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Optimal k-Fault-Tolerant Networks for Token Rings*

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Fault-tolerant multiprocessors are widely used in on-line transaction processing. Fault tolerance is also desirable in massively parallel systems that have a relatively high failure probability. Two types of failures in a multiprocessor system are of interest, processor failures and link failures. Most of the previous research in designing optimal fault-tolerant topologies was concentrated on the cases that only processor failures were allowed [1, 2, 4, 6], or the cases that only link failures were allowed [3, 5, 7, 8, 11-15]. In this paper, we discuss the case of a combination of processor failures and link failures for token rings. By "k faults" we mean k-component faults in any combination of processor faults and link faults. Designing an optimal k-fault-tolerant network for token rings is equivalent to constructing an optimal k-hamiltonian graph, where k is a positive integer and corresponds to the number of faults. A graph G is k-hamiltonian if $G - F$ is hamiltonian for any sets $F \subset V \cup E$ with $|F| \le k$. An n-node k-hamiltonian graph is optimal if it contains the least number of edges among all n-node k-hamiltonian graphs. In this paper, we construct optimal k-hamiltonian graphs with $k = 2$ and 3, which are optimal k-fault-tolerant networks with respect to token rings.

Keywords: distributed systems, fault tolerance, hamiltonian cycles, hamiltonian graphs, processor failures, link failures, token rings

1. INTRODUCTION AND DEFINITIONS

Fault-tolerant multiprocessors are widely used in on-line transaction processing. Fault tolerance is also desirable in massively parallel systems that have a relatively high failure probability. A number of fault-tolerant designs for specific multiprocessor architectures have been proposed based on graph theoretic models in which the processor-to-processor interconnection structure is represented by a graph and multiple edges are allowed.

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Let the graph $G = (V, E)$ represent an underlying interconnection network. Two types of failures in a multiprocessor system are of interest, processor failures and link failures. A link failure corresponds to the deletion of an edge from G, while a processor failure corresponds to the deletion of a node and all the edges incident on it from G. If F is a set of faulty components including faulty nodes and faulty edges in G, then $G - F$ denotes the graph obtained by deleting the fault set F from G, as explained above. Note that a link fault cannot be ascribed to a fault at one of the adjacent processors since two adjacent processors of a faulty link are still included in reconfigurations while faulty processors are not. Most of previous research in designing optimal fault-tolerant topologies was concentrated on the cases that only processor failures were allowed [1, 2, 4, 6] or the cases that only link failures were allowed [3, 5, 7, 8, 11-15]. In their constructions, a supergraph $G' = (V', E')$ with respect to G satisfying $V \subseteq V'$ and $E \subseteq E'$ is constructed such that $G' - F$ contains G as a subgraph where F is a set of faulty components with restriction to either $F \subset V'$ or $F \subset E'$. But in our model, given an underlying network $G = (V, E)$, we consider $F \subset V \cup E$ that is in any combination of processor failures and link failures. Our design concern is that $G - F$ contains a specified network topology that includes all nonfaulty processors. Henceforth, by "k faults", we mean k-component faults in any combination of processor failures and link failures. In this paper, we aim at designing *k-fault-tolerant* networks G for *token rings*; i.e., for any k-fault F, $G - F$ contains a token ring including all of the nonfaulty processors. Furthermore, our constructions are shown to be optimal in terms of the number of edges contained in G. Note that token rings contained in $G - F$ may contain different number of processors for different k-faults F.

A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A cycle is a path of at least three nodes such that the first and the last nodes are the same. A path is delimited by "$\langle$" and "$\rangle$", e.g., $\langle x_0, x_1, ..., x_{n-1}\rangle$. A cycle is called a *hamiltonian cycle*, if its nodes are distinct and they span all of the nonfaulty nodes. A graph having a hamiltonian cycle is called a *hamiltonian graph*. Let k be a positive integer. A graph G is *k-hamiltonian* if $G - F$ is hamiltonian for any sets $F \subset V \cup E$ with $|F| \leq k$. It follows that a *k-hamiltonian* graph is a k-fault-tolerant network for token rings since $G - F$ contains a token ring that includes all of the nonfaulty processors for any k-fault F. The design of *k-fault-tolerant* networks for token rings is equivalent to the design of k-hamiltonian graphs.

