Short Paper

k-Arbiter Join Operation*

YU-CHEN KUO
Department of Computer and Information Science
SooChow University
Taipei, Taiwan 100, R.O.C.
E-mail:yckuo@cis.scu.edu.tw

k-Arbiter is a useful concept for solving the distributed *h*-out of-*k* resources allocation problem. The distributed *h*-out of-*k* resources allocation algorithms based on *k*-arbiter have the benefits of high fault-tolerance and low communication cost. However, according to the definition of *k*-arbiter, it is required to have a non-empty intersection among any (k+1) quorums in a *k*-arbiter. Consequently, constructing *k*-arbiters is difficult. The coterie join operation proposed by Neilsen and Mizuno produces a new and larger coterie by joining known coteries. In this paper, by extending the coterie join operation, we first propose a *k*-arbiter *join operation* to construct a new and larger *k*-arbiter from known *k*-arbiters. Then, we derive a necessary and sufficient condition for the *k*-arbiter join operation to construct a nondominated composite *k*-arbiter. Moreover, we discuss the availability properties of the composite *k*-arbiters. We observe that by selecting the proper *k*-arbiters as join inputs, the composite *k*-arbiter can have a higher availability than that of the original inputs.

***Keywords*:** distributed systems, fault-tolerance, *h*-out of-*k* resources allocation, quorums, mutual exclusion

1. INTRODUCTION

The distributed mutual exclusion problem [2, 16] is to design algorithms such that concurrent nodes accessing a shared resource execute in a mutually exclusive way. The distributed *k*-mutual exclusion problem [14], an extension of the mutual exclusion problem, is a problem which concerns a distributed system has *k* identical shared resources and each resource can be accessed by only one node at a time. Therefore, any solution to the distributed *k*-mutual exclusion problem must guarantee that at most *k* concurrent nodes can access *k* shared resources, simultaneously.

The distributed *h*-out of-*k* mutual exclusion problem [15] is closely related to the distributed mutual exclusion problem and *k*-mutual exclusion problem. It considers a distributed system with *N* nodes and *k* identical shared resources, where each resource can be accessed by only one node at a time. However, the difference between the mutual exclusion problem and the *h*-out of-*k* mutual exclusion problem is as follows. In the *h*-out of-*k* mutual exclusion problem, a node may request h ($1 \le h \le k$) resources at a

Received March 13, 2000; revised June 29 & July 26, 2000; accepted August 16, 2000.
Communicated by Chu-Sing Yang.
*This research was supported in part by the National Science Council of the Republic of China under Grant No. NSC 88-2213-E-031-004.

time, whereas in the mutual exclusion problem or in the *k*-mutual exclusion problem, a node can request only one resource at a time. An example of the *h*-out of-*k* mutual exclusion problem is the sharing of a processor pool with *k* CPUs. Each node may request *h* CPUs at a time to execute its job. Since each job has its own requirement, the number of CPUs desired differs from request to request.

In order to avoid deadlocks, a node should request all the resources it needs at once, then blocks itself until it gets all its requested resources. A conflict may arise when a node tries to request a number of resources that is greater than the number of currently available resources. Thus, in the *h*-out of-*k* mutual exclusion problem, it has to control that at most *k* resources can be accessed by request nodes at any given time, where each resource can be accessed by only one node, and a node may access *h* ($1 \leq h \leq k$) resources at a time.

Numerous solutions for the distributed mutual exclusion problem and the *k*-mutual exclusion problem have been proposed [1-3, 5-8, 10, 12-14, 17, 18]. Among them, the *quorum-based* algorithm [10] is an important class of algorithms. They utilize *coterie* [2] or *k-coterie* [6, 7] quorum structures to achieve mutual exclusion or *k*-mutual exclusion, respectively, providing high fault-tolerance and low communication cost.

Consequently, in [11] a quorum structure, *k-arbiter*, is proposed for solving the *h*-out of-*k* mutual exclusion problem. In the algorithm based on *k*-arbiters (so called *quorum-based h*-out of-*k* mutual exclusion algorithm), each node has *k* permissions corresponding to *k* shared resources. Some nodes in the system constitute a quorum. According to the definition of *k*-arbiters, the intersection among any $k + 1$ quorums is required to be non-empty (viz., the intersection property of *k*-arbiters). Any node requesting to access *h* shared resources must collect the permissions of all members in a quorum (*h* permissions from each member). Since any $k + 1$ quorums have a non-empty intersection and each node in that intersection has only *k* votes, it guarantees that there are at most *k* resources that can be accessed at any given time. Thus, the *h*-out of-*k* mutual exclusion problem can be solved.

Due to the intersection property of *k*-arbiters, it becomes more difficult to construct *k*-arbiters as *k* increases. The coterie join operation proposed by Neilsen and Mizuno [12] can produce a new and larger coterie by joining known coteries. In this paper, by extending the coterie join operation, we propose a *k*-arbiter join operation to construct a new and larger *k*-arbiter from known *k*-arbiters. We derive a necessary and sufficient condition for the *k*-arbiter join operation to construct a nondominated composite *k*-arbiter. Moreover, we discuss the properties of the composite *k*-arbiters regarding availability. We observe that by selecting *k*-arbiters satisfying a sufficient condition, the composite *k*-arbiter can have a higher availability than that of the original inputs.

The rest of this paper is organized as follows. In Section 2, we introduce the definitions of quorum structures: coteries and *k*-arbiters, and discuss the relationship between coteries and *k*-arbiters. In Section 3, we introduce the notion of nondominated quorum structures, and present some theories for recognizing nondominated quorum structures. In Section 4, we propose a *k*-arbiter join operation for constructing new and larger *k*-arbiters from known *k*-arbiters. We also derive a necessary and sufficient condition for the *k*-arbiter join operation to construct nondominated *k*-arbiters. In Section 5, we discuss the availability properties of constructed *k*-arbiters on. Finally, in Section 6, we give the conclusion.

2. COTERIES AND *k*-ARBITERS

In this section, we introduce the definitions of coteries and *k*-arbiters, and discuss the relationship between coteries and *k*-arbiters.

2.1 Coteries

Let $U = \{1, 2, ..., N\}$ be a set of nodes in a system, where N denotes the total number of nodes. A non-empty set C of non-empty subsets of U is called a *coterie* [2] *iff* both of the following two properties hold:

1) *Intersection property*: $(\forall Q_1, Q_2 \in C, Q_1 \cap Q_2 \neq \emptyset)$.
2) *Minimality property*: $(\forall Q_1, Q_2 \in C, Q_1 \not\subset Q_2)$.

A member Q of coterie C is called a *quorum*.

For example, $C = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$ is a coterie under $U = \{1, 2, 3\}$, which contains three quorums $\{1, 2\}$, $\{2, 3\}$, and $\{1, 3\}$.

