JOURNAL OF INFORMATION SCIENCE AND ENGINEERING 18, 467-473 (2002)

Short Paper

Triangle Graphs and Simple Trapezoid Graphs

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In this paper, we present results on two subclasses of trapezoid graphs, including simple trapezoid graphs and triangle graphs (also known as *PI* graph in [3]). Simple trapezoid graphs and triangle graphs are proper subclasses of trapezoid graphs [3, 5]. Here we show that simple trapezoid graphs and triangle graphs are also two distinct subclasses of trapezoid graphs.

Keywords: triangle graphs, simple trapezoid graphs, parallelogram graphs, maximum clique, vertex coloring, maximum independent set, clique partition

1. INTRODUCTION

In the literature of graph recognition algorithms, two of the most extensively covered graphs are interval graphs and permutation graphs [6, 2]. One way of generalizing both interval graphs and permutation graphs is the intersection graph of a collection of trapezoids with corner points lying on two parallel lines. Such a graph is called the *trapezoid graph* [3, 4]; Ma and Spinrad [8] show that trapezoid graphs can be recognized in $O(n^2)$ time.

On the other hand, Golumbic and Monma [7] introduce the class of *tolerance graphs* that also generalizes permutation graphs and interval graphs. Each vertex $v \in V$ of a tolerance graph $G = (V, E)$ corresponds an interval I_v on a line and a positive real number t_v (the tolerance). Two vertices u, v are adjacent to each other, i.e., $uv \in E$ iff $|I_u \cap I_v| \geq \min\{t_u, t_v\}$. Furthermore, a tolerance graph is called *bounded* if $t_v \leq |I_v|$ for all $v \in V$. Bogart *et al.* [1] showed that bounded tolerance graphs are actually *parallelogram graphs;* i.e., intersection graphs of parallelograms each of which has its horizontal lines on twoparallel lines. The most interesting thing shown in the study of these subclasses of tolerance graphs is that, while they are all subclasses of trapezoid graphs, we still do not know how to efficiently recognize these graphs. This gives us a motivation for trying to find other properties of these subclasses of trapezoid graphs, which are still superclasses of permutation and interval graphs.

In this paper, the author presents results on two subclasses of trapezoid graphs including simple trapezoid graphs and triangle graphs. Here the intersection graph of rectangles and line segments whose two ends lie on two parallel lines is called the *simple*

Received February 1, 2000; revised November 30, 2000; accepted April 25, 2001.
Communicated by Jeremy P. Spinrad.

trapezoid graph. The intersection graph of triangles with top vertices on one line and bases on the other is called the *triangle graph*. The triangle graph were known by Corneil and Kamula as *PI* graphs [3, 2]. Surprisingly, such slight changes in the geometric structure render different intersection classes. In particular, we show that simple trapezoid graphs and triangle graphs are distinct from each other.

2. SIMPLE AND TRAPEZOID GRAPHS

We start by considering the simplest generalization of interval graphs and permutation graphs. One way to generalize these two different graphs is to directly simulate their topological structures using an interval/permutation model. So consider two parallel lines ℓ_1, ℓ_2 which lie in a plane. The line segment with one endpoint lies on ℓ_1 and the other lies on ℓ_1 is called a *stick*. The rectangle whose upper side lies on ℓ_1 and the lower side lies on ℓ_2 is called a *block*. It follows that the intersection graph of these sticks or blocks is called a *simple trapezoid graph*. Simple trapezoid graphs clearly generalize interval graphs and permutation graphs because interval graphs are the intersection graphs of blocks, and permutation graphs the intersection graph of sticks. Note that the graph F shown in Fig. 1 is an interval graph but not a permutation graph.

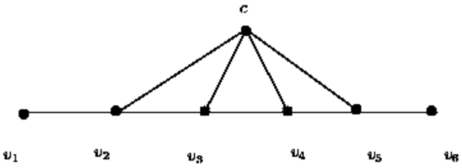

Fig. 1. The graph F is an interval graph but not a permutation graph.

Sticks and blocks in the models of simple trapezoid graphs do not have the same flexibility as the ordinary trapezoids. Consider a C_4, the chordless 4-cycle, in a simple trapezoid graph. It is clear that not every vertex of a C_4 can have a block representation because C_4 is not an interval graph.

Given a simple graph $G = (V, E)$ with $U \subset V$, the *subgraph induced by U* is denoted by $\langle U \rangle_G$; that is, $\langle U \rangle_G = (U, E \cap (U \times U))$. Two isomorphic graphs G and H are denoted by $G \cong H$. Let $N(v)$ denote the neighborhood of vertex v; i.e., $N(v) = \{u \mid uv \in E\}$. Furthermore, in the following discussion, we use the infix notation $u \sim v$ to denote two adjacent vertices u and v; in other words, $(u, v) \in E$.

Lemma 2.1 Given a simple trapezoid graph G, let vertices v_1, v_2, v_3, v_4 induce a C_4 in G, such that $v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_1$. Then no two adjacent elements can have a block representation in the underlying simple trapezoid model.

