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Short Paper

Novel Hierarchical Interconnection Networks for High-Performance Multicomputer Systems

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This paper proposes several novel hierarchical interconnection networks based on Heawood graphs, namely, *folded Heawood networks*, *root-folded Heawood networks*, *recursively expanded Heawood networks*, and *flooded Heawood networks*. Like hypercubes and networks extended from Petersen networks, these hierarchical Heawood networks have the following properties: regular topology, high scalability, and small diameter. Due to these important properties, these hierarchical Heawood networks seem to have potential as alternatives for future interconnection structures of multicomputer systems. Furthermore, this paper will demonstrate that the routing and broadcasting algorithms for these proposed networks are as elegant as the algorithms for hypercubes and Petersen-based networks.

Keywords: broadcasting algorithm, routing algorithm, Heawood graph, Heawood networks, folded Heawood networks, root-folded Heawood networks, recursively expanded Heawood networks, flooded Heawood networks

1. INTRODUCTION

Advanced computers employ parallel processing. One major way to achieve parallel processing is to integrate multiple computers through an interconnection network. The entire system performance is then determined not only by the computers but also by the underlying interconnection network. Hence, high performance interconnection networks are essential for multicomputer systems to achieve high performance.

Various high-performance interconnection networks have been extensively studied in the literature, including meshes, hypercubes, twisted hypercubes [1, 5, 6], recursive

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networks [2, 3], and pyramids [14]. Among these networks, the hypercube family has become popular due to the fact that hypercubes have several elegant properties: symmetry, regularity, high fault-tolerance, logarithmetic degree and diameter, self-routing, and simple broadcasting schemes [7, 8]. Nevertheless, new networks are being proposed and analyzed with regards to their applicability and enhanced topological or performance properties. The most popular ones are those based on Petersen graphs and their derivatives. These include folded Petersen cube networks [12], root-folded Petersen networks, recursively expanded Petersen networks [9], and hyper Petersen networks [4]. The major features of Petersen networks are: regular topology, high scalability, small diameter, and lower network cost compared to hypercubes.

This paper presents several new hierarchical interconnection networks based on Heawood graphs [10]: *folded Heawood networks*, *root-folded Heawood networks*, *recursively expanded Heawood networks*, and *flooded Heawood networks*. They all have basically the same features as Petersen networks and, hence, are perfect candidates for use as the interconnection structures of multicomputer systems.

In general, the following major criteria are commonly used to evaluate an interconnection network: diameter, degree, connectivity, and cost [12]. The diameter of a network is the maximum distance among all node-pairs. The degree of a node is the maximum number of links connected to it. The node connectivity (edge connectivity) of a graph is the minimum number of nodes (edges) whose removal results in a disconnected network. The product of degree and diameter is usually called the cost of the network. Consequently, any underlying interconnection networks of multicomputer systems must have the following properties: small diameter, reasonable degree, and low cost. This paper will evaluate these hierarchical Heawood networks based on these properties.

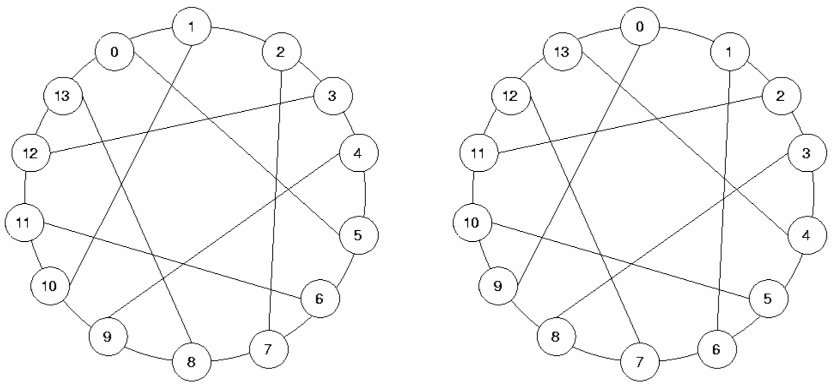
The rest of the paper is organized as follows. Section 2 describes the Heawood network and its addressing schemes. Section 3 extends Heawood networks to several *n*-dimensional hierarchical networks and presents routing and broadcasting algorithms for these hierarchical Heawood-based networks. Section 4 evaluates these hierarchical networks by comparing their topological properties. Section 5 concludes this paper.