It is obvious that a k-hamiltonian graph has at least $k + 3$ nodes. Moreover, the degree of any node in a k-hamiltonian graph is at least $k + 2$. An n-node k-hamiltonian graph is *optimal* if it contains the least number of edges among all the n-node k-hamiltonian graphs. Mukhopadhyaya and Sinha [9] worked on k-hamiltonian graphs with $k = 1$, i.e., 1-fault-tolerant networks for token rings.

An undirected graph G is called a *circulant graph* with distance sequence $\{d_1, d_2, ..., d_k\}$ if $V(G) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(G) = \{(v_i, v_j) \mid (i - j) \bmod n = d_l, \forall 1 \leq l \leq k\}$. Given two positive integers n and k with $n > 2k$, we construct a graph $G_{n,k}$ as follows: The nodes of $G_{n,k}$ are denoted by $x_0, x_1, ..., x_{n-1}$ and are arranged clockwise with ascending order of the indices. If k is even, $G_{n,k}$ is defined as a circulant graph with distance sequence $\{1, 2, ..., \frac{k}{2}+1\}$. If k is odd and n is even, $G_{n,k}$ is defined as a circulant graph with distance sequence $\{1, 2, ..., \frac{k+1}{2}, \frac{n}{2}\}$. Otherwise, $G_{n,k}$ is not a circulant graph but has the edge set $\{(x_i, x_{i+j}) \mid 0 \leq i \leq n - 1 \text{ and } 1 \leq j \leq \frac{k+1}{2}\} \cup \{(x_i, x_{i+\frac{n+1}{2}}) \mid 0 \leq i \leq \frac{n-3}{2}\} \cup \{(x_0, x_{\frac{n-1}{2}})\}$. Examples of $G_{n,k}$ for some n, k are illustrated in Fig. 1. The cycle $\langle x_0, x_1, \cdots, x_{n-1}, x_0\rangle$ is called the *elementary cycle* in $G_{n,k}$. In this paper, we will prove that any $G_{n,k}$ is an optimal k-hamiltonian graph, i.e., an optimal k-fault-tolerant design for token rings, where $k = 2, 3$.

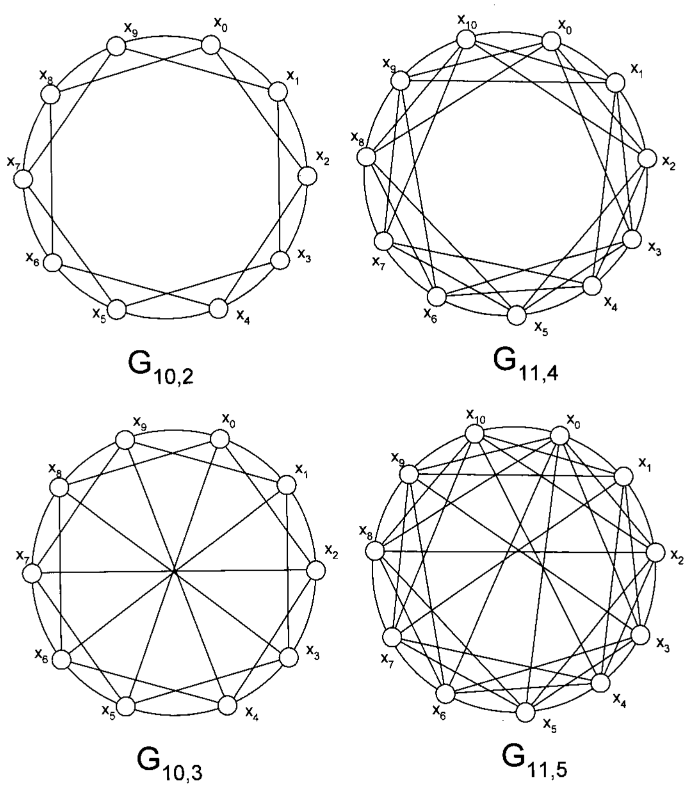

Fig. 1. Examples of $G_{n,k}$.