2.2 *k*-Arbiters

A non-empty set C of non-empty subsets of U is defined to be a *k-arbiter* [11] *iff* both of the following two properties hold:

1) *Intersection property*: $(\forall Q_1, Q_2, \ldots, Q_{k+1} \in C, Q_1 \cap Q_2 \cap \ldots \cap Q_{k+1})$.
2) *Minimality property*: $(\forall Q_1, Q_2 \in C, Q_1 \not\subset Q_2)$.

A member Q of *k*-arbiter C is also a *quorum*.

For example, under $U = \{1, 2, 3, 4\}$, $C = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ is a 2-arbiter containing four quorums $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$. The intersection among any three of those quorums is not empty.

Using the definition, the *k*-arbiter can be applied to develop algorithms for solving the distributed *h*-out of-*k* mutual exclusion problem. In the algorithm given in [11], each node has *k* permissions corresponding to *k* shared resources. Any node requesting to access *h* shared resources must send its requested messages to all members of a particular quorum, and then wait for the reception of the permissions from all the members, receiving *h* permissions from each member. When a member receives a request, it grants the request by sending its *h* permissions back only if the number of the available permissions it currently holds is not less than *h*. Note that each node is initially assigned by *k* permissions; each node can grant requests which request total no more than *k* permissions.

Since any $k + 1$ quorums in a *k*-arbiter have a non-empty intersection, and each node in the intersection can grant requests which request total no more than *k* permissions, at most *k* resources can be granted to access by the request nodes at any one time. Thus, the requirement of the *h*-out of-*k* mutual exclusion is satisfied. It should be noted that in this algorithm, a node requesting to access *h* resources is required to issue only one request to the members in one quorum. The minimality property is not necessary for the correctness of the requirement of the *h*-out of-*k* mutual exclusion, but rather to enhance efficiency.

The *h*-out of-*k* resource allocation algorithms based on *k*-arbiters are fault-tolerant in the sense that when some nodes in the system become unavailable, due to network partitioning and/or some other condition, then the quorums that don't contain unavailable nodes can still be successfully formed.

2.3 The Relationship Between Coteries and *k*-Arbiters

According to the definition of *k*-arbiters, it is obvious that *k*-arbiters have more restricted definitions than coteries. We show their relationships by the following corollaries. Each corollary is a direct consequence of the definitions of *k*-arbiters and coteries.

Corollary 1. Every *k*-arbiter is a coterie.

Corollary 2. Coterie is a special case of *k*-arbiters, where $k = 1$.

Corollary 3. Every *k*-arbiter is also a $(k - 1)$-arbiter.

3. NONDOMINATED COTERIES AND *k*-ARBITERS

In this section, we introduce the notion of nondominated coteries and nondominated *k*-arbiters.

For two coteries C and D, we say that *C dominates D* [2] if and only if $C \neq D$ and $(\forall Q \in D, (\exists Q' \in C, Q' \subseteq Q))$. A coterie that is not dominated by any other coterie is called a *nondominated* (*ND*) coterie; otherwise, it is called a *dominated* coterie.

Similarly, for two *k*-arbiters C and D, we say that *C dominates D* [11] if and only if $C \neq D$ and $(\forall Q \in D, (\exists Q' \in C, Q' \subseteq Q))$. A *k*-arbiter that is not dominated by any other *k*-arbiter is called a *nondominated* (*ND*) *k*-arbiter; otherwise, it is called a *dominated* *k*-arbiter.

In general, a *ND* *k*-arbiter has higher fault-tolerant ability and smaller quorum than those that it dominates because it has more chance for a quorum to be successfully formed in the face of node and/or communication link failures.

For example, under $U = \{1, 2, 3, 4, 5\}$, 2-arbiter $C = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}$ dominates another 2-arbiter $D = \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}$ because every quorum in D is a superset of some quorum in C. Thus, D is a dominated 2-arbiter. In addition, it can be verified that no other 2-arbiter under U can dominate C; i.e., C is a *ND* 2-arbiter. C has better fault-tolerance than the dominated 2-arbiter D because if node 4 is unavailable, the quorum $\{1, 2, 3, 5\}$ can still be formed in C while none can be formed in D. Thus, in order to have better fault-tolerance, we should always concentrate on finding *ND* *k*-arbiters. There are some theories [2, 11] for recognizing *ND* coteries and *ND* *k*-arbiters. Here, we repeat them as follows.

Theorem 1. A coterie C under U is dominated *iff* $\exists H \subseteq U$ such that both the following properties are satisfied:

[*R*1] $(\forall Q \in C, Q \not\subseteq H)$.
[*R*2] $(\forall Q \in C, Q \cap H \neq \varnothing)$.

Thus, for a particular coterie C, if there does not exist a set $H \subseteq U$ satisfying both [*R*1] and [*R*2], then C is *ND*; otherwise, it is dominated.

Theorem 2. A *k*-arbiter C under U is dominated *iff* $\exists H \subseteq U$ such that both the following properties are satisfied:

[*P*1] $(\forall Q \in C, Q \not\subseteq H)$.
[*P*2] $(\forall Q_1, Q_2, \ldots, Q_k \in C, Q_1 \cap Q_2 \cap \ldots \cap Q_k \cap H \neq \varnothing)$

Thus, for a particular *k*-arbiter C, if there does not exist a set $H \subseteq U$ satisfying both [*P*1] and [*P*2], then C is *ND*; otherwise, it is dominated.

4. *k*-ARBITER JOIN OPERATION

In [12], a coterie join operation is proposed for combining known coteries into a new and larger coterie. In this section, we extend the coterie join operation and propose a *k*-arbiter join operation for constructing a new and larger *k*-arbiter from known *k*-arbiters.

Let C_1 and C_2 be two *k*-arbiters under U_1 and U_2, respectively, where $U_1 \cap U_2 = \varnothing$. Let $x \in \cup Q$, where $Q \in C_1$ and $U_3 = (U_1 - \{x\}) \cup U_2$. The *k*-arbiter join operation $\oplus_x$ is defined by

$$C_1 \oplus_x C_2 = \{ CT_x(Q_1, Q_2) \mid Q_1 \in C_1, Q_2 \in C_2 \} \text{ where}$$

$$CT_x(Q_1, Q_2) = \begin{cases} (Q_1 - \{x\}) \cup Q_2 & \text{if } x \in Q_1 \quad \text{(Type -- I)} \\ Q_1 & \text{otherwise} \quad \text{(Type -- II)}. \end{cases}$$

That is, $C_1 \oplus_x C_2$ is defined by replacing each occurrence of x in quorums of C_1 by the nodes of any quorum in C_2. For example, consider the following two 2-arbiters:

C_1 = {{1, 2, 3}, {1, 2, a}, {1, 3, a}, {2, 3, a}} under U_1 = {1, 2, 3, a} and
C_2 = {{4, 5, 6}, {4, 5, 7}, {4, 6, 7}, {5, 6, 7}} under U_2 = {4, 5, 6, 7}.
$C_1 \oplus_a C_2$ = {{1, 2, 3}, {1, 2, 4, 5, 6}, {1, 2, 4, 5, 7}, {1, 2, 4, 6, 7}, {1, 2, 5, 6, 7},
{1, 3, 4, 5, 6}, {1, 3, 4, 5, 7}, {1, 3, 4, 6, 7}, {1, 3, 5, 6, 7},
{2, 3, 4, 5, 6}, {2, 3, 4, 5, 7}, {2, 3, 4, 6, 7}, {2, 3, 5, 6, 7}}.

