **value** $beta$, **integer** $depth$)
    - // min node
    - determine the successor positions $p_1, \ldots, p_b$;
    - if $b = 0$ // a terminal node
      or $depth = 0$ // remaining depth to search
      or time is running up // from timing control
      or some other constraints are met // add knowledge here
    - then return $f(p)$; else begin
        - $\triangleright$ $m := \infty$;
        - $\triangleright$ for $i := 1$ to b do
        - $\triangleright$ begin
        - $\triangleright$ if p_i is to play a chance node x
          then $t := Star1\_G3.1'(p_i, x, alpha, \min\{beta, m\}, depth - 1)$;
        - $\triangleright$ else $t := F3.1'(p_i, alpha, \min\{beta, m\}, depth - 1)$;
        - $\triangleright$ if $t < m$ then $m := t$;
        - $\triangleright$ if $m \leq alpha$ then return(m); // alpha cut off
        - $\triangleright$ end;
    - end;
    - return m;

Star1: uniform case (MAX)

- **Algorithm** $Star1\_EQU\_F3.1'$(**position** p, **node** x, **value** $alpha$, **value** $beta$, **integer** $depth$)
  - // **a max chance node** x **with** c **equal probability choices** k_1, ..., k_c
  - **determine the possible values of the chance node** x **to be** $k_1, \ldots, k_c$
  - $A_0 = c \cdot (alpha - v_{max}) + v_{max}$, $B_0 = c \cdot (beta - v_{min}) + v_{min}$;
  - $m_0 = v_{min}$, $M_0 = v_{max}$ // **initial lower and upper bounds**
  - $vsum = 0$; // **initial sum of expected values**
  - **for** $i = 1$ **to** c **do**
  - **begin**
    - ▷ *let p_i be the position of assigning k_i to x in p;*
    - ▷ $t := G3.1'(p_i, \mathbf{max}\{A_{i-1}, v_{min}\}, \mathbf{min}\{B_{i-1}, v_{max}\}, depth)$
    - ▷ $m_i = m_{i-1} + (t - v_{min})/c$, $M_i = M_{i-1} + (t - v_{max})/c$;
    - ▷ *if $t \geq B_{i-1}$ then return m_i; // failed high, chance node cut off I*
    - ▷ *if $t \leq A_{i-1}$ then return M_i; // failed low, chance node cut off II*
    - ▷ $vsum \mathrel{+}= t$;
    - ▷ $A_i = A_{i-1} + v_{max} - t$, $B_i = B_{i-1} + v_{min} - t$;
  - **end**
- **return** $vsum/c$;

Star1: uniform case (MIN)

- **Algorithm** $Star1\_EQU\_G3.1'$**(position** p**, node** x**, value** $alpha$**, value** $beta$**, integer** $depth$**)**
    - // **a min chance node** x **with** c **equal probability choices** k_1, ..., k_c
    - **determine the possible values of the chance node** x **to be** $k_1, \ldots, k_c$
    - $A_0 = c \cdot (alpha - v_{max}) + v_{max}$, $B_0 = c \cdot (beta - v_{min}) + v_{min}$;
    - $m_0 = v_{min}$, $M_0 = v_{max}$ // **initial lower and upper bounds**
    - $vsum = 0$; // **initial sum of expected values**
    - **for** $i = 1$ **to** c **do**
    - **begin**
        - ▷ *let p_i be the position of assigning k_i to x in p;*
        - ▷ $t := F3.1'(p_i,$**max**$\{A_{i-1}, v_{min}\},$**min**$\{B_{i-1}, v_{max}\}, depth)$
        - ▷ $m_i = m_{i-1} + (t - v_{min})/c$, $M_i = M_{i-1} + (t - v_{max})/c$;
        - ▷ *if $t \geq B_{i-1}$ then return m_i; // failed high, chance node cut off I*
        - ▷ *if $t \leq A_{i-1}$ then return M_i; // failed low, chance node cut off II*
        - ▷ $vsum \mathrel{+}= t$;
        - ▷ $A_i = A_{i-1} + v_{max} - t$, $B_i = B_{i-1} + v_{min} - t$;
    - **end**
- **return** $vsum/c$**;**

Illustration: Star1

Star1: general case (1/3)

- **Assume the search window entering a chance node with $N = c$ choices is $(alpha, beta)$.**
- **The ith choice happens with the probability $Pr(x = i) = Pr_i$.**
- **For all i, the evaluated value of the ith choice is v_i.**
- **The value of a chance node after the first i choices are explored can be expressed as**
  - **an expected value $E_i = vexp_i$;**
    - ▷ $vexp_i = \sum_{j=1}^{i} Pr_j * v_j$
    - ▷ *This value is returned **only** when all choices are explored.*
      ⇒ The expected value of an un-explored child shouldn't be $\frac{v_{min} + v_{max}}{2}$.
  - **a range of possible values $[m_i, M_i]$.**
    - ▷ $m_i = vexp_i + \sum_{j=i+1}^{c} Pr_j * v_{min}$
    - ▷ $M_i = vexp_i + \sum_{j=i+1}^{c} Pr_j * v_{max}$
  - **Invariants:**
    - ▷ $E_i \in [m_i, M_i]$
    - ▷ $E_c = m_c = M_c$

Star1: general case (2/3)

- **Let m_i and M_i be the current lower and upper bounds, respectively, of the expected value of this chance node immediately after the evaluation of the ith node.**
  - $m_i = vexp_{i-1} + Pr_i * v_i + \sum_{j=i+1}^{c} Pr_j * v_{min}$
  - $M_i = vexp_{i-1} + Pr_i * v_i + \sum_{j=i+1}^{c} Pr_j * v_{max}$
- **How to incrementally update m_i and M_i:**
  - $m_0 = v_{min}$
  - $M_0 = v_{max}$
  -
$$m_i = m_{i-1} + Pr_i * (v_i - v_{min}) \qquad (7)$$
  -
$$M_i = M_{i-1} + Pr_i * (v_i - v_{max}) \qquad (8)$$

Star1: general case (3/3)