Throughout this paper, all the addition and subtraction are carried out with modulo n. Let $P(x_i, x_j)$ denote a path in the elementary cycle traversing clockwise from x_i to x_j, i.e., $P(x_i, x_j) = \langle x_i, x_{i+1}, ..., x_j\rangle$. By convention, $P(x_i, x_j)$ defines a single vertex if $x_i = x_j$. For ease of exposition, we define the following three types of paths in $G_{n,k}$ that may be used in reconfigurations for a faulty edge (x_i, x_{i+1}):

$$\tilde{P}(x_i, x_j) = \langle x_i, x_{i-1}, \cdots, x_j \rangle,$$

$$T(x_i, x_j) = \begin{cases} \langle x_i, x_{i+2}, \cdots, x_j, x_{j-1}, x_{j-3}, \cdots, x_{i+1} \rangle & \text{when } n \text{ is even and } i+j \text{ is even,} \\ & \text{or } n \text{ is odd, } i > j \text{ and } i+j \text{ is odd,} \\ & \text{or } n \text{ is odd, } i < j \text{ and } i+j \text{ is even,} \\ \langle x_i, x_{i+2}, \cdots, x_{j-1}, x_j, x_{j-2}, \cdots x_{i+1} \rangle & \text{when } n \text{ is even and } i+j \text{ is odd,} \\ & \text{or } n \text{ is odd, } i > j \text{ and } i+j \text{ is even,} \\ & \text{or } n \text{ is odd, } i < j \text{ and } i+j \text{ is odd,} \end{cases}$$

$$X(x_i, x_{i+3}) = \langle x_i, x_{i+2}, x_{i+1}, x_{i+3} \rangle.$$

(See Fig. 2 for an illustration.) In other words, $\tilde{P}(x_i, x_j)$ is a path from x_i to x_j traversing counterclockwise; $X(x_i, x_{i+3})$ is a path from x_i to x_{i+3} without using (x_i, x_{i+1}); and $T(x_i, x_j)$ is a path from x_i to x_{i+1} traversing $x_i, x_{i+1}, x_{i+2}, ..., x_j$, first using edges (x_p, x_{p+2}) clockwise until