Proof. Suppose otherwise. Without loss of generality, assume that both v_1 and v_2 have block representations in the underlying model. None of the objects of v_1 or v_2 can contain each other in the model since $N(v_1) \not\subset N(v_2)$ and $N(v_2) \not\subset N(v_1)$. Without loss of generality, we can assume that some portion of the block corresponding to v_1 lies to the left of the block of v_2. Thus v_3 has to touch v_1 on the left, and now it is impossible for v_4 to touch both v_1 and v_3 without any contact with the block v_2.

❑

The *Join* of two graphs G and H, denoted as Join (G, H), is the graph with $V(G) \cup V(H)$ as its vertex set, and $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in H(G)\}$ as its edges set. Now we are ready to show that not every triangle graph is a simple trapezoid graph:

Theorem 2.2 Let F denote the graph illustrated in Fig. 1. Join (F, F) is a triangle graph but not a simple trapezoid graph.

Proof. It is not hard to show that Join (F, F) is a triangle graph. The triangle model of Join (F, F) is illustrated in Fig. 2. Now, partition the vertices of Join (F, F) into two groups $V = \{c, v_1, \dots , v_6\}$ and $V' = \{c', v'_1, \dots, v'_6\}$ such that $\langle V\rangle_{\text{Join}(F, F)} \cong \langle V'\rangle_{\text{Join}(F, F)} \cong F$, $v_1(v'_1) \sim \cdots \sim v_6(v'_6)$, and $c \sim \{v_2, \dots, v_5\}$ $(\{v'_2, \dots, v'_5\})$. Assume first that the object corresponding to v_1 in the simple trapezoid model is a block. Since $v_1 \nsim v_3$ in F, every two nonadjacent vertices of V' together with $\{v_1, v_3\}$ forms a C_4 in Join (F, F). Note that no vertex is adjacent to every other vertices in F. Thus, by Lemma 2.1, every vertex in V' must have a stick representation. However, since $\langle V'\rangle_{\text{Join}(F, F)} \cong F$ is not a permutation graph, we have to conclude that vertex v_1 cannot be represented as a block in the underlying simple trapezoid model.

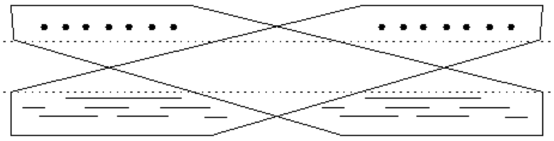

Fig. 2. A triangle model of the graph Join (F, F).

The argument that we used to conclude that v_1 has to be represented by a stick can be applied to every vertex of V (and V'). This leads to a contradiction because F is not a permutation graphs. Thus we conclude that Join (F, F) is not a simple trapezoid graph.

It is interesting to note that this result can be generalized to a family of triangle graphs. Graph F_n is defined by an n-vertex simple path plus one extra vertex c connecting to every vertex of the path except the two endpoints. It follows that graph F is exactly graph F_6. By similar arguments, it can be shown that each graph of $\{\text{Join }(F_{2k}, F_{2k}): k \geq 3\}$ is a triangle graph but not a simple trapezoid graph.

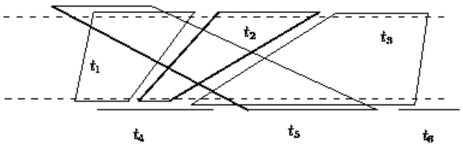

Fig. 3. A $K_{3, 3}$ induced subgraph in trapezoid graph shall force a "X" shape of two middle trapezoids t_2 and t_5.

3. TRIANGLE GRAPHS

A different approach to generalizing interval graphs and permutation graphs is to consider the objects in each of the two parallel lines. We can consider the model in which the objects in one parallel line are intervals and the objects in the other line are points. We call such an intersection graph a *triangle graph*, since the objects of this graph are triangles. Note that triangle graphs are also called *PI* graphs [3]. To show that triangle graphs are distinct from simple trapezoid graphs, we observe the following property for general trapezoid graphs:

Lemma 3.1 Let v_1, ……, v_6 be six distinct vertices of a trapezoid graph G such that their induced subgraph forms a $K_{3,3}$ and $\{v_1, v_2, v_3\} \sim \{v_4, v_5, v_6\}$. Let $S(T)$ be the middle trapezoid of the three trapezoids corresponding to vertices $\{v_1, v_2, v_3\}$ ($\{v_4, v_5, v_6\}$) in the trapezoid model of G. Then the upper (lower) interval of S is disjoint with the upper (lower) interval of T.

By the previous observation, we are able to construct a simple trapezoid graph that does not permit triangles representation:

Theorem 3.2 The graph G shown in Fig. 4 is a simple trapezoid graph, but not a triangle graph.

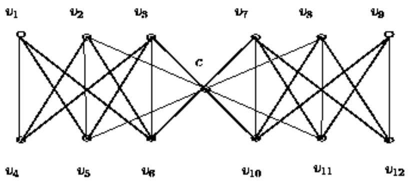

Fig. 4. The graph G is a simple trapezoid but not a triangle graph.