- **The current search window is** $(alpha, beta)$**.**
- **No more searching is needed when**
  - $m_i \geq beta$, **chance node cut off I**;
    $\Rightarrow$ **The lower bound found so far is good enough.**
    $\Rightarrow$ **Similar to a beta cut off.**
    $\Rightarrow$ **The returned value is** m_i**.**
  - $M_i \leq alpha$, **chance node cut off II**.
    $\Rightarrow$ **The upper bound found so far is bad enough.**
    $\Rightarrow$ **Similar to an alpha cut off.**
    $\Rightarrow$ **The returned value is** M_i**.**

Star1 cut off: general case (1/2)

- **When $m_i \geq beta$, chance node cut off I,**
    - **which means** $vexp_{i-1} + Pr_i * v_i + \sum_{j=i+1}^{c} Pr_j * v_{min} \geq beta$
    - $\Rightarrow v_i \geq B_{i-1} = \frac{1}{Pr_i} \cdot (beta - (vexp_{i-1} + \sum_{j=i+1}^{c} Pr_j * v_{min}))$
- **When $M_i \leq alpha$, chance node cut off II,**
    - **which means** $vexp_{i-1} + Pr_i * v_i + \sum_{j=i+1}^{c} Pr_j * v_{max} \leq alpha$
    - $\Rightarrow v_i \leq A_{i-1} = \frac{1}{Pr_i} \cdot (alpha - (vexp_{i-1} + \sum_{j=i+1}^{c} Pr_j * v_{max}))$
- **Hence set the window for searching the ith choice to be (A_{i-1}, B_{i-1}) which means no further search is needed if the result is not within this window.**

Star1 cut off: general case (2/2)

- **How to incrementally update A_i and B_i?**

  -
  $$A_0 = \frac{1}{Pr_1} \cdot (alpha - v_{max} * \sum_{i=1}^{c} Pr_i) + v_{max} \qquad (9)$$

  -
  $$B_0 = \frac{1}{Pr_1} \cdot (beta - v_{min} * \sum_{i=1}^{c} Pr_i) + v_{min} \qquad (10)$$

  -
  $$A_i = \frac{1}{Pr_{i+1}} * (Pr_i * A_{i-1} + Pr_{i+1} * v_{max} - Pr_i * v_i) \qquad (11)$$

  -
  $$B_i = \frac{1}{Pr_{i+1}} * (Pr_i * B_{i-1} + Pr_{i+1} * v_{min} - Pr_i * v_i) \qquad (12)$$

Star1: general case (MAX)

- **Algorithm** $Star1\_F3.1'$**(position** p**, node** x**, value** $alpha$**, value** $beta$**, integer** $depth$**)**
    - // **a max chance node** x **with** c **choices** k_1**, ...,** k_c
    - // **the** i**th choice happens with the probability** Pr_i
    - **determine the possible values of the chance node** x **to be** $k_1, \ldots, k_c$
    - **initialize** A_0 **and** B_0 **using formulas (9) and (10)**
    - $m_0 = v_{min}$**,** $M_0 = v_{max}$ // **initial lower and upper bounds**
    - $vexp = 0$**;** // **initial weighted sum of expected values**
    - **for** $i = 1$ **to** c **do**
    - **begin**
        - $\triangleright$ **let** P_i **be the position of assigning** k_i **to** x **in** p**;**
        - $\triangleright$ $t := G3.1'(p_i,$**max$\{A_{i-1}, v_{min}\}$,min$\{B_{i-1}, v_{max}\}$,**$depth)$
        - $\triangleright$ **incrementally update** m_i **and** M_i **using formulas (7) and (8)**
        - $\triangleright$ **if** $t \geq B_{i-1}$ **then return** m_i**;** // **failed high, chance node cut off I**
        - $\triangleright$ **if** $t \leq A_{i-1}$ **then return** M_i**;** // **failed low, chance node cut off II**
        - $\triangleright$ $vexp \mathrel{+}= Pr_i * t$**;**
        - $\triangleright$ **incrementally update** A_i **and** B_i **using formulas (11) and (12)**
    - **end**
- **return** $vexp$**;**

Star1: general case (MIN)

- **Algorithm** $Star1\_G3.1'$**(position** p**, node** x**, value** $alpha$**, value** $beta$**, integer** $depth$**)**
  - // **a min chance node** x **with** c **choices** k_1**,** $\ldots$**,** k_c
  - // **the** i**th choice happens with the probability** Pr_i
  - **determine the possible values of the chance node** x **to be** $k_1, \ldots, k_c$
  - **initialize** A_0 **and** B_0 **using formulas (9) and (10)**
  - $m_0 = v_{min}$**,** $M_0 = v_{max}$ // **initial lower and upper bounds**
  - $vexp = 0$**;** // **initial weighted sum of expected values**
  - **for** $i = 1$ **to** c **do**
  - **begin**
    - $\triangleright$ *let* P_i *be the position of assigning* k_i *to* x *in* p*;*
    - $\triangleright$ $t := F3.1'(p_i,$**max**$\{A_{i-1}, v_{min}\},$**min**$\{B_{i-1}, v_{max}\}, depth)$
    - $\triangleright$ *incrementally update* m_i *and* M_i *using formulas (7) and (8)*
    - $\triangleright$ *if* $t \geq B_{i-1}$ *then return* m_i*;* // *failed high, chance node cut off I*
    - $\triangleright$ *if* $t \leq A_{i-1}$ *then return* M_i*;* // *failed low, chance node cut off II*
    - $\triangleright$ $vexp \mathrel{+}= Pr_i * t$*;*
    - $\triangleright$ *incrementally update* A_i *and* B_i *using formulas (11) and (12)*
  - **end**
- **return** $vexp$**;**