2. THE HEAWOOD GRAPH AND NETWORK

This section reviews the definition and important properties of the Heawood graph [10]. Since the interconnecting network of a parallel computer system based on the Heawood graph is called a *Heawood network*, it has the same properties as the Heawood graph. In addition, this section will present routing and broadcasting algorithms for Heawood networks. Please note that the Heawood networks defined in this section will be called the *basic* Heawood network in order to differentiate it from the hierarchical Heawood-based networks described in section 3.

2.1 Definition and Properties

A Heawood graph $H = (V_H, E_H)$ has fourteen nodes with twenty-one links connecting them. Because of the symmetric property of the Heawood graph, the nodes of the graph can be named consecutively counterclockwise or clockwise from any node and starting from 0 in an arbitrary manner, as shown in Fig. 1. For convenience, this paper will use the addressing scheme II shown in Fig. 1 (b) because it seems to be more symmetrical although both structures are isomorphic.



(a) Addressing scheme I. (b) Addressing scheme II.

Fig. 1. The Heawood graph.

Based on node addressing scheme II, the Heawood graph can then be defined as follows.

Definition 1 (Heawood Graph) A Heawood graph has fourteen nodes and twenty-one edges, and is defined as $H = (V_H, E_H)$, where $V_H = \{i \mid 0 \le i < 14\}$ and $E_H = \{(i, j) \mid |i - j| = 1 \bmod 14, \forall 0 \le i, j < 14\} \cup \{(i, j) \mid j = (i + 5) \bmod 14, \forall i < 14$ and i is odd.$\} \cup \{(i, j) \mid j = (i - 5) \bmod 14, \forall i < 14$ and i is even.$\}$

Since the basic Heawood network is constructed based on the Heawood graph, it has the same properties as the Heawood graph:

1. Each node X has three neighboring nodes with addresses: $(X + 1) \bmod 14$, $(X - 1) \bmod 14$, and $(X + 5) \bmod 14$ (if X is odd) or $(X - 5) \bmod 14$ (if X is even).
2. For any pair of nodes, there are three paths for routing a message between them. Consequently, the diameter of the network is three, which is the same as that of the three dimensional hypercube but with more nodes.
3. The minimum length of cycles containing any pair of nodes is 6. That is, the paths between any two nodes have the following properties:
   (a) If the shortest path has a length of 1, then the length of the rest of paths on the cycle is 5.
   (b) If the shortest path has a length of 2, then the length of the rest of paths on the cycle is 4.
   (c) If the shortest path has a length of 3, then the length of the rest of paths on the cycle is 3.

2.2 Basic Routing and Broadcasting Algorithms

Due to the symmetric topology of the basic Heawood network, the routing and broadcasting algorithms for the basic Heawood network can be easily developed. In order to give a more concise presentation of the routing and broadcasting algorithms, the following functions are defined:

$N_{plus}(X)$: return the neighboring node of X with node address $(X + 1) \bmod 14$;
$N_{minus}(X)$: return the neighboring node of X with node address $(X - 1) \bmod 14$;
$N_{distance}(X)$: return the neighboring node of X with node address $(X + 5) \bmod 14$, if X is odd; otherwise return $(X - 5) \bmod 14$;
adjacent(X, Y): return true if X and Y are neighbors; otherwise return false.

