

## Short Paper

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# A Numerical Method for Solving Time-Invariant System by The Generalized Inverse Vandermonde Matrix

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A simple numerical algorithm based on the generalized Inverse Vandermonde matrix for evaluation of time response to a time-invariant system is proposed. The system is assumed to be governed by a high order linear differential equation with constant coefficients. The technique involves determination of the partial-fraction expansion of rational functions. Only synthetic division and longhand division are required, which makes the process very suitable for computer programming. Meanwhile, because the solution is directly related to the system's initial conditions, the proposed approach only requires computation of the inverse generalized Vandermonde matrix.

**Keywords:** inverse Vandermonde matrix, ordinary differential equations, time-response, time-invariant system

## 1. INTRODUCTION

In the analysis of a time-invariant system, one often needs to solve the time response function  $y(t)$  from the  $n$ -th order linear differential equation with prescribed initial conditions at  $t = 0$ . The systematic approach to the solution is usually to use time-frequency transformation. The transfer function is derived by the Laplace transformation, and the resolved time response is then determined by the inverse transformation. However, if the initial values are altered for some reason, the whole calculation process has to be repeated from the very beginning. The calculation procedure becomes very complicated when the order is large and the characteristic values have high multiplicities. Furthermore, in solving the real-time time response, an accurate and fast method which is suitable for computer programming is essential for some applications. The objective of this article is to derive an algebraic, from approach based on some simple operations for solving the time response function of a time-invariant system.

For illustration purposes, there is a Laplace transfer function:

$$Y(s) = \frac{A(s)}{B(s)} = \frac{A(s)}{(s + s_1)(s + s_2)\dots(s + s_{n-m})(s + s_i)^m}. \quad (1)$$

Then  $Y(s)$  can be expanded as

$$Y(s) = \frac{K_1}{s+s_1} + \frac{K_2}{s+s_2} + \dots + \frac{K_{(n-m)}}{s+s_{n-m}} + \frac{X_1}{s+s_i} + \frac{X_2}{(s+s_i)^2} + \dots + \frac{X_m}{(s+s_i)^m}. \quad (2)$$

The  $(n-m)$  coefficients, which correspond to simple poles,  $K_1, K_2, \dots, K_{(n-m)}$ , may be evaluated simply by

$$\begin{aligned} K_j &= \left[ (s+s_j) \frac{A(s)}{B(s)} \right]_{s=-s_j}, j=1 \sim (n-m) \\ &= \frac{A(-s_j)}{(s_2-s_j)(s_3-s_j)\dots(s_{n-m}-s_j)}. \end{aligned} \quad (3)$$

The determination of the coefficients that correspond to the multiple-order poles can be described as follows:

$$\begin{aligned} X_m &= [(s+s_i)^m Y(s)]_{s=-s_i}, \\ X_{m-1} &= \frac{d}{ds} [(s+s_i)^m Y(s)]_{s=-s_i}, \\ X_{m-2} &= \frac{1}{2!} \frac{d^2}{ds^2} [(s+s_i)^m Y(s)]_{s=-s_i}, \\ &\vdots \\ &\vdots \\ &\vdots \\ X_1 &= \frac{1}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} [(s+s_i)^m Y(s)]_{s=-s_i}. \end{aligned}$$

The desired time response  $y(t)$  is then determined by the inverse Laplace transformation[2]. The inverse Laplace transform operation involving rational functions can be carried out using a Laplace transfer table. Although some numerical methods of partial-fraction expansion, which can be carried out using digital computer, has been proposed, but most of them are limited to allowing lower multiple-order poles of transformation or calculation precision [6, 9, 10].

On the other hand, to deal with higher order homogeneous differential equations with constant coefficients, such as the function  $Y(s)$  in Eq.(1) in the time domain, we have

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y^1 + a_0y = 0 \quad (4)$$

For simplicity, Eq.(4) can rewritten as

$$L[y] = [D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0]y. \quad (5)$$

One can derive the solution from the characteristic equation:

$$\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0 = 0. \tag{6}$$

Based on the roots of Eq. (6), the solution is found to be an exponential function:

$$L[e^{\alpha x}] = (\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0)e^{\alpha x}. \tag{7}$$