- **Assume the ith choice happens with a chance w_i/c where $c = \sum_{i=1}^{N} w_i$ and N is the total number of choices.**
  - $m_0 = v_{min}$
  - $M_0 = v_{max}$
  - $m_i = (\sum_{j=1}^{i-1} w_j \cdot v_j + w_i \cdot v_i + v_{min} \cdot (c - \sum_{j=1}^{i} w_j))/c$
    - $\triangleright$

$$m_i = m_{i-1} + (w_i/c) \cdot (v_i - v_{min}) \tag{13}$$

  - $M_i = (\sum_{j=1}^{i-1} w_j \cdot v_j + w_i \cdot v_i + v_{max} \cdot (c - \sum_{j=1}^{i} w_j))/c$
    - $\triangleright$

$$M_i = M_{i-1} + (w_i/c) \cdot (v_i - v_{max}) \tag{14}$$

- **Assume the ith choice happens with a chance w_i/c where $c = \sum_{i=1}^{N} w_i$ and N is the total number of choices.**
  -
  $$A_0 = (c/w_1) \cdot (alpha - v_{max}) + v_{max} \tag{15}$$
  -
  $$B_0 = (c/w_1) \cdot (beta - v_{min}) + v_{min} \tag{16}$$
  - $A_{i-1} = (c \cdot alpha - (\sum_{j=1}^{i-1} w_j \cdot v_j - v_{max} \cdot (c - \sum_{j=1}^{i} w_j)))/w_i$
    - $\triangleright$
    $$A_i = (w_i/w_{i+1}) \cdot (A_{i-1} - v_i) + v_{max} \tag{17}$$
  - $B_{i-1} = (c \cdot beta - (\sum_{j=1}^{i-1} w_j \cdot v_j - v_{min} \cdot (c - \sum_{j=1}^{i} w_j)))/w_i$
    - $\triangleright$
    $$B_i = (w_i/w_{i+1}) \cdot (B_{i-1} - v_i) + v_{min} \tag{18}$$

Remarks

- **To know what operations are simplified from the general case to special cases, compare these formulas**

	general case	GCD case	uniform case
m_i	7	13	1
M_i	8	14	2
a_0	9	15	3
b_0	10	16	4
A_i	11	17	5
B_i	12	18	6

Comments (1/2)

- **Star0.5 finishes searching a choice using the maximum window size and then decide whether to go on searching the next choice or not, while Star1 can use sharper window size to end searching a choice earlier.**

- **We illustrate the ideas using a fail soft version of the alpha-beta algorithm.**
  - Original (hallow) and fail hard version have a simpler logic in maintaining the search interval.
  - The semantic of comparing an exact return value with an expected returning value is something that needs careful thinking.
  - May want to pick a chance node with a lower expected value but having a hope of winning, not one with a slightly higher expected value but having no hope of winning when you are in disadvantageous.
  - May want to pick a chance node with a lower expected value but having no chance of losing, not one with a slightly higher expected value but having a chance of losing when you are in advantage.
  - Do not always pick one with a slightly larger expected value. Give the second one some chance to be selected.

Comments (2/2)

- **Need to revise algorithms carefully when dealing with the original, fail hard or NegaScout version.**
  - What does it mean to combine bounds from a fail hard version?
- **The lower and upper bounds of the expected score can be used to do alpha-beta pruning.**
  - Nicely fit into the alpha-beta search algorithm.
  - Not only we can terminate the searching of choices earlier, but also we can terminate the searching of a particular choice earlier.
- **Exist other improvements by searching choices of a chance node "in parallel".**

Implementation hints (1/2)

- **Fully unwrap a chance node takes more time than that of a non-chance node.**
  - **If you set your depth limit to d for a game without chance nodes, then the depth limit should be lower for that game when chance node is introduced.**
  - **Technically speaking, a chance node adds at least one level.**
    - ▷ *Depending on the number of choices you have compared to the number of non-chance children, you may need to reduce the search depth limit by at least 3 or 5, and maybe 7.*
    - ▷ *Estimate the complexity of a chance node by comparing the number of choices of a chance node and the number of non-chance-node moves.*

- **Without searching a chance node, it is easy to obtain not enough progress by just searching a long sequence of non-chance nodes.**
  - **In CDC, when there are only a limited number of revealed pieces, there is not much you can do by just moving around.**

Implementation hints (2/2)

- **Practical considerations, for example in Chinese Dark Chess (CDC), are as follows.**
  - **You normally do not need to consider the consequence of flipping more than 2 dark pieces.**
    - ▷ *Set a maximum number of chance node searching in any DFS search path.*
  - **It makes little sense to consider ending a search with exploring a chance node.**
    - ▷ *When depth limit left is less than 3 or 4, stop exploring chance nodes.*
  - **It also makes little sense to consider the consequence of exploring 2 chance nodes back to back.**
    - ▷ *Make sure two chance nodes in a DFS search path is separated by at least 3 or 4 non-chance nodes.*
  - **It is rarely the case that a chance node exploration is the first ply to consider in move ordering unless it is recommended by a prior knowledge or no other non-chance-node moves exists.**

More ideas for improvements

- **Notations**
  - **Assume p is a chance node with the tree T.**
    - ▷ *T_i is the tree of p when for the ith choice.*
    - ▷ *$T_{i,j}$ is the jth branch of T_i, namely, with the root $p_{i,j}$.*
    - ▷ *v_i is the evaluated value of T_i.*
   - ▷ *$v_{i,j}$ is the evaluated value of $T_{i,j}$.*

- **An exact probe of a tree rooted at r is thus to fully search a subtree rooted at a child of r.**
  - ▷ *An exact probe of T is thus to fully search T_i for some i and then obtain v_i.*
  - ▷ *An exact probe of T_i is to fully search $T_{i,j}$ for some j and then obtain $v_{i,j}$.*