Based on these functions, the routing algorithm for the basic Heawood network can be presented as follows:

Algorithm Basic-Routing(S, D, M)
{To route the message M from node S to node D}
begin
  while $D \neq S$ **do**
    if adjacent(S, D) **then** forward M to D; Set $S = D$
    else if adjacent$(D, N_{plus}(S))$ **then** forward M to $N_{plus}(S)$; Set $S = N_{plus}(S)$
    else if adjacent$(D, N_{minus}(S))$ **then** forward M to $N_{minus}(S)$; Set $S = N_{minus}(S)$
    else if adjacent$(D, N_{distance}(S))$ **then** forward M to $N_{distance}(S)$; Set $S = N_{distance}(S)$
    else Set $S = N_{plus}(S)$.
end {Algorithm Basic-Routing}

Since the longest length between any two nodes on the basic Heawood network is 3, the algorithm Basic-Routing takes at most 3 steps to route a message M from the source node S to the destination node D.

Similarly the broadcasting algorithm for the basic Heawood network can be easily designed based on the topological properties the Heawood graph. Broadcasting a message on the basic Heawood network is essentially equivalent to sending a message from the root node (i.e., the source node of the message) of a minimum spanning tree of the basic Heawood network to all other nodes. Following is the broadcasting algorithm for the basic Heawood network.

Algorithm Basic-Broadcasting(S, M)
{To broadcast the message M from the source node S}
begin
1: S copies M to all of its neighboring nodes:
  $N_{p1} = N_{plus}(S)$, $N_{m1} = N_{minus}(S)$, and $N_{d1} = N_{distance}(S)$.
2: Execute the following steps in parallel.
  2.1: N_{p1} copies M to $N_{m2(p1)} = N_{minus}(N_{p1})$ and $N_{d2(p1)} = N_{distance}(N_{p1})$

2.2: N_{m1} copies M to $N_{p2(m1)} = N_{plus}(N_{m1})$ and $N_{d2(m1)} = N_{distance}(N_{m1})$
2.3: N_{d1} copies M to $N_{m2(d1)} = N_{minus}(N_{d1})$ and $N_{p2(d1)} = N_{plus}(N_{d1})$
3: Execute the following steps in parallel.
3.1: $N_{m2(p1)}$ copies M to $N_{p3} = N_{plus}(N_{m2(p1)})$ and $N_{d3} = N_{distance}(N_{m2(p1)})$
3.2: $N_{d2(p1)}$ copies M to $N_{p3} = N_{plus}(N_{p2(p1)})$ and $N_{m3} = N_{minus}(N_{p2(p1)})$
end {Algorithm Basic-Broadcasting}

Fig. 2 shows an example illustrating the process of broadcasting a message from node 0. It is easy to show that both the **Basic-Routing** and **Basic-Broadcasting** algorithms have constant time complexity $O(1)$.

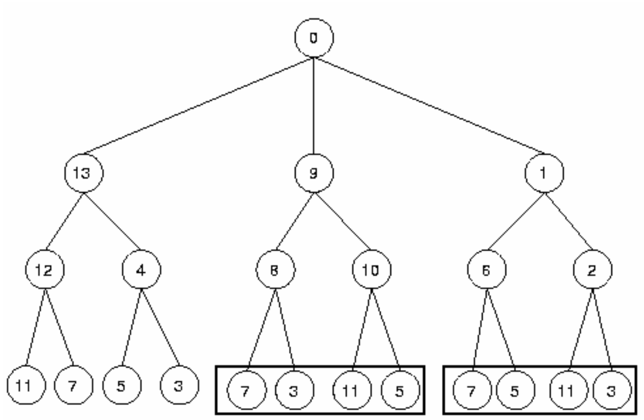


Fig. 2. A minimum spanning tree of the Heawood network.

3. HIERARCHICAL HEAWOOD NETWORKS

This section presents several hierarchical extensions of the basic Heawood network: folded Heawood networks, root-folded Heawood networks, recursively expanded Heawood networks, and flooded Heawood networks.

3.1 Folded Heawood Networks

To generalize the Heawood network into n dimensions, several schemes can be used. Among these, the following one, called the folded Heawood network, is the most popular due to the fact that it possesses the node-symmetric and edge-symmetric properties. It is defined based on the same concept used in the definition of the folded Petersen network [11, 12]. In this section, we will define folded Heawood networks and present their routing and broadcasting algorithms.

3.1.1 Definition and properties

The formal definition of the folded Heawood network is given as follows.