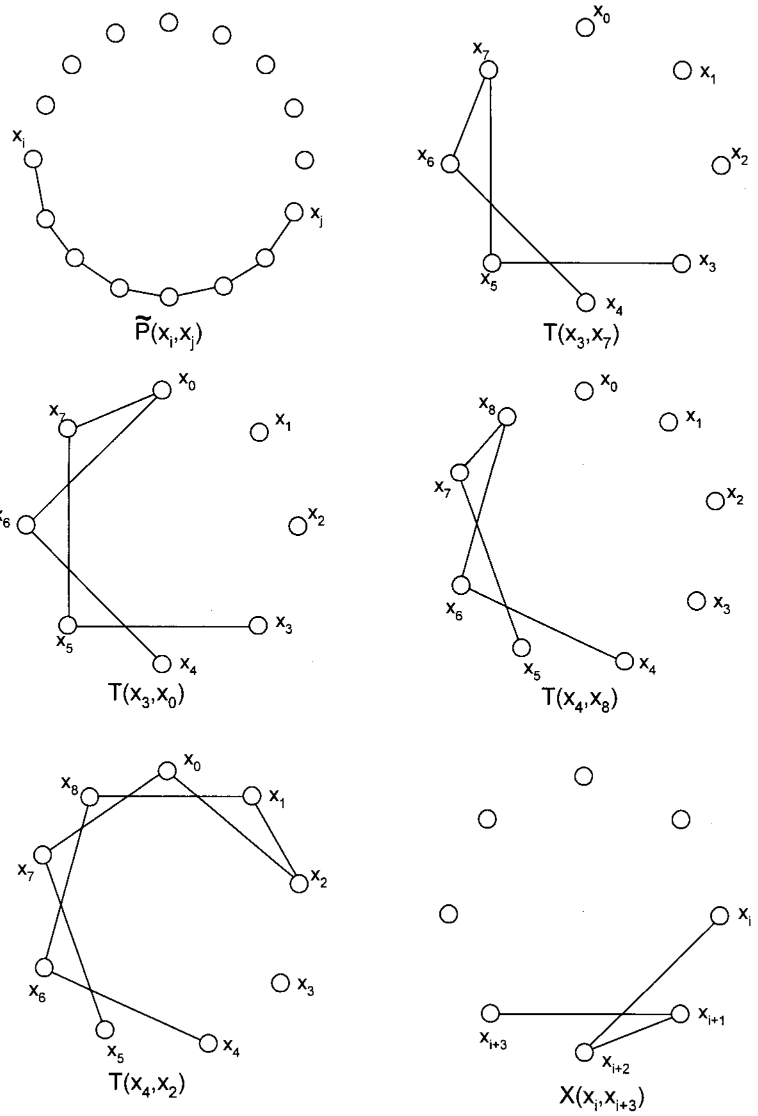

Fig. 2. Various types of paths.

reaching x_{j-1} or x_j, depending on n, i, j, and then using the edge (x_{j-1}, x_j) and edges (x_q, x_{q+2}) counterclockwise until reaching x_{i+1}. By convention, $T(x_i, x_{i+1})$ denotes a path of only one edge (x_i, x_{i+1}).

In addition, we define two types of paths, $J(x_i, x_j)$ and $\tilde{J}(x_r, x_s)$. $J(x_i, x_j)$ denotes the path $\langle x_i, x_{i+2}, ..., x_j\rangle$, where $j - i$ is even. The path $\tilde{J}(x_r, x_s)$ traverses counterclockwise from x_r to x_s using edges (x_i, x_{i+2}), where $r - s$ is even. Equivalently, $\tilde{J}(x_j, x_i) = J(x_i, x_j)$ if $j - i$ is even. Let Q_j be a path from x_{j_1} to x_{j_2}, $1 \le j \le r$. We use $\langle Q_p, Q_q\rangle$ to denote a path from x_{p_1} to x_{q_2}, defined as follows:

$$\langle Q_p, Q_q \rangle = \begin{cases} Q_p \cup Q_q & \text{if } x_{p_2} = x_{q_1}, \\ Q_p \cup \{(x_{p_2}, x_{q_1})\} \cup Q_q & \text{if } (x_{p_2}, x_{q_1}) \in E, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

For example, $\langle P(x_1, x_{i-1}), P(x_{i+1}, x_{n-1})\rangle$ in $G_{n,2}$ denotes the path $\langle x_1, x_2, ..., x_{i-1}, x_{i+1}, x_{i+2}, ..., x_{n-1}\rangle$ since $(x_{i-1}, x_{i+1}) \in E$, and $\langle P(x_0, x_{n-1}), x_0\rangle$ denotes the elementary cycle. Therefore, we can say $\langle Q_k, Q_{k+1}, ..., Q_r\rangle$ defines a path from x_{k_1} to x_{r_2} if $\langle Q_j, Q_{j+1}\rangle$ well defines a path for $k \le j \le r-1$.

In the proofs in the following sections, we always assume faulty nodes or faulty edges to be in a certain range by relabelling of vertices since $G_{n,3}$ for n even and $G_{n,2}$ are node-symmetric.

2. OPTIMAL 2-FAULT-TOLERANT NETWORKS

Since $G_{n,2}$ is a circulant graph with distance sequence $\{1, 2\}$, we decompose the edges of $G_{n,2}$ into two classes: (x_i, x_{i+1}) called class-1 edges, and (x_i, x_{i+2}) called class-2 edges.

Lemma 1: $G_{n,2}$ is a 2-hamiltonian graph for $n \ge 5$.