- **Can do better by not searching the DFS order.**
  - **It is not necessary to search completely T_1 and then start to look at the subtree of T_2, ... etc.**
    - ▷ *The approach used by Star1.*
  - **Probe T_i gives you some information about the possible range of v_i.**
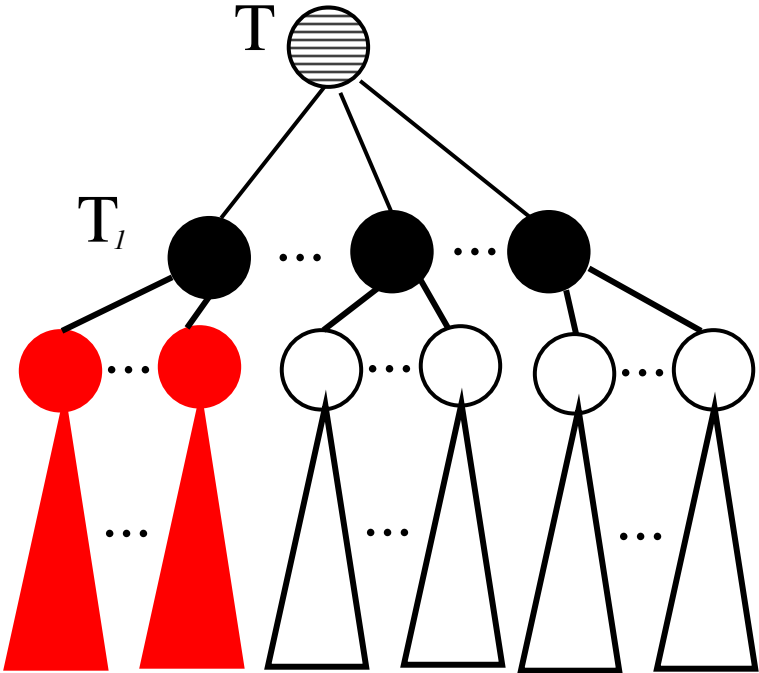
Illustration: Probe
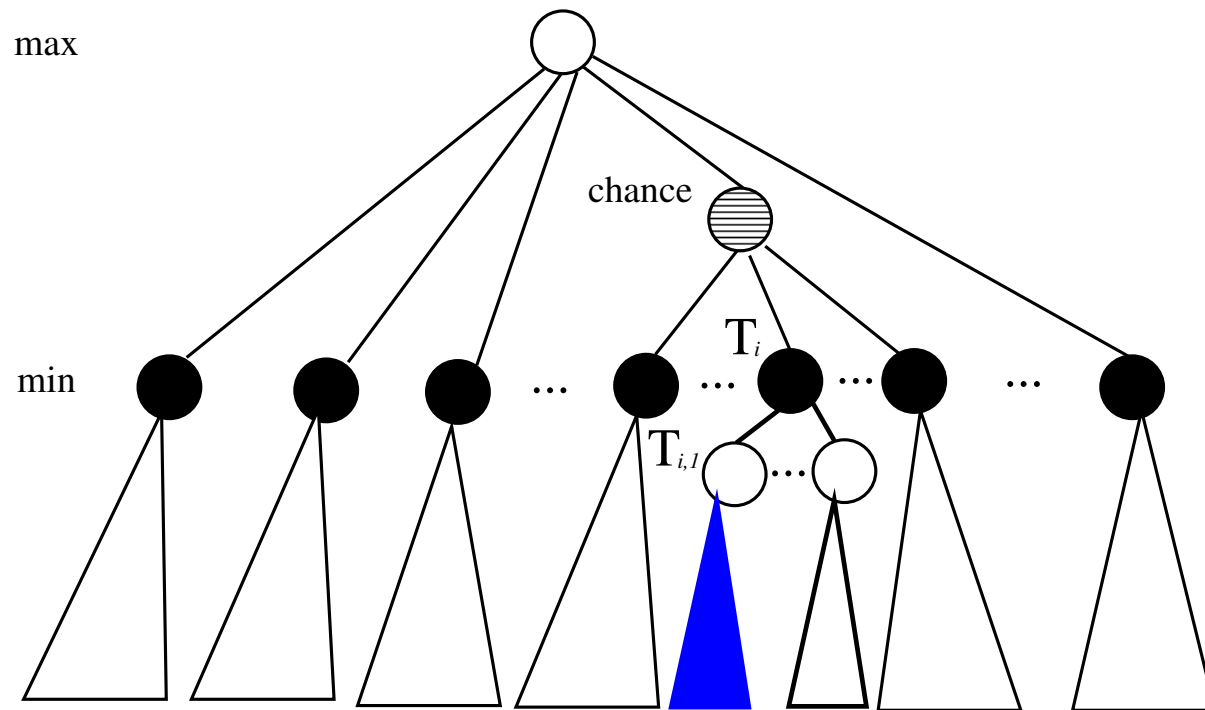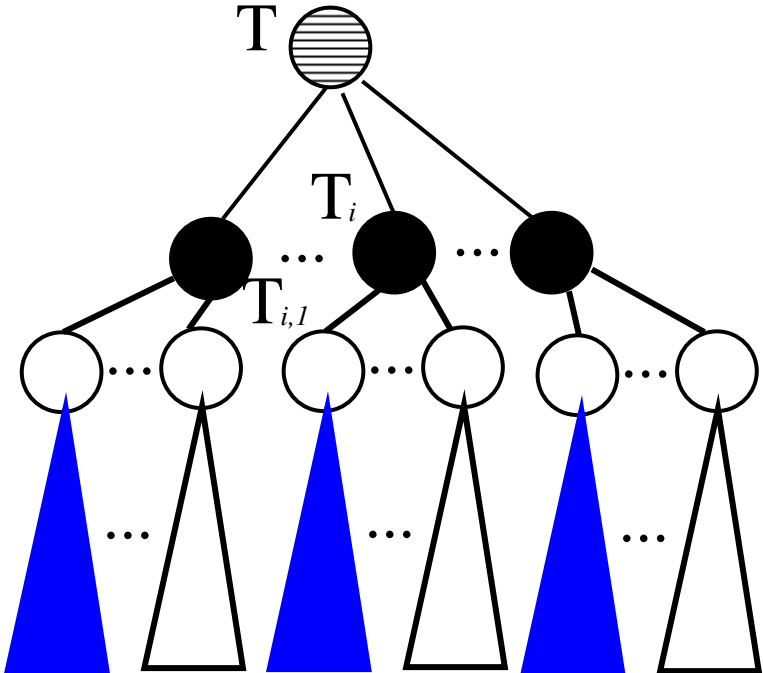

The first child of Ti is probed.