Definition 2 (Folded Heawood Network) An n-dimensional folded Heawood graph is defined as $FH_n = (V_{FH_n}, E_{FH_n})$, where $V_{FH_n} = \{(V_{n-1}, V_{n-2}, \ldots, V_0) \mid V_i \in V_H\}$ and $E_{FH_n} = \{\{(U_{n-1}, U_{n-2}, \ldots, U_i, \ldots, U_0), (V_{n-1}, V_{n-2}, \ldots, V_i, \ldots, V_0)\} \mid U_j = V_j, \forall j \neq i \wedge (U_i, V_i) \in E_H$, for $0 \leq i, j \leq n-1\}$.

A two-dimensional folded Heawood network FH_2 is shown in Fig. 3. There are 14 edges between any neighboring subnetworks. However, the figure only shows in detail the connections between subnetworks FH_1^0 and FH_1^{13}, where FH_{n-1}^i denotes the ith subnetwork of an n-dimensional folded Heawood network FH_n^i. The detailed connections between other subnetworks are left out for the sake of brevity.

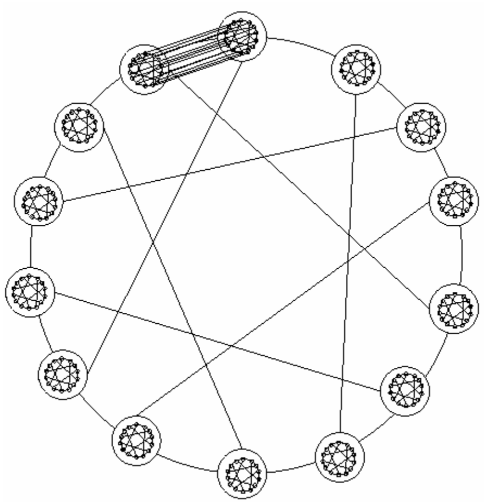


Fig. 3. An example of a two-dimensional folded Heawood network.

3.1.2 Routing and broadcasting algorithms

The routing and broadcasting algorithms for basic Heawood networks described in the previous section can be extended to route and broadcast messages on folded Heawood networks.

Algorithm FH_n-**Routing**(S, D, M)
{The addresses of source and destination nodes S and D are represented as $S \equiv (S_{n-1}S_{n-2} \ldots S_0)$ and $D \equiv (D_{n-1}D_{n-2} \ldots D_0)$, respectively.}
begin
  for $i = n-1$ **to** 0 **step** – 1 **do**
    FH_n-**Basic-Routing**(S_i, D_i, M)
  end {for}
end {Algorithm FH_n-Routing}

The FH_n-Basic-Routing(S_i, D_i, M) algorithm is modified from the Basic-Routing(S, D, M) algorithm presented in the previous section and is shown as follows:

Algorithm FH_n**-Basic-Routing**(S_i, D_i, M)
begin
  while $D_i \neq S_i$ **do**
    if adjacent(S_i, D_i) **then** forward M to D_i; Set $S_i = D_i$
    else if adjacent(D_i, $N_{plus}(S_i)$) **then** forward M to $N_{plus}(S_i)$; Set $S_i = N_{plus}(S_i)$
    else if adjacent(D_i, $N_{minus}(S_i)$) **then** forward M to $N_{minus}(S_i)$; Set $S_i = N_{minus}(S_i)$
    else if adjacent(D_i, $N_{distance}(S_i)$) **then** forward M to $N_{distance}(S_i)$; Set $S_i = N_{distance}(S_i)$
    else Set $S_i = N_{plus}(S_i)$.
end {Algorithm FH_n-Basic-Routing}

It is easy to see that the above algorithm has time complexity $O(n)$, where n is the dimension of the network.

Similarly, the broadcasting algorithm for the folded Heawood networks can be extended from the Basic-Broadcasting algorithm for Heawood networks and is shown as follows:

Algorithm FH_n**-Broadcasting**(S, M)
begin
  for $i = n - 1$ **to** 0 **step** – 1 **do**
    Basic-Broadcasting(S_i, M)
  end {for}
end {Algorithm FH_n-Broadcasting}

Since constant time is required for the **Basic-Broadcasting**(S_i, M) algorithm to execute, the time complexity of the above algorithm is $O(n)$ as well.