Obviously, $C_1 \oplus_a C_2$ is essentially identical with C_1 except that node a in $C_1 \oplus_a C_2$ is a *logical* node, which is replaced by the nodes of any quorum in C_2. In the following theorem, we will show that if both C_1 and C_2 are k-arbiters, then $C_3 = C_1 \oplus_x C_2$ is a k-arbiter, too.

Theorem 3. Let C_1 and C_2 be two k-arbiters under U_1 and U_2, respectively, where $U_1 \cap U_2 = \varnothing$. $C_3 = C_1 \oplus_x C_2$ is a k-arbiter under $U_3 = (U_1 - \{x\}) \cup U_2$ for some $x \in U_1$.

Proof: First we want to show that C_3 satisfies the intersection property of k-arbiters; $\forall Q_1^{(3)}, Q_2^{(3)}, ..., Q_{k+1}^{(3)} \in C_3$, $Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_{k+1}^{(3)} \neq \varnothing$.

We consider the following three cases.

(1) $Q_1^{(3)}, Q_2^{(3)}, ..., Q_{k+1}^{(3)}$ are all Type-II quorums.
Then, by the definition of the k-arbiter join function, we observe that $Q_1^{(3)}, Q_2^{(3)}, ..., Q_{k+1}^{(3)} \in C_1$. Since C_1 is a k-arbiter, $Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_{k+1}^{(3)} \neq \varnothing$.

(2) For some j, $Q_1^{(3)}, Q_2^{(3)}, ..., Q_j^{(3)}$ are Type-I quorums and $Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_{k+1}^{(3)}$ are Type-II quorums.
Then, $\exists\ Q_1^{(1)}, Q_2^{(1)}, ..., Q_j^{(1)} \in C_1$, $Q_1^{(2)}, Q_2^{(2)}, ..., Q_j^{(2)} \in C_2$ such that

$$Q_1^{(3)} = (Q_1^{(1)} - \{x\}) \cup Q_1^{(2)},$$

$$Q_2^{(3)} = (Q_2^{(1)} - \{x\}) \cup Q_2^{(2)},$$

$$\vdots$$

$$Q_j^{(3)} = (Q_j^{(1)} - \{x\}) \cup Q_j^{(2)}.$$

By the definition of Type-II quorums, we know that $x \notin Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_{k+1}^{(3)}$ and $Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_{k+1}^{(3)} \in C_1$. Since C_1 is a k-arbiter,

$$Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_j^{(1)} \cap Q_{j+1}^{(3)} \cap Q_{j+2}^{(3)} \cap ... \cap Q_{k+1}^{(3)} \neq \varnothing. \quad (1)$$

And since $x \notin Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_{k+1}^{(3)}$,

$$x \notin Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_j^{(1)} \cap Q_{j+1}^{(3)} \cap Q_{j+2}^{(3)} \cap ... \cap Q_{k+1}^{(3)}. \quad (2)$$

By (1) and (2), we observe that

$$(Q_1^{(1)} - \{x\}) \cap (Q_2^{(1)} - \{x\}) \cap ... \cap (Q_j^{(1)} - \{x\}) \cap Q_{j+1}^{(3)} \cap Q_{j+2}^{(3)} \cap ... \cap Q_{k+1}^{(3)} \neq \varnothing$$

Thus,

$[(Q_1^{(1)}-\{x\})\cup Q_1^{(2)}]\cap[(Q_2^{(1)}-\{x\})\cup Q_2^{(2)}]\cap\ldots\cap[(Q_j^{(1)}-\{x\})\cup Q_j^{(2)}]\cap Q_{j+1}^{(3)}\cap Q_{j+2}^{(3)}\cap\ldots\cap Q_{k+1}^{(3)}\neq\varnothing$

; $Q_1^{(3)}\cap Q_2^{(3)}\cap\ldots\cap Q_j^{(3)}\cap Q_{j+1}^{(3)}\cap\ldots\cap Q_{k+1}^{(3)}\neq\varnothing$.

(3) $Q_1^{(3)},Q_2^{(3)},\ldots,Q_{k+1}^{(3)}$ are all Type-I quorums.

Then, $\exists$ $Q_1^{(1)},Q_2^{(1)},\ldots,Q_{k+1}^{(1)}\in C_1$, $Q_1^{(2)},Q_2^{(2)},\ldots,Q_{k+1}^{(2)}\in C_2$ such that

$Q_1^{(3)}=(Q_1^{(1)}-\{x\})\cup Q_1^{(2)}$,

$Q_2^{(3)}=(Q_2^{(1)}-\{x\})\cup Q_2^{(2)}$,

$\vdots$

$Q_{k+1}^{(3)}=(Q_{k+1}^{(1)}-\{x\})\cup Q_{k+1}^{(2)}$.

Since C_2 is a *k*-arbiter, $Q_1^{(2)}\cap Q_2^{(2)}\cap\ldots\cap Q_{k+1}^{(2)}\neq\varnothing$.

Thus,

$[(Q_1^{(1)}-\{x\})\cup Q_1^{(2)}]\cap[(Q_2^{(1)}-\{x\})\cup Q_2^{(2)}]\cap\ldots\cap[(Q_{k+1}^{(1)}-\{x\})\cup Q_{k+1}^{(2)}]\neq\varnothing$

; $Q_1^{(3)}\cap Q_2^{(3)}\cap\ldots\cap Q_{k+1}^{(3)}\neq\varnothing$.

Next, we show that C_3 satisfies the minimality property: $\forall Q_1^{(3)},Q_2^{(3)}\in C_3$, $Q_1^{(3)}\not\subset Q_2^{(3)}$.

We have the following four cases to consider:

(1) Both $Q_1^{(3)}$ and $Q_2^{(3)}$ are Type-II quorums.

Then, by the definition of *k*-arbiter join function, $Q_1^{(3)}\in C_1$ and $Q_2^{(3)}\in C_1$. Since C_1 is a *k*-arbiter, $Q_1^{(3)}\not\subset Q_2^{(3)}$.

(2) $Q_1^{(3)}$ is a Type-I quorum and $Q_2^{(3)}$ is a Type-II quorum.

Then, $\exists Q^{(1)}\in C_1$, $Q^{(2)}\in C_2$ such that $Q_1^{(3)}=(Q^{(1)}-\{x\})\cup Q^{(2)}$ and $Q_2^{(3)}\in C_1$.