***Proof*:** Let $F \subset V(G_{n,2}) \cup E(G_{n,2})$ be an arbitrary set with $|F| \le 2$. It suffices to show that $G_{n,2} - F$ is hamiltonian. In [2, 15], it was proved that $G_{n,2} - F$ is hamiltonian if either $F \subset E(G_{n,2})$ or $F \subset V(G_{n,2})$. Hence, we consider the remaining case that F is composed of a node and an edge. We assume without loss of generality that the faulty node is x_0, and that the faulty edge e is not incident with x_0. If the faulty edge e is in class 2, $G_{n,2} - F$ possesses a hamiltonian cycle $\langle P(x_1, x_{n-1}), x_1\rangle$ for $e \ne (x_1, x_{n-1})$ and $\langle T(x_1, x_{n-1}), x_1\rangle$ for $e = (x_1, x_{n-1})$. If the faulty edge e is in class 1, say $e = (x_i, x_{i+1})$ with $1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1$, then $G_{n,2} - F$ comprises a hamiltonian cycle $\langle P(x_1, x_i), X(x_i, x_{i+3}), P(x_{i+3}, x_{n-1}), x_1\rangle$ or $\langle x_1, x_2, x_4, x_3, x_1\rangle$. Hence, the lemma follows. □

Theorem 1: $G_{n,2}$ is an optimal 2-fault-tolerant network for token rings for $n \ge 5$.

***Proof*:** It is observed that the degree of any node in a 2-hamiltonian graph is at least 4, and that the degree of any node in $G_{n,2}$ is 4. Thus, $G_{n,2}$ is optimal 2-hamiltonian and furthermore, optimal 2-fault-tolerant for token rings. □

3. OPTIMAL 3-FAULT-TOLERANT NETWORKS

Edges of $G_{n,3}$ are divided into three classes: class-1 edges, class-2 edges as defined in section 2, and the remaining edges are called class-3 edges.

Lemma 2: $G_{n,3}$ is a 3-hamiltonian graph for $n \ge 6$.

***Proof*:** Since $G_{n,2}$ is a subgraph of $G_{n,3}$, $G_{n,3} - F$ contains a hamiltonian cycle for every set $F \subset V(G_{n,3}) \cup E(G_{n,3})$ with $|F| \leq 2$ or at least one class-3 edge in F. It suffices to show for $|F| = 3$ and no class-3 edge in F. In [2, 11], it is proved that $G_{n,3} - F$ is hamiltonian if either $F \subset E(G_{n,3})$ or $F \subset V(G_{n,3})$ with $|F| \leq 3$. First, we prove that every $G_{n,3}$ is 3-hamiltonian when n is **even**. We distinguish the remaining cases as follows:

Case 1: F consists of a node, say x_0, and two edges.

Suppose all of the faulty edges are in class 2. If F does not contain the edge (x_{n-1}, x_1), then $\langle P(x_1, x_{n-1}), x_1\rangle$ is a hamiltonian cycle in $G_{n,3} - F$. Otherwise, we assume without loss of generality that the other faulty edge in F is (x_i, x_{i+2}) for $i \leq \frac{n}{2} - 1$. Then $G_{n,3} - F$ includes a hamiltonian cycle $\langle P(x_1, x_{\frac{n}{2}}), T(x_{\frac{n}{2}}, x_{n-1}), x_1\rangle$.

Suppose all of the faulty edges are in class 1. If neither (x_1, x_2) nor (x_{n-2}, x_{n-1}) is in F, $G_{n,3} - F$ possesses a hamiltonian cycle $\langle T(x_1, x_{n-1}), x_1\rangle$. If both (x_1, x_2) and (x_{n-2}, x_{n-1}) are in F, then $G_{n,3} - F$ contains a hamiltonian cycle $\langle X(x_1, x_4), P(x_4, x_{n-4}), X(x_{n-4}, x_{n-1}), x_1\rangle$ when $n \geq 8$, and $\langle x_1, x_4, x_3, x_2, x_5, x_1\rangle$ when $n = 6$. Consider that one of $\{(x_1, x_2), (x_{n-2}, x_{n-1})\}$ is in F. Without loss of generality, we assume (x_1, x_2) and (x_i, x_{i+1}) are in F. When $n = 6$, $G_{n,3} - F$ contains a hamiltonian cycle $\langle x_1, x_3, x_4, x_2, x_5, x_1\rangle$ for $i = 2, 4$, and $\langle x_1, x_3, x_2, x_4, x_5, x_1\rangle$ for $i = 3$. When $n \geq 8$, $G_{n,3} - F$ possesses a hamiltonian cycle:

$$\begin{array}{ll}
\left\langle T(x_1, x_{\frac{n}{2}+1}), P(x_{\frac{n}{2}+2}, x_{n-1}), x_1 \right\rangle & \text{for } i = 2, \\
\left\langle X(x_1, x_4), P(x_4, x_{n-1}), x_1 \right\rangle & \text{for } i = 3, \\
\left\langle X(x_1, x_4), P(x_4, x_i), X(x_i, x_{i+3}), P(x_{i+3}, x_{n-1}), x_1 \right\rangle & \text{for } 4 \leq i \leq n-4, \\
\left\langle X(x_1, x_4), P(x_4, x_{n-5}), X(x_{n-5}, x_{n-2}), x_{n-1}, x_1 \right\rangle & \text{for } i = n-3 \text{ and } n \geq 10, \\
\left\langle x_1, \tilde{P}(x_5, x_2), x_6, x_7, x_1 \right\rangle & \text{for } i = n-3 \text{ and } n = 8.
\end{array}$$

Suppose that one faulty edge is in class 1, and that the other is in class 2. We assume without loss of generality that $e_i = (x_i, x_{i+1})$ for $1 \leq i \leq \frac{n}{2} - 1$ and $e_j = (x_j, x_{j+2})$ are in F. First, we consider that (x_1, x_{n-1}) is not in F. $G_{n,3} - F$ possesses a hamiltonian cycle $\langle P(x_1, x_i), \tilde{P}(x_{\frac{n}{2}+i}, x_{i+1}), P(x_{\frac{n}{2}+i+1}, x_{n-1}), x_1\rangle$ if $1 \leq i \leq \frac{n}{2} - 2$, either $\langle P(x_1, x_{\frac{n}{2}-3}), X(x_{\frac{n}{2}-3}, x_{\frac{n}{2}}), P(x_{\frac{n}{2}}, x_{n-1}), x_1\rangle$, or $\langle P(x_1, x_{\frac{n}{2}-1}), X(x_{\frac{n}{2}-1}, x_{\frac{n}{2}+2}), P(x_{\frac{n}{2}+2}, x_{n-1}), x_1\rangle$ if $i = \frac{n}{2} - 1$ and $n \geq 8$, and either $\langle x_1, x_3, x_4, x_5, x_2, x_1\rangle$ or $\langle x_1, x_2, x_5, x_3, x_4, x_1\rangle$ if $i = \frac{n}{2} - 1$ and $n = 6$. Next, we consider that (x_1, x_{n-1}) is in F. We can attain in $G_{n,3} - F$ a hamiltonian cycle $\langle T(x_1, x_{n-1}), x_1\rangle$ if $i \neq 1$, $\langle X(x_1, x_4), P(x_4, x_{\frac{n}{2}}), T(x_{\frac{n}{2}}, x_{n-1}), x_1\rangle$ if $i = 1$ and $n \geq 8$, and $\langle x_1, x_3, x_2, x_5, x_4, x_1\rangle$ if $i = 1$ and $n = 6$.

Case 2: F consists of two nodes, say x_0 and x_i with $1 \leq i \leq \frac{n}{2}$, and one edge.

Let $i = 1$. Consider that the faulty edge is in class 1. If (x_{n-2}, x_{n-1}) is not faulty, then $G_{n,3} - F$ possesses a hamiltonian cycle:

$\langle T(x_2, x_{n-1}), x_2 \rangle$ for (x_2, x_3) nonfaulty and $n \geq 8$,
$\left\langle X(x_2, x_5), P(x_5, x_{\frac{n}{2}+1}), T(x_{\frac{n}{2}+1}, x_{n-1}), x_2 \right\rangle$ for (x_2, x_3) faulty and $n \geq 8$,
$\langle x_2, x_4, x_3, x_5, x_2 \rangle$ for (x_3, x_4) nonfaulty and $n = 6$,
$\langle x_2, x_4, x_5, x_3, x_2 \rangle$ for (x_3, x_4) faulty and $n = 6$.