Star2: MAX node, general case

- p **is a MAX node. Thus each child** p_i **is a MIN node.**
- **We have probed the first child of** T_i **and obtained** $v_{i,1}$**.**
  - **Since** p_i **is a MIN node,** $v_{i,1}$ **is an upper bound of** v_i **which is usually lower than the maximum possible value** v_{max}**.**
  - **The upper bound of** v_1 **is thus lowered.**
  - **It is possible because of this probe, an alpha cut can be performed.**
- **Notations**
  - $v \in [m_i, M_i]$ **which are the lower and upper bounds of** v **after the** i **probe.**
  - $v_j \in [L_j, U_j]$ **which are the lower and upper bounds of** v_j**.**
- **Formulas for Star2**
  - $v_j \in [v_{min}, v_{j,1}]$ **after** T_j **is probed.**
  - **After the** i**th probe,** $v \in [m_0, M_{i-1} + Pr_i \times (v_{i,1} - v_{max})]$**.**
    - ▷ m_i *is unchanged, but* $M_i = M_{i-1} + Pr_i \times (v_{i,1} - v_{max})]$
    - ▷ A_i *is updated according to 11, but* B_i *is unchanged.*
  - **In comparison, for Star1**
    - ▷ $m_i = m_{i-1} + Pr_i \times (v_i - v_{min})]$
    - ▷ $M_i = M_{i-1} + Pr_i \times (v_i - v_{max})]$
    - ▷ *Both* A_i *and* B_i *are updated.*

Illustration: Star1 and Star2 probing

Star1: Probe the first child of T

Star2: Probe the first child of each Ti

Star2: MIN node, general case

- p is a MIN chance node. Thus each child p_i is a MAX node.
- We have probed the first child of T_i and obtained $v_{i,1}$.
  - Since p_i is a MAX node, $v_{i,1}$ is a lower bound of v_i which is usually larger than the minimum possible value v_{min}.
  - The lower bound of v_i is thus raised.
  - It is possible because of this probe, a beta cut can be performed.
- **Notations**
  - $v \in [m_i, M_i]$ which are the lower and upper bounds of v after the i probe.
  - $v_j \in [L_j, U_j]$ which are the lower and upper bounds of v_j.
- **Formulas for Star2**
  - $v_j \in [v_{j,1}, v_{max}]$ after T_j is probed.
  - **After the ith probe, $v \in [m_{i-1} + Pr_i \times (v_{i,1} - v_{min}), M_0$.**
    - ▷ $m_i = m_{i-1} + Pr_i \times (v_{i,1} - v_{max})]$, but M_i is unchanged.
    - ▷ A_i is unchanged, but B_i is updated according to 12.
  - **In comparison, for Star1**
    - ▷ $m_i = m_{i-1} + Pr_i \times (v_i - v_{min})]$
    - ▷ $M_i = M_{i-1} + Pr_i \times (v_i - v_{max})]$
    - ▷ Both A_i and B_i are updated.

Algorithm: Chance_Search with Star2 (MAX)

- **Algorithm** $F3.2'$(**position** p, **value** $alpha$, **value** $beta$, **integer** $depth$)
    - **// max node**
    - **determine the successor positions** $p_1, \ldots, p_b$;
    - **if** $b = 0$ **// a terminal node**
      or $depth = 0$ **// remaining depth to search**
      or time is running up // from timing control
      or some other constraints are met // add knowledge here
    - **then return** $f(p)$; **else begin**
        - $\triangleright$ $m := -\infty$;
        - $\triangleright$ *for* $i := 1$ *to* b *do*
        - $\triangleright$ *begin*
        - $\triangleright$ *if* p_i *is to play a chance node* x
          then $t := Star2\_F3.2'(p_i, x, \max\{alpha, m\}, beta, depth - 1)$;
        - $\triangleright$ *else* $t := G3.2'(p_i, \max\{alpha, m\}, beta, depth - 1)$;
        - $\triangleright$ *if* $t > m$ *then* $m := t$;
        - $\triangleright$ *if* $m \geq beta$ *then return*(m); **// beta cut off**
        - $\triangleright$ *end*;
    - **end**;
    - **return** m;

Algorithm: Chance_Search with Star2 (MIN)

- **Algorithm** $G3.2'$(**position** p, **value** $alpha$, **value** $beta$, **integer** $depth$)
    - **// min node**
    - **determine the successor positions** $p_1, \ldots, p_b$;
    - **if** $b = 0$ **// a terminal node**
      or $depth = 0$ **// remaining depth to search**
      or time is running up // from timing control
      or some other constraints are met // add knowledge here
    - **then return** $f(p)$; **else begin**
        - ▷ $m := \infty$;
        - ▷ *for* $i := 1$ *to* b *do*
        - ▷ *begin*
        - ▷   *if* p_i *is to play a chance node* x
          then $t := Star2\_G3.2'(p_i, x, alpha, \min\{beta, m\}, depth - 1)$;
        - ▷   *else* $t := F3.2'(p_i, alpha, \min\{beta, m\}, depth - 1)$;
        - ▷   *if* $t < m$ *then* $m := t$;
        - ▷   *if* $m \leq alpha$ *then return*(m); **// alpha cut off**
        - ▷ *end*;
    - **end;**
    - **return** m;

Star2: MAX

- **Algorithm** $Star2\_F3.2'$(**position** p, **node** x, **value** $alpha$, **value** $beta$, **integer** $depth$)
  - // a max chance node x with c choices k_1, ..., k_c
  - // the ith choice happens with the probability $\Pr_i$
  - determine the possible values of the chance node x to be $k_1, \ldots, k_c$
  - initialize A_0, B_0, m_0 and M_0 as in $Star1\_F3.1'$
  - // Do an exact probing for each choice to find cut off's.
  - for each choice i from 1 to c do
    - ▷ Let p_i be the position obtained from p by making x the choice k_i. // p is MAX, p_i is MIN, $p_{i,j}$ is MAX
    - ▷ // do an exact probe on the first MAX child of p_i $v := F3.2'(p_{i,1}, \max\{A_{i-1}, v_{min}\}, \min\{B_{i-1}, v_{max}\}, depth)$
    - ▷ update A_i and M_i as in $Star1\_F3.1'$
    - ▷ If $M_i \leq alpha$ then return M_i; // alpha cut off
  - // normal exhaustive search phase // no cut off is found in the above, do the normal Star1 search. // Chance node cut off I may happen. // Chance node cut off II may happen.
  - return $vexp = Star1\_F3.1(p, x, alpha, beta, depth)$;