3.2 Root-Folded Heawood Networks

Since every node, say FH_{n-1}^0, of an n-dimensional folded Heawood network FH_n is connected to any of its neighboring nodes, say FH_{n-1}^1, with 14 edges, the cost of connections will become high as n grows larger. This section presents a new class of hierarchical Heawood networks, called Root-Folded Heawood Networks (Fig. 4), as a remedy. The main difference is that any two neighboring nodes are now connected by a single edge.

3.2.1 Definition and properties

Definition 3 (Root-Folded Heawood Networks) An n-dimensional Root-Folded Heawood network is defined as $RFH_n = (V_{RFH_n}, E_{RFH_n})$, where $V_{RFH_n} = \{(V_{n-1}, V_{n-2}, \ldots, V_0) \mid V_i \in V_H\}$ and $E_{RFH_n} = \{\{(U_{n-1}, U_{-n-2}, \ldots, U_i, \ldots, U_0), (V_{n-1}, V_{n-2}, \ldots, V_i, \ldots, V_0)\} \mid U_j = V_j = (0, 0), \forall j \neq i \wedge (U_i, V_i) \in E_H$, for $0 \leq i, j \leq n - 1\}$.

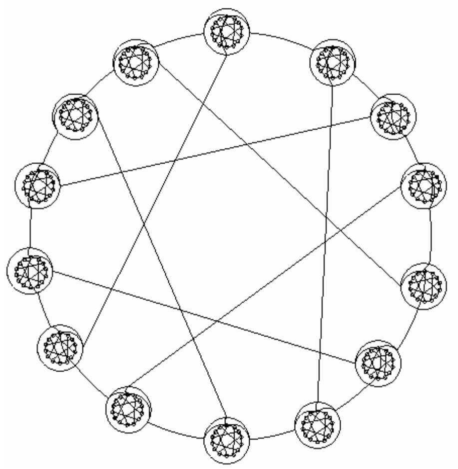


Fig. 4. A two-dimensional root-folded Heawood network.

3.2.2 Routing and broadcasting algorithms

The routing and broadcasting algorithms are listed as follows:

Algorithm RFH_n**-Routing**(S, D, M)
begin
  $i = n - 1$
  while $D_i = S_i$ **do**
    $i = i - 1$
    if $i = -1$ **then break** {S and D are the same node}
    else
      begin
        for $j = 0$ **to** i **do Basic-Routing**(S_i, 0_j, M)
        for $j = i$ **to** 0 **do Basic-Routing**(0_j, D_i, M)
      end {else}
  end {while}
end {Algorithm RFH_n-Routing}

Here, 0_j is defined as $(\overbrace{xxx\cdots x}^{n-j}\overbrace{000\cdots 0}^{j})$. The time complexity of the above routing algorithm for n-dimensional root-folded Heawood networks is $O(n)$.

Algorithm RFH_n**-Broadcasting**(S, M)
begin
  for $i = 0$ **to** $n - 1$ **do Basic-Routing**(S_i, 0_i, M)
  for $i = n - 1$ **to** 0 **do Basic-Broadcasting**(S_i, M)
end {Algorithm RFH_n-Broadcasting}

Since the **Basic-Broadcasting** algorithm takes a constant time to execute, the time complexity of the above algorithm is $O(n)$.

3.3 Recursively Expanded Heawood Networks

The recursive expansion (RE) method has been applied to the Petersen networks [13]. The same method can be applied to the Heawood networks as well, and the new hierarchical networks can be called *recursively expanded Heawood networks* (*RE Heawood networks*). Each n-dimensional RE Heawood network can have up to 14^n ($1 \le n \le 15$) nodes. Furthermore, the degree of each node is 6. Consequently, RE Heawood networks can avoid the bottlenecks caused by the root nodes of root-folded Heawood networks.