Let  $\alpha_1$  be an  $m$ -th order root of Eq. (6), and let  $\alpha_{m+1}, \dots, \alpha_n$  be the other roots, all different from  $\alpha_1$ , when  $m < n$ . In the product form, it can be expressed as

$$L[e^{\alpha x}] = (\alpha - \alpha_1)^m h(\alpha)e^{\alpha x} \tag{8}$$

with  $h(\alpha)=1$  if  $m = n$  and  $h(\alpha)=(\alpha-\alpha_{m+1})\dots(\alpha-\alpha_n)$  if  $m < n$ . By differentiating on both sides with respect to  $\alpha$ , the following equation can be obtained:

$$\frac{\partial}{\partial \alpha} L[e^{\alpha x}] = m(\alpha - \alpha_1)^{m-1} h(\alpha)e^{\alpha x} + (\alpha - \alpha_1)^m \frac{\partial}{\partial \alpha} h(\alpha)e^{\alpha x}. \tag{9}$$

Since  $x$  and  $\alpha$  are independent, the above equation can be expressed as

$$\frac{\partial}{\partial \alpha} L[e^{\alpha x}] = L[\frac{\partial}{\partial \alpha} e^{\alpha x}] = L[xe^{\alpha x}]. \tag{10}$$

Now the right side of Eq. (9) is zero for  $\alpha = \alpha_1$  because of the factor  $\alpha - \alpha_1$  and  $m \geq 2$ . Hence Eq. (10) shows that  $xe^{\alpha_1 x}$  is a solution of Eq. (4). Repeating this step, another  $m-2$  solutions,  $x^2e^{\alpha_1 x}, \dots, x^{m-1}e^{\alpha_1 x}$ , can be obtained. The solution obtained by iteratively differentiating Eq. (10) reveals that the function is in the form of a Vandermonde matrix [5,7].

On the basis of the computational characteristics of the time response function illustrated above, a simple technique for solving the time response function for a time-invariant system can be derived by evaluating the inverse generalized Vandermonde matrix [1,3,4].

## 2. THEORETICAL DERIVATION

Let there be a proper rational function,  $Y(s) = \frac{A(s)}{B(s)}$ , where

$$A(s) = \sum_{p=0}^{n-1} a_p s^p, \tag{11}$$

$$B(s) = \sum_{p=0}^n b_p s^{n-p} = \prod_{k=1}^m (s - s_k)^{r_k}, \tag{12}$$

where  $b_0 = 1$ ,  $\sum_{k=1}^m r_k = n$ . We wish to find the matrix  $H$  such that the following relation exists:

$$c = Ha, \tag{13}$$

where  $a$  is a column vector with element  $a_i$ ,  $i = 0, 1, 2, \dots, n-1$ ;  $c$  is a partitioned column vector with subvectors  $c^{(k)}$ , whose elements  $c_i^{(k)}$ ,  $k = 1, 2, 3, \dots, m$ , and  $i = 0, 1, 2, \dots, r_k-1$ , are the coefficients of the partial-fraction expansion of  $Y(s)$ ; thus,

$$Y(s) = \frac{A(s)}{B(s)} = \sum_{k=1}^m \sum_{i=0}^{r_k-1} \frac{c_i^{(k)}}{(s-s_k)^{i+1}}. \quad (14)$$

Let the matrix  $H$  be partitioned as

$$H = \begin{bmatrix} H^{(1)} \\ H^{(2)} \\ H^{(3)} \\ \vdots \\ H^{(m)} \end{bmatrix}, \quad (15)$$

where the submatrix  $H^{(k)}$  is  $r_k \times n$  with element  $h_{ij}^{(k)}$ ,  $k = 1, 2, 3, \dots, m$ ;  $i = 0, 1, 2, \dots, r_k-1$ ;  $j = 0, 1, 2, \dots, n-1$ . Eqs. (13) and (14) imply that the elements  $h_{ij}^{(k)}$  are the coefficients of the partial-fraction expansion of the function  $s^j/B(s)$ :

$$\frac{s^j}{B(s)} = \sum_{k=1}^m \sum_{i=0}^{r_k-1} \frac{h_{ij}^{(k)}}{(s-s_k)^{i+1}} \quad j = 0, 1, 2, \dots, n-1. \quad (16)$$

The expansion coefficients  $h_{ij}^{(k)}$  may be obtained by Eq. (4), i.e.,

$$h_{ij}^{(k)} = \frac{1}{(r_k-1-i)!} \frac{d^{r_k-1-i}}{ds^{r_k-1-i}} [(s-s_k)^{r_k} \frac{s^j}{B(s)}]_{s=s_k},$$

$$k = 1, 2, 3, \dots, m,$$

$$i = 0, 1, 2, \dots, r_k-1,$$

$$j = 0, 1, 2, \dots, n-1, \quad (17)$$

which is, in general, not simple in computation. However, from Eq.(17), we see that the value of  $h_{ij}^{(k)}$  is equal to the coefficient of  $(s-s_k)^{r_k-1-i}$  when  $s^j / [(s-s_k)^{r_k} B(s)]$  is expanded into a Taylor's series at  $s = s_k$ . Therefore, if we expand  $s^j$  and  $(s-s_k)^{-r}$   $B(s)$  into a series of  $(s-s_k)$ , that is,