Star2: MIN

- **Algorithm** $Star2\_G3.2'$(**position** p, **node** x, **value** $alpha$, **value** $beta$, **integer** $depth$)
  - // a min chance node x with c choices k_1, ..., k_c
  - // the ith choice happens with the probability $\Pr_i$
  - determine the possible values of the chance node x to be $k_1, \ldots, k_c$
  - initialize A_0, B_0, m_0 and M_0 as in $Star1\_G3.1'$
  - // Do an exact probing for each choice to find cut off's.
  - for each choice i from 1 to c do
    - ▷ Let p_i be the position obtained from p by making x the choice k_i. // p is MIN, p_i is MAX, $p_{i,j}$ is MIN
    - ▷ // do an exact probe on the first MIN child of p_i $v := G3.2'(p_{i,1},\max\{A_{i-1},v_{min}\},\min\{B_{i-1},v_{max}\},depth)$
    - ▷ update B_i and m_i as in $Star1\_G3.1'$
    - ▷ If $m_i \geq beta$ then return m_i; // beta cut off
  - // normal exhaustive search phase
    // no cut off is found in the above, do the normal Star1 search.
    // Chance node cut off I may happen.
    // Chance node cut off II may happen.
  - return $vexp = Star1\_G3.1(p, x, alpha, beta, depth)$;

Comments for Star2

- **NOTE:**
  - **In** $Star2\_F3.2'(p, x, alpha, beta, depth)$**, we do a probe on** $p_{i,1}$**,** $1 \le i \le c$**.**

    - ▷ *Positions p and $p_{i.1}$ are played by the same player. Hence we both use max-node algorithms.*
  - **Similarly for** $Star2\_G3.2'(p, x, alpha, beta, depth)$**, min-node algorithms are used.**
  - **This is different from Star0.5 and Star1.**
- **During the exact probe phase, some bounds are known which can be used to update the search window.**
- **If no cut off is found in the probing phase, then we need to do the exhaustive searching phase.**
  - **The searched branches in the probing phase do not need to be researched again.**

More ideas for probes

- **Move ordering in exploring the choices is critical in having a good performance.**
- **Picking which child to do the probe is also critical.**
- **Can do exact probes on h children, called probing factor $h > 1$, of a choice instead of just exactly one.**
- **May decide to probe different number of children for each choice.**

Probing orders

- **Two types of probing orders with a probing factor h**
  - **Cyclic probing**
    - ▷ *Probe one child of a choice at one time for all choices, and do this for h rounds.*
    - ▷ *for $j = 1$ to h do*
          for $i = 1$ to c do
              probe the jth child of the ith choice
  - **Sequential probing**
    - ▷ *Probe h children of a choice at one time and then do it for each choice in sequence*
    - ▷ *for $i = 1$ to c do*
          probe h children of the ith choice
    - ▷ *Switch lines 6 and 7 in algorithms Star2.5_F3.2.5' and Star2.5_G3.2.5'.*

- **Special cases**
  - ▷ *When $h = 0$, Star2 == Star1.*
  - ▷ *When $h = 1$, cyclic probing == sequential probing and also Star2 == Star2.5.*

Illustration: Star2.5 probing

sequential probing: 1   h $h*i$ $h*(i+1)$ $h*(c-1)$ $h*c$

(order of probing)

cyclic probing: 1 $h*(c-1)$ 2 $h*c-1$ c $h*c$
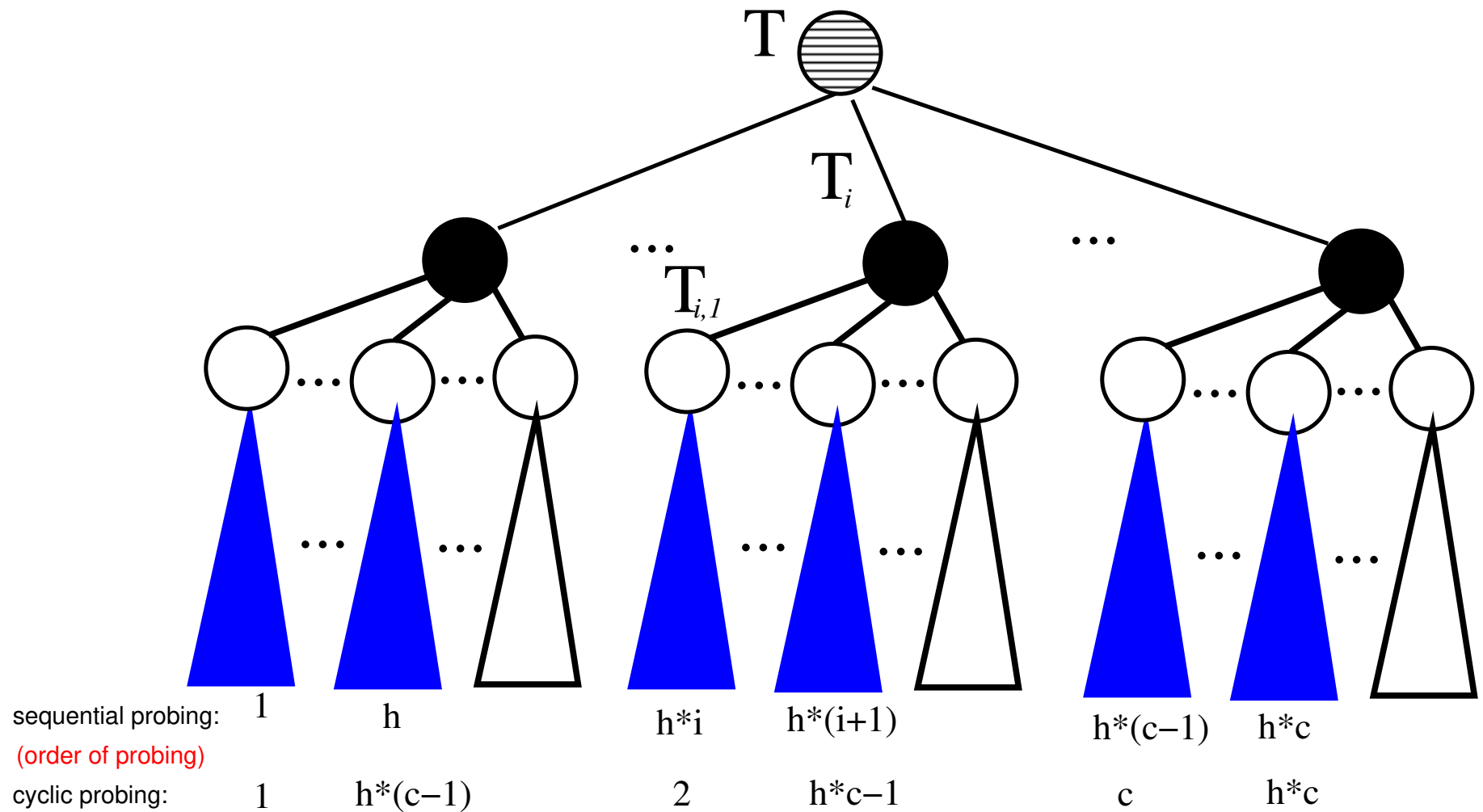
Star2.5: Probe the first h children of each Ti