Since $U_1\cap U_2=\varnothing$, $Q^{(2)}$ $(\in C_2)$ $\not\subset$ $Q_2^{(3)}$ $(\in C_1)$. Thus, $Q_1^{(3)}\not\subset Q_2^{(3)}$.

(3) $Q_1^{(3)}$ is a Type-II quorum and $Q_2^{(3)}$ is a Type-I quorum.

Then, $Q_1^{(3)}\in C_1$ and $\exists Q^{(1)}\in C_1$, $Q^{(2)}\in C_2$ such that $Q_2^{(3)}=(Q^{(1)}-\{x\})\cup Q^{(2)}$ and $Q_1^{(3)}\not\subset Q_2^{(3)}$; otherwise, the minimality property of C_1 will be violated.

(4) $Q_1^{(3)}$ and $Q_2^{(3)}$ are both Type-I quorums.

Then, $\exists Q_1^{(1)}$, $Q_2^{(1)}\in C_1$ and $Q_1^{(2)}$, $Q_2^{(2)}\in C_2$ such that

$Q_1^{(3)}=(Q_1^{(1)}-\{x\})\cup Q_1^{(2)}$ and $Q_2^{(3)}=(Q_2^{(1)}-\{x\})\cup Q_2^{(2)}$.

Since $U_1\cap U_2=\varnothing$ and both C_1 and C_2 are *k*-arbiters,

$(Q_1^{(1)}-\{x\})\not\subset(Q_2^{(1)}-\{x\})$ and $Q_1^{(2)}\not\subset Q_2^{(2)}$. Thus, $Q_1^{(3)}\not\subset Q_2^{(3)}$.

Therefore, C_3 is a *k*-arbiter under U_3. ❑

In the following theorems, we will show that C_3 is *ND* if and only if both C_1 and C_2 are *ND*.

Theorem 4. $C_3 = C_1 \oplus_x C_2$ is a *ND* *k*-arbiter if both C_1 and C_2 are *ND* *k*-arbiters.

Proof: On the basis of Theorem 3, we know that C_3 is a *k*-arbiter. In the rest of this proof, we show that C_3 is *ND* if both C_1 and C_2 are *ND*.

By contraposition, suppose C_3 is dominated. According to Theorem 2, we observe that $\exists$ H such that ($\forall Q^{(3)} \in C_3,\ Q^{(3)} \not\subseteq H$) and

$$(\forall Q_1^{(3)}, Q_2^{(3)}, ..., Q_k^{(3)} \in C_3,\ Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_k^{(3)} \cap H \neq \varnothing).$$

We consider the relation between H and quorums in C_2. There are two cases to consider: either (1) ($\forall Q_1^{(2)}, Q_2^{(2)}, ..., Q_k^{(2)} \in C_2,\ Q_1^{(2)} \cap Q_2^{(2)} \cap ... \cap Q_k^{(2)} \cap H \neq \varnothing$) or (2) ($\exists Q_1^{(2)}, Q_2^{(2)}, ..., Q_k^{(2)} \in C_2,\ Q_1^{(2)} \cap Q_2^{(2)} \cap ... \cap Q_k^{(2)} \cap H = \varnothing$).

(1) ($\forall Q_1^{(2)}, Q_2^{(2)}, ..., Q_k^{(2)} \in C_2,\ Q_1^{(2)} \cap Q_2^{(2)} \cap ... \cap Q_k^{(2)} \cap H \neq \varnothing$).

Let $H_1 = (H \cup \{x\}) \cap U_1$ and $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ be any k quorums in C_1. There are three sub-cases to consider: either (a) none of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contains x, (b) some of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contain x or (c) all of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contain x.

(a) none of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contains x.

We have $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)} \in C_3$. Thus, $Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_k^{(1)} \cap H \neq \varnothing$ and then $Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_k^{(1)} \cap H_1 \neq \varnothing$.

(b) some of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contain x.

Without loss of generality, we assume $x \in Q_1^{(1)}, Q_2^{(1)}, ..., Q_j^{(1)}$ and $x \notin Q_{j+1}^{(1)}, Q_{j+2}^{(1)}, ..., Q_k^{(1)}$ for some $1 \leq j < k$. Then, $\exists$ $Q_1^{(3)}, Q_2^{(3)}, ..., Q_k^{(3)} \in C_3$, $Q_1^{(2)}, Q_2^{(2)}, ..., Q_j^{(2)} \in C_2$ such that

$$Q_1^{(3)} = (Q_1^{(1)} - \{x\}) \cup Q_1^{(2)},$$

$$Q_2^{(3)} = (Q_2^{(1)} - \{x\}) \cup Q_2^{(2)},$$

$$\vdots$$

$$Q_j^{(3)} = (Q_j^{(1)} - \{x\}) \cup Q_j^{(2)} \text{ and}$$

$$Q_i^{(3)} = Q_i^{(1)} \text{ for } j+1 \leq i \leq k.$$

Since $H \cap Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing$ and $x \notin Q_{j+1}^{(1)}, Q_{j+2}^{(1)}, ..., Q_k^{(1)}$,

$$H \cap [(Q_1^{(1)} - \{x\}) \cup Q_1^{(2)}] \cap ... \cap [(Q_j^{(1)} - \{x\}) \cup Q_j^{(2)}] \cap Q_{j+1}^{(1)} \cap ... \cap Q_k^{(1)} \neq \varnothing$$

$\rightarrow H \cap (Q_1^{(1)} \cup Q_1^{(2)}) \cap ... \cap (Q_j^{(1)} \cup Q_j^{(2)}) \cap Q_{j+1}^{(1)} \cap ... \cap Q_k^{(1)} \neq \emptyset$.

By the definition of the *k*-arbiter join operation, we know that $Q^{(1)} \cap Q^{(2)} = \emptyset$, $\forall Q^{(1)} \in C_1$, $Q^{(2)} \in C_2$. Thus, $H \cap Q_1^{(1)} \cap ... \cap Q_j^{(1)} \cap Q_{j+1}^{(1)} \cap ... \cap Q_k^{(1)} \neq \emptyset$ and then $H_1 \cap Q_1^{(1)} \cap ... \cap Q_j^{(1)} \cap Q_{j+1}^{(1)} \cap ... \cap Q_k^{(1)} \neq \emptyset$.

(c) all of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contain *x*.

It is obvious that $\{x\} \subseteq H_1 \cap Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_k^{(1)} \neq \emptyset$.

Consequently, $\forall Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)} \in C_1$, $H_1 \cap Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_k^{(1)} \neq \emptyset$. We conclude that either $\exists Q^{(1)} \in C_1$, $Q^{(1)} \subseteq H_1$ or H_1 satisfies [*P*1] and [*P*2] for C_1, and, therefore, C_1 is dominated.