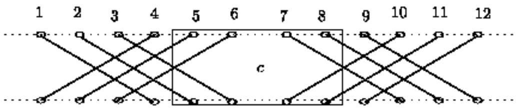

Fig. 5. A simple trapezoid model of the graph G.

Proof. We first show that G is a simple trapezoid graph (and thus a trapezoid graph) by presenting its model in Fig. 5. Now we show that G is not a triangle graph by contradiction. Suppose that G is a triangle graph and let $\triangle_i$ represent the triangle corresponding to vertex v_i for $i \in [1..12]$ while $\triangle_c$ represents the triangle of vertex c. Note that vertices v_1 (or v_4), c, and v_9 (or v_{12}) form an independent set of size 3, and the vertex v_5 is adjacent to vertices v_1 and c but not to v_9 (or v_{12}). We conclude that $\triangle_c$ lies between vertices v_1 (or v_4) and v_9 (or v_{12}). Let $\triangle_i < \triangle_j$ denote that triangle $\triangle_i$ lies on the left hand side of triangle $\triangle_j$. We can now deduce the following relationships between these triangles:

$$\begin{array}{ccc} \Delta_1<\Delta_2<\Delta_3 & & \Delta_7<\Delta_8<\Delta_9 \\ & < & \\ \Delta_4<\Delta_5<\Delta_6 & & \Delta_{10}<\Delta_{11}<\Delta_{12} \end{array}$$

Recall that, by Lemma 3.1, the base of triangle Δ_2 (Δ_8) does not intersect the base of triangle Δ_5 (Δ_{11}). We can then assume that the base of Δ_2 (Δ_8) lies on the right of the base of Δ_5 (Δ_{11}) by symmetry.

This situation is demonstrated by Fig. 6. Note that $\Delta_5 < \Delta_6$ (and $\Delta_7 < \Delta_8$) and $v_1 \sim v_6$ ($v_{12} \sim v_7$). By the same reasoning, the left edge of Δ_{12} lies to the left of Δ_8. In short, since $\Delta_1 < \Delta_c < \Delta_{12}$, the base of triangle Δ_c cannot intersect the bases of triangles Δ_5 or Δ_8. But this is impossible, since that it implies that Δ_c can intersect at most one of Δ_5 or Δ_8, but not both. However, in *G*, we have $v_5 \sim c \sim v_8$. Thus we reach the contradiction, and we conclude that *G* cannot be a triangle graph. ❑

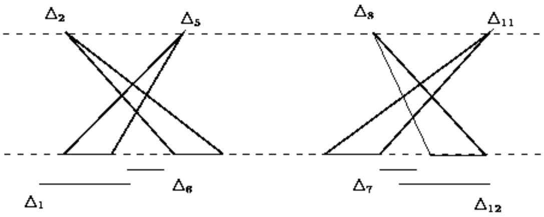

Fig. 6. *G* is not a triangle graph.

Combining with Theorem 2.2, we have:

Corollary 3.3 Triangle graphs and simple trapezoid graphs are two distinct proper subclasses of trapezoid graphs.

We mentioned that Bogart *et al.* [1] showed that bounded tolerance graphs are equivalent to the class of *parallelogram graphs*, which is a subclass of trapezoid graphs with the restriction that all trapezoids are parallelograms. Felsner [5] showed that the parallelogram graph is a proper subclass of the trapezoid graph.

It is easy to see that simple trapezoid graphs are a subclass of parallelogram graphs, because rectangles or line segments are just special forms of parallelograms. It is interesting to known that triangle graphs are also a subclass of parallelogram graphs: Given a triangle graph with its model, note that the top vertices of these triangle graphs are all isolated vertices, i.e., they do not intersect each other. Thus topologically we can transform each vertex into an interval; as long as these transformed intervals do not intersect each other and still maintain their ordering, they give the same intersection graph as the original triangle graph. Since we can transform the vertices into intervals regardless of the length of the intervals and still maintain the intersection graph relationship, we can certainly transform these vertices so that they have the same length as their bottom bases. Note that these transformed triangles become parallelograms, and thus their intersection graphs are parallelogram graphs.

Since simple trapezoid graphs and triangle graphs are both subclasses of parallelogram graphs, and they are distinguishable from each other by Theorem 2.2 and Theorem 3.2, it is clear that parallelogram graphs properly contains these two classes.

Corollary 3.4 Triangle graphs and simple trapezoid graphs are two distinct proper subclasses of parallelogram graphs.

4. CONCLUDING REMARKS

Recall that our original motivation in finding properties of these subclasses of trapezoid graphs is to try to have a better understanding of ways to recognize these graphs. We are still seeking solutions to these recognition problems.

The author would like to thank Steven S. Skiena for the discussions about the properties of the subclasses of parallelogram graphs, and for discussions of the optimization problems of triangle graphs. He suggested the name simple trapezoid graphs to the author. The author thanks Jeremy P. Spinrad and Tze-Heng Ma for valuable discussions about the recognition of two dimensional interval orders and trapezoid graphs. The author would also like to thank Kenneth P. Bogart and Stefan Felsner for discussions of the properties of bounded tolerance graphs and two dimensional interval orders.

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