If (x_{n-2}, x_{n-1}) is faulty, we can obtain in $G_{n,3} - F$ a hamiltonian cycle:

$\left\langle J(x_2, x_{\frac{n}{2}-1}), \widetilde{J}(x_{n-1}, x_{\frac{n}{2}}), J(x_{\frac{n}{2}+1}, x_{n-2}), \widetilde{J}(x_{\frac{n}{2}-2}, x_3), x_2 \right\rangle$ for $\frac{n}{2}$ odd and $n \geq 8$,
$\left\langle J(x_2, x_{\frac{n}{2}-2}), \widetilde{J}(x_{n-2}, x_{\frac{n}{2}}), J(x_{\frac{n}{2}+1}, x_{n-1}), \widetilde{J}(x_{\frac{n}{2}-1}, x_3), x_2 \right\rangle$ for $\frac{n}{2}$ even and $n \geq 8$,
$\langle x_2, x_4, x_3, x_5, x_2 \rangle$ for $n = 6$.

Now consider that the faulty edge is in class 2, say (x_j, x_{j+2}) with $2 \leq j \leq \frac{n}{2} - 1$. $G_{n,3} - F$ possesses a hamiltonian cycle $\langle P(x_2, x_{\frac{n}{2}+1}), T(x_{\frac{n}{2}+1}, x_{n-1}), x_2 \rangle$ for $n \geq 8$, and $\langle x_2, x_3, x_4, x_5, x_2 \rangle$ for $n = 6$.

Let $i = 2$. If the faulty edge is in class 1, let the edge be (x_j, x_{j+1}), where $3 \leq j \leq \frac{n}{2}$. $G_{n,3} - F$ includes a hamiltonian cycle $\langle x_1, P(x_3, x_j), X(x_j, x_{j+3}), P(x_{j+3}, x_{n-1}), x_1 \rangle$ when $n \geq 8$, and $\langle x_1, x_4, x_5, x_3, x_1 \rangle$ when $n = 6$. Now consider the case that the faulty edge is in class 2. If none of $\{(x_1, x_3), (x_1, x_{n-1})\}$ is faulty, then $G_{n,3} - F$ possesses a hamiltonian cycle $\langle x_1, P(x_3, x_{n-1}), x_1 \rangle$. If one of $\{(x_1, x_3), (x_1, x_{n-1})\}$ is faulty, we assume without loss of generality that the faulty edge is (x_1, x_{n-1}). Then, $G_{n,3} - F$ possesses a hamiltonian cycle $\langle x_1, P(x_3, x_{\frac{n}{2}}), T(x_{\frac{n}{2}}, x_{n-1}), x_1 \rangle$.

Let $3 \leq i \leq \frac{n}{2}$. Consider that the faulty edge is in class 1, say (x_j, x_{j+1}), and is not incident with x_0 or x_i. $G_{n,3} - F$ contains a hamiltonian cycle:

$\left\langle P(x_1, x_j), \widetilde{P}(x_{j+\frac{n}{2}}, x_{i+1}), \widetilde{P}(x_{i-1}, x_{j+1}), P(x_{\frac{n}{2}+j+1}, x_{n-1}), x_1 \right\rangle$ for $1 \leq j \leq i - 2$,
$\langle P(x_1, x_{i-1}), P(x_{i+1}, x_j), X(x_j, x_{j+3}), P(x_{j+3}, x_{n-1}), x_1 \rangle$ for $i + 1 \leq j \leq n - 4$,
$\left\langle P(x_1, x_{x-1}), P(x_{i+1}, x_{\frac{n}{2}}), T(x_{\frac{n}{2}}, x_{n-1}), x_1 \right\rangle$ for $j = n - 3,\ 3 \leq i \leq \frac{n}{2} - 1$,
$\left\langle T(x_1, x_{\frac{n}{2}-1}), T(x_{\frac{n}{2}+2}, x_{n-1}), x_{\frac{n}{2}+1}, x_1 \right\rangle$ for $j = n - 3,\ i = \frac{n}{2}$,
$\langle P(x_1, x_{i-1}), P(x_{i+1}, x_{n-4}), X(x_{n-4}, x_{n-1}), x_1 \rangle$ for $j = n - 2,\ 3 \leq i \leq \frac{n}{2} - 1$,
$\left\langle P(x_1, x_{\frac{n}{2}-1}), P(x_{\frac{n}{2}+1}, x_{n-4}), X(x_{n-4}, x_{n-1}), x_1 \right\rangle$ for $j = n - 2,\ i = \frac{n}{2},\ n \geq 10$,
$\langle x_1, x_3, x_2, x_6, x_5, x_7, x_1 \rangle$ for $j = n - 2,\ i = \frac{n}{2},\ n = 8$,
$\langle x_1, x_4, x_2, x_5, x_1 \rangle$ for $j = n - 2,\ i = \frac{n}{2},\ n = 6$,