Let $H_2 = H \cap U_2$ and $Q_1^{(2)}, Q_2^{(2)}, ..., Q_k^{(2)}$ be any *k* quorums in C_2. By hypothesis, we have $H \cap Q_1^{(2)} \cap Q_2^{(2)} \cap ... \cap Q_k^{(2)} \neq \emptyset$ and then $H_2 \cap Q_1^{(2)} \cap Q_2^{(2)} \cap ... \cap Q_k^{(2)} \neq \emptyset$. We conclude that either $\exists Q^{(2)} \in C_2$, $Q^{(2)} \subseteq H_2$ or H_2 satisfies [*P*1] and [*P*2] for C_2, and, therefore, C_2 is dominated.

Finally, we want to show that either $\exists Q^{(1)} \in C_1$, $Q^{(1)} \subseteq H_1$ or $\exists Q^{(2)} \in C_2$, $Q^{(2)} \subseteq H_2$ will be true, but not both. Suppose that both $\exists Q^{(1)} \in C_1$, $Q^{(1)} \subseteq H_1$ and $\exists Q^{(2)} \in C_2$, $Q^{(2)} \subseteq H_2$ are true. We further consider the following two cases: (i) $x \in Q^{(1)}$ or (ii) $x \notin Q^{(1)}$. For Case (i) $\exists Q^{(3)} \in C_3$, $Q^{(3)} = (Q^{(1)} - \{x\}) \cup Q^{(2)}$ and for Case (ii) $\exists Q^{(3)} \in C_3$, $Q^{(3)} = Q^{(1)}$. It is obvious that for both cases, $Q^{(3)} \subseteq H$. It is a contradiction to ($\forall Q^{(3)} \in C_3$, $Q^{(3)} \not\subseteq H$). Consequently, either H_1 satisfies [*P*1] and [*P*2] for C_1, and, therefore C_1 is dominated or H_2 satisfies [*P*1] and [*P*2] for C_2, and, therefore, C_2 is dominated.

(2) ($\exists Q_1^{(2)}, Q_2^{(2)}, ..., Q_k^{(2)} \in C_2$, $Q_1^{(2)} \cap Q_2^{(2)} \cap ... \cap Q_k^{(2)} \cap H = \emptyset$).

Let $H_1 = H \cap U_1$ and $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ be any *k* quorums in C_1. There are three cases to consider: either (a) none of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contains *x*, (b) some of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contain *x* , or (c) all of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contain *x*.

Since Cases (a) and (b) are similar to those in Case (1), we omit them.

(c) all of $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)}$ contain *x*.

Since $\exists Q_1^{(2)}, Q_2^{(2)}, ..., Q_k^{(2)} \in C_2$, $Q_1^{(2)} \cap Q_2^{(2)} \cap ... \cap Q_k^{(2)} \cap H = \emptyset$,

$\exists$ $Q_1^{(3)}, Q_2^{(3)}, ..., Q_k^{(3)} \in C_3$ such that

$$Q_1^{(3)} = (Q_1^{(1)} - \{x\}) \cup Q_1^{(2)},$$
$$Q_2^{(3)} = (Q_2^{(1)} - \{x\}) \cup Q_2^{(2)},$$
$$\vdots$$
$$Q_k^{(3)} = (Q_k^{(1)} - \{x\}) \cup Q_k^{(2)}$$

Since $H \cap Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_k^{(3)} \neq \emptyset$ and $H \cap Q_1^{(2)} \cap Q_2^{(2)} \cap ... \cap Q_k^{(2)} = \emptyset$,

$H \cap Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_k^{(1)} \neq \varnothing$ and then $H_1 \cap Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_k^{(1)} \neq \varnothing$.

Consequently, $\forall\, Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)} \in C_1$, $H_1 \cap Q_1^{(1)} \cap Q_2^{(1)} \cap ... \cap Q_k^{(1)} \neq \varnothing$. We conclude that either $\exists\, Q^{(1)} \in C_1$, $Q^{(1)} \subseteq H_1$ or H_1 satisfies [P1] and [P2] for C_1, and, therefore, C_1 is dominated.

Suppose $\exists Q^{(1)} \in C_1$, $Q^{(1)} \subseteq H_1$. Since $H_1 \subseteq H$ and $H \subseteq U_3$ (= $(U_1 - \{x\}) \cup U_2$), $x \notin H_1$. We have $x \notin Q^{(1)}$ because $x \notin H_1$ and $Q^{(1)} \subseteq H_1$. Then, $\exists\, Q^{(3)} \in C_3$, $Q^{(3)} = Q^{(1)}$ Hence, $\exists\, Q^{(3)} \in C_3$, $Q^{(3)} = Q^{(1)} \subseteq H_1 \subseteq H$. It is a contradiction to ($\forall\, Q^{(3)} \in C_3$, $Q^{(3)} \not\subseteq H$). Consequently, H_1 satisfies [P1] and [P2] for C_1, and, therefore, C_1 is dominated. ❑

As a consequence of Theorem 4, we can construct a new *ND k*-arbiter by joining known *ND k*-arbiters.

Theorem 5. $C_3 = C_1 \oplus_x C_2$ is a dominated k-arbiter if either C_1 or C_2 is dominated.

***Proof*:** We want to show that $\exists$ H such that ($\forall\, Q^{(3)} \in C_3$, $Q^{(3)} \not\subseteq H$) and ($H \cap Q_1^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing$).

(A) Assume C_1 is dominated.

Then, $\exists$ H such that ($\forall Q^{(1)} \in C_1$, $Q^{(1)} \not\subseteq H$) and ($H \cap Q_1^{(1)} \cap ... \cap Q_k^{(1)} \neq \varnothing$).

Let $H' = (H - \{x\}) \cup Q_i^{(2)}$ for some $Q_i^{(2)} \in C_2$.

Here, we show that H' satisfies [P1] and [P2] for C_3.

We consider the following two cases:

(1) Suppose for some j, $Q_1^{(3)}, Q_2^{(3)}, ..., Q_j^{(3)}$ are Type-I quorums and $Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_k^{(3)}$ are Type-II quorums.

Then, $\exists$ $Q_1^{(1)}, Q_2^{(1)}, ..., Q_j^{(1)} \in C_1$, $Q_1^{(2)}, Q_2^{(2)}, ..., Q_j^{(2)} \in C_2$ such that

$$Q_1^{(3)} = (Q_1^{(1)} - \{x\}) \cup Q_1^{(2)},$$

$$Q_2^{(3)} = (Q_2^{(1)} - \{x\}) \cup Q_2^{(2)},$$

$$\vdots$$

$$Q_j^{(3)} = (Q_j^{(1)} - \{x\}) \cup Q_j^{(2)}.$$

Since $Q_1^{(1)}, Q_2^{(1)}, ..., Q_j^{(1)}$, $Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_k^{(3)} \in C_1$ and [P1],

$Q_1^{(1)}, Q_2^{(1)}, ..., Q_j^{(1)}, Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_k^{(3)} \not\subseteq H$.