Chance_Search with Star2.5 (MAX)

- **Algorithm** $F3.2.5'$(**position** p, **value** $alpha$, **value** $beta$, **integer** $depth$, **integer** h)
  - // max node
  - determine the successor positions $p_1, \ldots, p_b$;
  - **if** $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return** $f(p)$; **else begin**
    - ▷ $m := -\infty$;
    - ▷ **for** $i := 1$ **to** b **do**
    - ▷ **begin**
    - ▷   **if** p_i **is to play a chance node** x
      then $t := Star2.5\_F3.2.5'(p_i, x, \max\{alpha, m\}, beta, depth - 1, h)$;
    - ▷   **else** $t := G3.2.5'(p_i, \max\{alpha, m\}, beta, depth - 1, h)$;
    - ▷   **if** $t > m$ **then** $m := t$;
    - ▷   **if** $m \geq beta$ **then** $return(m)$; // beta cut off
    - ▷ **end**;
  - **end**;
  - **return** m;

Chance_Search with Star2.5 (MIN)

- **Algorithm** $G3.2.5'$(**position** p, **value** $alpha$, **value** $beta$, **integer** $depth$, **integer** h)
  - // **min node**
  - **determine the successor positions** $p_1, \ldots, p_b$;
  - **if** $b = 0$ // **a terminal node**
    or $depth = 0$ // **remaining depth to search**
    or time is running up // **from timing control**
    or some other constraints are met // **add knowledge here**
  - **then return** $f(p)$; **else begin**
    - ▷ $m := \infty$;
    - ▷ **for** $i := 1$ **to** b **do**
    - ▷ **begin**
    - ▷ **if** p_i **is to play a chance node** x
      then $t := Star2.5\_G3.2.5'(p_i, x, alpha, \min\{beta, m\}, depth - 1, h)$;
    - ▷ **else** $t := F3.2.5'(p_i, alpha, \min\{beta, m\}, depth - 1, h)$;
    - ▷ **if** $t < m$ **then** $m := t$;
    - ▷ **if** $m \leq alpha$ **then return**(m); // **alpha cut off**
    - ▷ **end;**
  - **end;**
  - **return** m;

Star2.5: cyclic probing (MAX)

- **Algorithm** $Star2.5\_F3.2.5'$(**position** p, **node** x, **value** $alpha$, **value** $beta$, **integer** h) // h **is the probing factor**
  - // a MAX chance node x with c choices k_1, ..., k_c
  - // the ith choice happens with the probability Pr_i
  - determine the possible values of the chance node x to be $k_1, \ldots, k_c$
  - initialize A_0, B_0, m_0 and M_0 as in $Star1\_F3.1'$
  - // Do a cyclic probing to decide whether some cut off can be performed.
  - 6: for j from 1 to h do
    7:     for each choice i from 1 to c do
    - $\triangleright$ Let p_i be the position obtained from p by making x the choice k_i. // p is MAX, p_i is MIN, $p_{i,j}$ is MAX
    - $\triangleright$ // do an exact probe on the jth MAX child of p_i. $v := F3.2.5'(p_{i,j}, \max\{A_{i-1}, v_{min}\}, \min\{B_{i-1}, v_{max}\}, depth)$
    - $\triangleright$ update A_i and M_i as in $Star1\_F3.1'$
    - $\triangleright$ If $M_i \leq alpha$ then return M_i; // alpha cut off
  - // normal exhaustive search phase // no cut off is found in the above, do the normal Star1 search. // Chance node cut off I may happen. // Chance node cut off II may happen.
  - return $vexp = Star1\_F3.1(p, x, alpha, beta, depth)$;

Star2.5: cyclic probing (MIN)

- **Algorithm** $Star2.5\_G3.2.5'$(**position** p, **node** x, **value** $alpha$, **value** $beta$, **integer** h) **//** h **is the probing factor**
    - **// a MIN chance node** x **with** c **choices** k_1, ..., k_c
    - **// the** i**th choice happens with the probability Pr**$_i$
    - **determine the possible values of the chance node** x **to be** $k_1, \dots, k_c$
    - **initialize** A_0, B_0, m_0 **and** M_0 **as in** $Star1\_G3.1'$
    - **// Do a cyclic probing to decide whether some cut off can be performed.**
    - **6: for** j **from** 1 **to** h **do**
      7:   for each choice i **from** 1 **to** c **do**
        - ▷ *Let p_i be the position obtained from p by making x the choice k_i.*
          // p is MIN, p_i is MAX, $p_{i,j}$ is MIN
        - ▷ *// do an exact probe on the jth MIN child of p_i.*
          $v := G3.2.5'(p_{i,j},\mathbf{max}\{A_{i-1},v_{min}\},\mathbf{min}\{B_{i-1},v_{max}\},depth)$
        - ▷ *update B_i and m_i as in $Star1\_G3.1'$*
        - ▷ *If $m_i \geq beta$ then return m_i; // beta cut off*
    - **// normal exhaustive search phase**
      // no cut off is found in the above, do the normal Star1 search.
      // Chance node cut off I that is similar to beta cut off may happen.
      // Chance node cut off II that is similar to alpha cut off may happen.
    - **return** $vexp = Star1\_G3.1(p, x, alpha, beta, depth)$;