Consider that the faulty edge is in class 2, say (x_j, x_{j+2}), and is not incident with x_0 or x_i. When $j \neq n-1, i-1$, $G_{n,3} - F$ possesses a hamiltonian cycle $\langle P(x_1, x_{i-1}), P(x_{i+1}, x_{n-1}), x_1\rangle$. When $j = n-1$, $G_{n,3} - F$ possesses a hamiltonian cycle $\langle P(x_1, x_{i-1}), P(x_{i+1}, x_{\frac{n}{2}}), T(x_{\frac{n}{2}}, x_{n-1}), x_1\rangle$ for $i \leq \frac{n}{2} - 1$, and $\langle P(x_1, x_{\frac{n}{2}-1}), \tilde{P}(x_{n-1}, x_{\frac{n}{2}+1}), x_1\rangle$ for $i = \frac{n}{2}$. The case $j = i - 1$ can be treated similarly. Thus, we have proven the lemma for even n.

Now consider $G_{n,3}$ with odd n. It is observed that for any even n, every node x_i in $G_{n,3}$ is incident with a unique edge in class 3. However, for odd n, every node x_i with $i \neq 0$ in $G_{n,3}$ is incident with a unique edge in class 3, and the node x_0 is incident with exactly two edges in class 3, namely, $(x_0, x_{\frac{n-1}{2}})$ and $(x_0, x_{\frac{n+1}{2}})$. By replacing edge $(x_0, x_{\frac{n}{2}})$ used in reconfigurations discussed for even n with $(x_0, x_{\frac{n-1}{2}})$ or $(x_0, x_{\frac{n+1}{2}})$, we can obtain a hamiltonian cycle in $G_{n,3} - F$ with $|F| \leq 3$ for odd n.

Hence, the lemma follows. □

Theorem 2: $G_{n,3}$ is an optimal 3-fault-tolerant network for token rings for $n \geq 6$.

***Proof*:** It is observed that the degree of any node in a 3-hamiltonian graph is at least 5. Since the degree of any node in $G_{n,3}$ for even n is 5, $G_{n,3}$ is optimal for even n. Next, we consider when n is odd. The degree of x_i is 5 if $i \neq 0$, and 6 otherwise. Futhermore, since the sum of degree over all nodes in every graph is even, $G_{n,3}$ is optimal 3-hamiltonian for odd n with $n \geq 7$. Thus, the theorem follows. □

4. DISCUSSION

In this paper, we have constructed a family of graphs $G_{n,k}$ and shown that $G_{n,k}$ is optimal k-hamiltonian for $k = 2, 3$, i.e., an optimal k-fault-tolerant design for token rings while k faults allow to be any combination of processor failures and link failures. Note that $G_{n,k}$ is not always circulant; e.g., $G_{11,5}$ shown in Fig. 1. However, we are unable to show that $G_{n,k}$ is k-hamiltonian for large k since there are too many cases to verify hamiltonicity in $G_{n,k} - F$ for any fault set F satisfying $|F| \leq k$. Nevertheless, we conjecture that every $G_{n,k}$ is optimal k-fault-tolerant for token rings.

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