$\rightarrow (Q_1^{(1)} - \{x\}), (Q_2^{(1)} - \{x\}), ..., (Q_j^{(1)} - \{x\}), Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_k^{(3)} \not\subseteq (H - \{x\})$

$\rightarrow Q_1^{(3)}, Q_2^{(3)}, ..., Q_k^{(3)} \not\subseteq H'$

Thus, H' satisfies [P1] for C_3.

By hypothesis, we know that $H \cap Q_1^{(1)} \cap ... \cap Q_j^{(1)} \cap Q_{j+1}^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing$

$\rightarrow H \cap (Q_1^{(1)} \cup Q_1^{(2)}) \cap ... \cap (Q_j^{(1)} \cup Q_j^{(2)}) \cap Q_{j+1}^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing$

$\rightarrow H \cap [(Q_1^{(1)} - \{x\}) \cup Q_1^{(2)}] \cap ... \cap [(Q_j^{(1)} - \{x\}) \cup Q_j^{(2)}] \cap Q_{j+1}^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing$

$\rightarrow H \cap Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing$

Since $x \notin Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_k^{(3)}$, $H' \cap Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing$.

Consequently, H' satisfies [*P*2] for C_3, and hence C_3 is dominated.

(2) Suppose $Q_1^{(3)}, Q_2^{(3)}, ..., Q_k^{(3)}$ are all Type-I quorums.

Then, $\exists\ Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)} \in C_1$, $Q_1^{(2)}, Q_2^{(2)}, ..., Q_k^{(2)} \in C_2$ such that

$Q_1^{(3)} = (Q_1^{(1)} - \{x\}) \cup Q_1^{(2)}$,

$Q_2^{(3)} = (Q_2^{(1)} - \{x\}) \cup Q_2^{(2)}$,

$\vdots$

$Q_k^{(3)} = (Q_k^{(1)} - \{x\}) \cup Q_k^{(2)}$.

Since $Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)} \in C_1$ and [*P*1], we have ($Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)} \not\subseteq H$).

$\rightarrow ((Q_1^{(1)} - \{x\}), (Q_2^{(1)} - \{x\}), ..., (Q_k^{(1)} - \{x\}) \not\subseteq (H - \{x\})$

$\rightarrow Q_1^{(3)}, Q_2^{(3)}, ..., Q_k^{(3)} \not\subseteq H'$

Thus, H' satisfied [*P*1] for C_3.

Since C_2 is a *k*-arbiter, $Q_i^{(2)} \cap Q_1^{(2)} \cap ... \cap Q_k^{(2)} \neq \varnothing$.

$\rightarrow H' \cap Q_1^{(2)} \cap ... \cap Q_k^{(2)} \neq \varnothing$

$\rightarrow H' \cap (Q_1^{(1)} \cup Q_1^{(2)}) \cap ... \cap (Q_k^{(1)} \cup Q_k^{(2)}) \neq \varnothing$

$\rightarrow H' \cap [(Q_1^{(1)} - \{x\}) \cup Q_1^{(2)}] \cap ... \cap [(Q_k^{(1)} - \{x\}) \cup Q_k^{(2)}] \neq \varnothing$

$\rightarrow H' \cap Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing$

Consequently, H' satisfies [*P*2] for C_3, and hence C_3 is dominated.

(B) Assume C_2 is dominated.

$\exists\ H$ such that ($\forall Q^{(2)} \in C_2$, $Q^{(2)} \not\subseteq H$) and ($H \cap Q_1^{(2)} \cap ... \cap Q_k^{(2)} \neq \varnothing$).

Let $H' = H \cup (Q_i^{(1)} - \{x\})$ for some $Q_i^{(1)} \in C_1$ and $x \in Q_i^{(1)}$.

Here, we show that H' satisfies [*P*1] and [*P*2] for C_3.

We consider the following two cases:

(1) Suppose for some *j*, $Q_1^{(3)}, Q_2^{(3)}, ..., Q_j^{(3)}$ are Type-I quorums and $Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_k^{(3)}$ are Type-II quorums.

Then, $\exists\ Q_1^{(1)}, Q_2^{(1)}, ..., Q_j^{(1)} \in C_1$, $Q_1^{(2)}, Q_2^{(2)}, ..., Q_j^{(2)} \in C_2$ such that

$Q_1^{(3)} = (Q_1^{(1)} - \{x\}) \cup Q_1^{(2)}$,

$Q_2^{(3)} = (Q_2^{(1)} - \{x\}) \cup Q_2^{(2)}$,

$\vdots$

$Q_j^{(3)} = (Q_j^{(1)} - \{x\}) \cup Q_j^{(2)}$.

Since ($\forall Q^{(2)} \in C_2$, $Q^{(2)} \not\subseteq H$), we observe that ($Q_1^{(3)}, Q_2^{(3)}, ..., Q_j^{(3)} \not\subseteq H'$).

Furthermore, since C_1 is a *k*-arbiter, $Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_k^{(3)} \not\subseteq Q_i^{(1)} \not\subseteq (Q_i^{(1)} - \{x\}) \not\subseteq H'$. Thus, H' satisfies [*P*1] for C_3.

On the other hand, since C_3 is a *k*-arbiter, and $x \notin Q_{j+1}^{(3)}, Q_{j+2}^{(3)}, ..., Q_k^{(3)}$,

$$(Q_i^{(1)} - \{x\}) \cap [(Q_1^{(1)} - \{x\}) \cup Q_1^{(2)}] \cap .. \cap [(Q_j^{(1)} - \{x\}) \cup Q_j^{(2)}] \cap Q_{j+1}^{(3)} \cap .. \cap Q_k^{(3)} \neq \varnothing$$

$$\rightarrow H' \cap Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing .$$

Consequently, H' satisfies [*P*2] for C_3, and hence C_3 is dominated.

(2) Suppose $Q_1^{(3)}, Q_2^{(3)}, ..., Q_k^{(3)}$ are all Type-I quorums.

Then, $\exists \; Q_1^{(1)}, Q_2^{(1)}, ..., Q_k^{(1)} \in C_1$, $Q_1^{(2)}, Q_2^{(2)}, ..., Q_k^{(2)} \in C_2$ such that

$Q_1^{(3)} = (Q_1^{(1)} - \{x\}) \cup Q_1^{(2)}$,

$Q_2^{(3)} = (Q_2^{(1)} - \{x\}) \cup Q_2^{(2)}$,

⋮

$Q_k^{(3)} = (Q_k^{(1)} - \{x\}) \cup Q_k^{(2)}$.

Since ($\forall Q^{(2)} \in C_2$, $Q^{(2)} \not\subseteq H$), we observe that ($Q_1^{(3)}, Q_2^{(3)}, ..., Q_k^{(3)} \not\subseteq H'$). Thus, H' satisfies [*P*1] for C_3.