Comments

- **Experimental results provided in [Ballard '83] on artificial game trees.**
  - **Star1 may not be able to cut more than 20% of the leaves.**
  - **Star2.5 with $h = 1$, i.e. Star2, cuts more than 59% of the nodes and is about twice better than Star1.**
  - **Sequential probing is best when $h = 3$ which cuts more than 65% of the nodes and roughly cut about the same nodes as Star2.5 using the same probing factor.**
  - **Sequential probing gets worse when $h > 4$. For example, it only cut 20% of the leaves when $h = 20$.**
  - **Star2.5 continues to cut more nodes when h gets larger, though the gain is not that great. At $h = 3$, about 70% of the nodes are cut. At $h = 20$, about 72% of the nodes are cut.**
- **Need to store the bounds and when the bounds produces cuts in the hash table for later to resume searching if needed later when the node is revisited.**
- **Better move ordering is also needed to get a fast cut off.**

Approximated Probes

- **We can also have heuristics for issuing approximated probes which returns approximated values.**
- **Strategy I: random probing of some promising choices**
  - **Do a move ordering heuristic to pick one or some promising choices to expand first.**
  - **These promising choices can improve the lower or upper bounds and can cause beta or alpha cut off.**
- **Strategy II: fast probing of all choices**
  - **Possible implementations**
    - ▷ *do a static evaluation on all choices*
    - ▷ *do a shallow alpha-beta searching on each choice*
    - ▷ *do a MCTS-like simulation on the choices*
  - **Use these information to decide whether you have enough confidence to do a cut off.**

Using MCTS with chance nodes (1/2)

- **Assume a chance node x has c choices $k_1, \ldots, k_c$ and the ith choice happens with the probability Pr_i**
- **Selection**
  - **If x is picked in the PV during selection, then a random coin tossing according to the probability distribution of the choices is needed to pick which choice to descent.**
    - ▷ *It is better to even the number of simulations done on each choice.*
    - ▷ *Use random sampling without replacement. When every one is picked once, then start another round of picking.*

- **Expansion**
  - **If the last node in the PV is x, then expand all choices and simulate each choice some number of times.**
    - ▷ *Watch out the discuss on maxing chance nodes in a searching path such as whether it is desirable to have 2 chance nodes in sequence ... etc.*

Using MCTS with chance nodes (2/2)

- **Simulation**
  - **When a chance node is to be simulated, then be sure to randomly, according to the probability distribution, pick a choice.**
    - ▷ *Use some techniques to make sure you are doing an effective sampling when the number of choices is huge*
    - ▷ *Watch out what are "reasonable" in a simulated plyout on the mixing of chance nodes.*

- **Back propagation**
  - **The UCB score of x is $w_i + c\sqrt{(lnN/N_i)}$ where w_i is the weighted winning rate, or score, of the children, N_i is the total number of simulations done on all choices. and N is the total number of simulations done on the parent of x.**

Sparse sampling: intuition

- **Assume in searching the number of possible outcomes in a, maybe chance, node is too large. A technique called sparse sampling can be used [Kearns et al 2002] [Lanctot et al 2013].**
  - Can also be used in the expansion phase of MCTS.
- **Ideas:**
  - Assume the sample space $\mathcal{A}$ is too large.
    - ▷ *The sample space used is enlarged as the number of visits to the node increases.*
    - ▷ *Only consider a reduced sample space with k_t randomly selected choices from $\mathcal{A}$ called $\mathcal{S}_t$, in the first t visits where $k_t = \lceil c * t^\alpha \rceil$, and c and α are constants chosen by experiments.*
    - ▷ *Use the current choice set as an estimation of its goodness.*

Sparse sampling: algorithm

- **Algorithm SS for sparse sampling**
  - $t := 1$
  - **Initial k_t to be a small constant, say 1.**
  - **Initial the reduced candidate set $\mathcal{S}_t$ to be an empty set.**
  - **Randomly add k_t children from $\mathcal{A}$ into $\mathcal{S}_t$**
  - **loop: Performs some h samplings from $\mathcal{S}_t$.**
    - ▷ $t \mathrel{+}= h$
    - ▷ **Add randomly $k_{t+h} - k_t$ new children from $\mathcal{A}$ into $\mathcal{S}_t$**
  - **goto loop**
- **usually h is 1.**

Sparse sampling: analysis

- **The estimated value is accurate with a high probability [Kearns et al 2002] [Lanctot et al 2013]**
- **Theorem:**

$$Pr(|\tilde{V} - V| \le \lambda \cdot d) \ge 1 - (2 \cdot k_t \cdot c)^d exp\{\frac{-\lambda^2 \cdot k_t}{2 \cdot v_{max}^2}\},$$

where

> k_t *is the number of choices considered with* t *samplings,*
> $\tilde{V}$ *is the estimation considering only* k_t *choices,*
> V *is the value considering all choices,*
> c *is the actual number of choices,*
> d *is the depth simulated,*
> $\lambda \in (0, 2 \cdot v_{max}]$ *is a parameter chosen, where* v_{max} *is the maximum possible value.*

- **Note: the proof is done by making sampling with replacement, while the algorithm asks for sampling without replacement.**

Comments

- **Chance node introduces a large searching space that needs a careful treatment.**
    - **Need information in every possible branch to come out with a good strategy.**
- **Suppose that in each move,**
    - **on**
        - ▷ *a prior chance node: you have m possible moves followed by r different random outcomes.*
        - ▷ *a posteriori chance node: there are r different random outcomes from the coin toss and m possible moves followed.*
    - **Depending on r and m, good search algorithms can be designed.**
        - ▷ *When $m >> r$, you may plainly enumerate all r alternatives.*
        - ▷ *When $m << r$, may need to devise good strategies.*
- **Instead of looking for something that is sure-not-to-loss, may want something that is have-a-chance-to-win.**

References and further readings (1/2)

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