$$s^j = [s_k + (s-s_k)]^j = \sum_{i=0}^j v_{ji}^k (s-s_k)^i, \quad (18)$$

$$(s-s_k)^{-r_k} B(s) = \prod_{\substack{p=1 \\ p \neq k}}^m [(s_k - s_p) + (s-s_k)]^{r_p} = \sum_{i=0}^{n-r_k} d_i^{(k)} (s-s_k)^i,$$

$$k = 1, 2, 3, \dots, m,$$

$$j = 0, 1, 2, \dots, n-1, \quad (19)$$

then  $h_{ij}^{(k)}$ ,  $i = 0, 1, \dots, r_k-1$  may be obtained by directly dividing Eq. (18) by Eq. (19) through longhand division. Further simplification is also possible for division through merely using the first  $r_k$  terms of the series (18) and (19). That is,

$$\begin{aligned} & \frac{v_{j,0}^{(k)} + v_{j,1}^{(k)}(s - s_k) + \dots + v_{j,r_k-1}^{(k)}(s - s_k)^{r_k-1}}{d_0^{(k)} + d_1^{(k)}(s - s_k) + \dots + d_{r_k-1}^{(k)}(s - s_k)^{r_k-1}} \\ & = h_{r_k-1,j}^{(k)} + h_{r_k-2,j}^{(k)}(s - s_k) + \dots + h_{0,j}^{(k)}(s - s_k)^{r_k-1}, \end{aligned}$$

$$\begin{aligned} k &= 1, 2, 3, \dots, m, \\ j &= 0, 1, 2, \dots, n-1. \end{aligned} \tag{20}$$

Furthermore, in the non-zero initial state response, there exists a relationship between the coefficients of polynomial equation  $B(s)$  and the initial states at  $t = 0$  [8].

$$a = \Lambda y_0, \tag{21}$$

where  $y_0$  is a vector with entries equal to the initial values  $y^{(i)}(0)$ ,  $i = 0, 1, 2, \dots, n-1$ ;  $\Lambda$  is a symmetrical matrix whose  $ij$ -th entries are equal to  $b_{n-1-i,j}$  in Eq. (12); and the subscript  $i, j = 0, 1, 2, \dots, n-1$ ,  $i \neq j$ . The expansion coefficient is thus directly related to the initial value:

$$c = (H\Lambda)y_0 = V^{-1}y_0, \tag{22}$$

where  $V^{-1} = H\Lambda$  is the well-known inverse Vandermonde matrix [8].

The time response function  $y(t)$  of the system is then obtained by means of the inverse Laplace transform of function  $Y(s)$ . If the force function,  $g(t)$ , is assumed to be non-zero, the final expression can be written in the following compact matrix form:

$$y(t) = e(t)V^{-1}y_0 + \int_0^t g(t - \tau)e(\tau)q d\tau, \tag{23}$$

where  $e(t)$  is a partitioned row matrix with submatrix  $e_k(t)$ , whose entries are equal to  $e^{sk} t^j / j!$ ,  $j = 0, 1, 2, \dots, r_k - 1$ ,  $k = 1, 2, \dots, m$ ;  $q$  is a column matrix corresponding to the last column of  $V^{-1}$ .

For a given linear different equation with prescribed initial conditions, the solution  $y(t)$  can be readily computed using the above formula, provided that  $V^{-1}$  is known. Therefore, the formula is useless unless  $V^{-1}$  can be simply determined. Evaluation of the inverse Vandermonde matrix has been proposed in [1] using repeated synthetic divisions and long-hand divisions. This approach is especially suitable for programming algorithms.

### 3. CONCLUSIONS

A simple procedure for finding the matrix  $H$  and its derivative  $V^{-1}$ , which are especially useful for programming, has been presented. The matrices  $H$  and  $V^{-1}$  play a very significant role in the partial-fraction expansion of a proper rational function,  $Y(s) = \frac{A(s)}{B(s)}$ . Because  $H$  is determined uniquely by means of the given denominator  $B(s)$ , the partial-fraction expansion of  $Y(s)$  for any numerator  $A(s)$  can readily be found once  $H$  is known.

Furthermore, from matrix  $H$ , a procedure for deriving an inverse generalized Vandermonde matrix,  $V^{-1}$ , which describes the transition matrix of a time-invariant system has been proposed. In the case where the system might be activated with some other initial conditions, the proposed algorithm can systematically solve the problem by simply replacing the old initial vectors with a new one. This might shed light on the real-time control problem.

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