In addition, since ($H \cap Q_1^{(2)} \cap ... \cap Q_k^{(2)} \neq \varnothing$),

$$H \cap [(Q_1^{(1)} - \{x\}) \cup Q_1^{(2)}] \cap ... \cap [(Q_k^{(1)} - \{x\}) \cup Q_k^{(2)}] \neq \varnothing$$

$$\rightarrow H' \cap Q_1^{(3)} \cap Q_2^{(3)} \cap ... \cap Q_k^{(3)} \neq \varnothing .$$

Consequently, H' satisfies [*P*2] for C_3, and hence C_3 is dominated. ❑

As a consequence of Theorem 5, the joined *k*-arbiters are dominated if one of the joining *k*-arbiters is dominated.

For example, consider two *k*-arbiters, C_1 ={{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}}} under U_1 = {1, 2, 3, 4} and C_2 = {{a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}} under U_2 = {a, b, c, d}, which are constructed from the uniform *k*-arbiters proposed in [11]. From the result shown in [11], we know that both C_1 and C_2 are *ND* *k*-arbiters. Then,

$C_3 = C_1 \oplus_4 C_2$
={{1, 2, 3}, {1, 2, a, b, c}, {1, 2, a, b, d}, {1, 2, a, c, d}, {1, 2, b, c, d},
{1, 3, a, b, c}, {1, 3, a, b, d}, {1, 3, a, c, d}, {1, 3, b, c, d},
{2, 3, a, b, c}, {2, 3, a, b, d}, {2, 3, a, c, d}, {2, 3, b, c, d}}.

The reader can verify that C_3 is a *ND* *k*-arbiter under U_3 = {1, 2, 3, a, b, c, d}.

Corollary 4. $C_3 = C_1 \oplus_x C_2$ is a *ND* *k*-arbiter if and only if both C_1 and C_2 are *ND* *k*-arbiters.

5. AVAILABILITY OF JOINED *k*-ARBITERS

In this section, we discuss the availability properties of joined *k*-arbiters. By selecting *k*-arbiters satisfying a sufficient condition as join inputs, the joined *k*-arbiter could have a higher availability than that of the original inputs. We also investigate the availability properties of *ND* *k*-arbiters. We propose a way to recognize *ND* *k*-arbiters.

Availability is a widely accepted measure for analyzing coteries. The availability of a coterie C, $AV_C(p)$, is defined as the probability that a quorum in it can be successfully formed under the probability $p(v)$, $\forall v \in U$, that node v is operational. For example, the availability of the majority coterie [18] is given by the probability that at least a majority of nodes in the system are operational. High availability of a coterie exhibits a high fault-tolerant ability in the face of node and communication link failures. For simplicity, we usually assume that all nodes in the system have the same probability to be operational, i.e., $p(v)$ is assumed to be a constant function.

Similarly, we define the availability of a *k*-arbiter C, $AV_C^k(p)$, to be the probability that a quorum in it can be successfully formed under the probability $p(v)$, $\forall v \in U$, that node v is operational.

Obviously, the definition of the availability of coteries is essentially identical to the definition of the availability of *k*-arbiters; they both concern evaluating the probability of successfully forming a quorum. The only difference is that quorums in a system may satisfy the coterie requirements or the *k*-arbiter requirements.

In [3], the author analyzed the availability of the *k*-coterie join operation, which was proposed by Jiang and Huang [5]. Under a sufficient condition, the *k*-coterie join operation can generate a new *k*-coterie with higher availability. In the following, we derive a sufficient condition for the *k*-arbiter join operation such that the *k*-arbiter join operation can also generate a new *k*-arbiter with higher availability.

Let $p(v)$ be the probability that node v is operational. We define a probability function $p'(v)$ as follows:

$$p'(v) = \begin{cases} p(v) & \text{if} \quad v \in U_3 - \{x\} \\ AV_{C_2}^k(p) & \text{if} \quad v = x \end{cases} .$$

Corollary 5. Let C_1 and C_2 be two *k*-arbiters and $C_3 = C_1 \oplus_x C_2$. Then,

$$AV_{C_3}^k(p) = AV_{C_1}^k(p') .$$

Proof: The proof is obvious because C_3 is essentially identical to C_1 except that node x is a *logical* node in C_3 where it is replaced by the nodes of any quorum in C_2. Thus, the probability that node x is operational is equal to the probability that a quorum in C_2 is operational. ❑

Corollary 6. Let C_1 and C_2 be two k-arbiters and $C_3 = C_1 \oplus_x C_2$. If $\exists p_0$, $0 < p_0 < 1$ and $AV^k_{C_2}(p_0) \geq p_0$, $AV^k_{C_3}(p) \geq AV^k_{C_1}(p)$, for all $p \geq p_0$.

Proof: The proof is obvious because the availability of a k-arbiter increases as the probability that the node is operational increases. Thus, on the basis of Corollary 5,

$$AV^k_{C_3}(p) = AV^k_{C_1}(p') \geq AV^k_{C_1}(p), \text{ for all } p \geq p_0. \qquad \square$$

On the basis of Corollary 6, we could recursively apply the k-arbiter join operation to construct joined k-arbiters with higher and higher availability by selecting inputs as k-arbiters which satisfy the sufficient condition of Corollary 6.

For example, consider the uniform k-arbiter [11] under $N = 7$ and $k = 2$. Its availability under $p = 0.85$ is

$$AV^k_{uniform}(p) = \sum_{i=w}^{7} C(7,i)\, p^i (1-p)^{(7-i)}, \text{ where } w = \lceil {}^{kN+1}\!/\!_{k+1} \rceil = 5$$

$$= 0.8663\ (\geq 0.85).$$

That is, any k-arbiter C joined with the uniform k-arbiter will produce a new k-arbiter with availability higher than C for all $p \geq 0.85$.

In [9], a theory to recognize *ND* coterie by using availability is proposed. Here, we rewrite it as follows.

Theorem 6. A coterie C is *ND iff* $AV_C(p) = 0.5$ when $p = 0.5$.

On the basis of Theorem 6, we observe that by evaluating the availability of a coterie under $p = 0.5$, the coterie can be recognized as a *ND* coterie or not.

In the following, we propose a way to recognize *ND* k-arbiters by using availability.

Lemma 1. A dominated k-arbiter C is a dominated coterie.

Proof: By corollary 1, we know that C is a coterie. Next, we are going to show that C is not only a coterie but also a dominated coterie. Since C is a dominated k-arbiter, by Theorem 2, there must exist a set H satisfying [P1] and [P2]. According to Theorem 1, it is obvious that the set H also satisfies both [R1] and [R2] for C. Thus, C is a dominated coterie, too. $\square$

Lemma 2. If C is a dominated k-arbiter, then $AV^k_C(p) \neq 0.5$ when $p = 0.5$.

Proof: By Lemma 1, we know that C is a dominated coterie. According to Theorem 6, we observe that $AV_C(p) \neq 0.5$ when $p = 0.5$; thus, $AV^k_C(p) \neq 0.5$ when $p = 0.5$. $\square$

The following theorem is a direct consequence of Lemma 2.

Theorem 7. For a *k*-arbiter *C*, if $AV_C^k(p)$ = 0.5 when *p* = 0.5, then *C* is a *ND* *k*-arbiter.

For example, the uniform *k*-arbiter [11], under *N* = 7 and *k* = 1, is a *ND* *k*-arbiter because the availability of the uniform *k*-arbiter under *p* = 0.5 is

$$AV_{uniform}^k(p) = \sum_{i=w}^{7} C(7,i)p^i(1-p)^{(7-i)} = 0.5 \text{ where } w = \left\lceil \frac{1\times 7+1}{1+1} \right\rceil = 4.$$

Thus, on the basis of Theorem 7, we can check whether or not a *k*-arbiter is *ND* by evaluating its availability. However, the contraposition of Theorem 7 may not be true; that is, a *ND* *k*-arbiter *C* may not have $AV_C^k(p)$=0.5 when *p*=0.5. For example, consider the uniform *k*-arbiter under *N* = 4 and *k* = 2. According to the result shown in [11], we know that the uniform *k*-arbiter under *N* = 4 and *k* = 2 is a *ND* *k*-arbiter because $w = \frac{kN+1}{k+1} = 3$. However, its availability under *p* = 0.5 is

$$AV_{uniform}^k(p) = \sum_{i=w}^{4} C(4,i)p^i(1-p)^{(4-i)} \quad (= 0.3125) \neq 0.5.$$

Consequently, if the availability of a *k*-arbiter is 0.5 when *p* = 0.5, then we can instantly claim that the *k*-arbiter is *ND* instead of using a formal proof of Theorem 2. A way to recognize *ND* *k*-arbiters by using availability is proposed.

6. CONCLUSIONS

k-Arbiter is a useful concept for solving the distributed *h*-out of-*k* resources allocation problem. The distributed *h*-out of-*k* resources allocation algorithms based on *k*-arbiter can have the benefits of high fault-tolerant ability and low communication cost. However, according to the definition of *k*-arbiter, it is required to have a non-empty intersection among any (*k*+1) quorums in a *k*-arbiter. Consequently, to construct *k*-arbiters is difficult.

In this paper, we proposed a *k*-arbiter join operation to construct the new *k*-arbiter from known *k*-arbiters. By recursively applying the *k*-arbiter join operation, new and better *k*-arbiters are generated. Furthermore, we observe that by selecting the proper *k*-arbiters as inputs, the joined *k*-arbiter can be nondominated, and its availability can be higher than that of the original inputs.

REFERENCES

1. H. K. Chang and S. M. Yuan, “Message complexity of hierarchical quorum consensus algorithm,” *Information Processing Letters*, Vol. 50, 1994, pp. 69-73.
2. H. Garcia-Molina and D. Barbara, “How to assign votes in a distributed systems,” *JACM.*, Vol. 32, No. 4, 1985, pp. 841-860.

3. T. Harada and M. Yamashita, "Coterie join operation and tree structured *k*-coteries," Technical Report, Information Processing Center, Hiroshima University, Japan, 1999.
4. T. Ibaraki and T. Kameba, "A theory of coterie: Mutual exclusion in distributed systems," *IEEE Transactions on Parallel and Distributed System*, Vol. 4, No. 7, 1993, pp. 779-793.
5. J. R. Jiang and S. T. Huang, "Obtaining nondominated *k*-coteries for fault-tolerant distributed *k*-mutual exclusion," in *Proceedings of IEEE International Conf. on Parallel and Distributed Systems*, 1994, pp. 582-587.
6. J. R. Jiang, S. T. Huang, and Y. C. Kuo, "Cohorts structures for fault-tolerant *k* entries to a critical section," *IEEE Transactions on Comput*ing, Vol. 46, No. 2, 1997, pp. 222-228.
7. H. Kakugawa, S. Fujita, M. Yamashita, and T. Ae, "A distributed *k*-mutual exclusion algorithm using *k*-coterie," *Information Processing Letter*, Vol. 49, 1994, pp. 213-218.
8. Y. C. Kuo and S. T. Huang, "A geometric approach for constructing coteries and *k*-coteries," *IEEE Transactions on Parallel and Distributed Syst*em, Vol. 8, No. 4, 1997, pp. 402-411.
9. Y. C. Kuo and S. T. Huang, "Recognizing nondominated coteries and *wr*-coteries by availability," *IEEE Transactions on Parallel and Distributed Syst*em, Vol. 9, No. 8, 1998, pp. 721-728.
10. M. Maekawa, "A $\sqrt{N}$ algorithm for mutual exclusion in decentralized systems," *ACM Transactions on Computer System*, Vol. 3, No. 2, 1985, pp. 145-159.
11. Y. Manabe, R. Baldoni, M. Raynal, and S. Aoyagi, "*k*-Arbiter: A safe and general scheme for *h*-out of-*k* mutual exclusion," *Theoretical Computer Science*, Vol. 193, No. 1-2, 1998, pp. 97-112.
12. M. L. Neilsen and M. Mizuno, "Coterie join algorithm," *IEEE Transactions on Parallel and Distributed System*, Vol. 3, No. 5, 1992, pp. 582-590.
13. M. L. Neilsen and M. Mizuno, "Nondominated *k*-coteries for multiple mutual exclusion," *Information Processing Letter*, Vol. 50, 1994, pp. 247-252.
14. K. Raymond, "A distributed algorithm for multiple entries to a critical section," *Information Processing Letter*, Vol. 30, No. 4, 1989, pp. 189-193.
15. M. Raynal, "A distributed solution to the *k*-out of-*M* resources allocation problem," *Lecture Notes of Computer Science* 497, Springer-Verlag, 1991, pp. 599-609.
16. M. Raynal, "A simple taxonomy for distributed mutual exclusion algorithm," *ACM Operation System Review*, Vol. 25, No. 1, 1991, pp. 47-50.
17. D. Shou and S. D. Wang, "An efficient quorum generating approach for distributed mutual exclusion," *Journal of Information Science and Engineering*, Vol. 9, No. 2, 1993, pp. 201-227.
18. R. H. Thomas, "A majority consensus approach to concurrency control," *ACM Transactions on Database Syst*em, Vol. 4, No. 2, 1979, pp. 180-209.

Yu-Chen Kuo (郭育政) received the B.S. degree in applied mathematics from National Chung-Hsing University in June 1991, and the Ph.D. degree in computer science from National Tsing-Hua University in January 1997. In July 1997, he joined the faculty of the Department of Applied Mathematics, TaTung Institute of Technology, Taipei, Taiwan. Since August 1998, he has been an assistant professor in the Department of Computer and Information Science, SooChow University, Taipei, Taiwan. His research interests include fault-tolerant distributed systems, database management systems, computer networks, and operation systems.