3.3.1 Definition and properties

Let the basic Heawood network be H. An n-dimensional RE Heawood network σH_n can be defined by means of the following recursive expansion method:

Definition 4 (Recursively Expanded Heawood Networks)

- Let ${}^{\sigma}H_1 = H$.
- An n-dimensional network ${}^{\sigma}H_n$ is formed by connecting the nodes with the address $n - 2$ of the 14 subnetworks ${}^{\sigma}H_{n-1}^{j}$ ($0 \le j < 14$) to form a Heawood network.

Consequently, every node of an n-dimensional RE Heawood network ${}^{\sigma}H_i$ with the address i ($0 \le i \le n$) has 6 edges, 3 of which are edges of level 1 (i.e., within the same ${}^{\sigma}H_1$), and the other 3 are connections within ${}^{\sigma}H_i$. On the other hand, all the nodes with addresses greater than n have only 3 edges. Fig. 5 shows a 2-dimensional RE Heawood network.

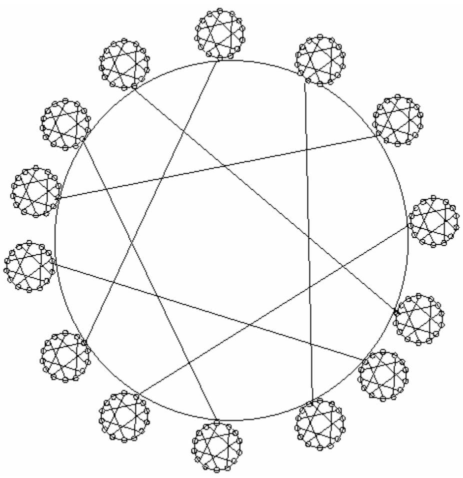


Fig. 5. A two-dimensional RE Heawood network σH_2.

3.3.2 Routing and broadcasting algorithms

The routing algorithm for RE Heawood networks can be adapted from the routing algorithm of the basic Heawood network.

Algorithm σH_n**-Routing**(S, D, M)
{The addresses of source and destination node S and D are represented as $S \equiv (S_{n-1}, S_{n-2}, \ldots, S_1, S_0)$ and $D \equiv (D_{n-1}, D_{n-2}, \ldots, D_1, D_0)$, respectively ($0 \leq n \leq 14$)}
begin
  for $i = n - 1$ **to** 0 **do** σH**-Basic-Routin**g(S_i, D_i, M)
end {Algorithm σH_n-Routing}

Algorithm σH**-Basic-Routing**(S_i, D_i, M)
begin
  while $D_i \neq S_i$ **do**
    begin
      Basic-Routing(S_i, $N_0(i)$, M) [$N_0(i)$: the node at level 0 with links to nodes of σH_i]
      if adjacent(S_i, D_i) **then** forward M to D_i; Set $S_i = D_i$
      else if adjacent(D_i, $N_{plus}(S_i)$) **then** forward M to $N_{plus}(S_i)$; Set $S_i = N_{plus}(S_i)$
      else if adjacent(D_i, $N_{minus}(S_i)$) **then** forward M to $N_{minus}(S_i)$; Set $S_i = N_{minus}(S_i)$
      else if adjacent(D_i, $N_{distance}(S_i)$) **then** forward M to $N_{distance}(S_i)$; Set $S_i = N_{distance}(S_i)$
      else Set $S_i = N_{plus}(S_i)$.
    end {while}
end {Algorithm σH_n-Basic-Routing}

The time complexity of the above routing algorithm for n-dimensional root-folded Heawood networks is $O(n)$.

Similarly, the broadcasting algorithm for an n-dimensional RE Heawood network can be obtained by extending the broadcasting algorithm for basic Heawood networks.

Algorithm σH_n**-Broadcasting**(S, M)
begin
  σH**-Basic-Broadcasting-All**(S_i, k, M) (diameter $k = 6n + 3$)
  parallel do
    begin
      σH_n**-Broadcasting**(**the first neighbor of** S_i, M)
      σH_n**-Broadcasting**(**the second neighbor of** S_i, M)
        .
        .
        .
      σH_n**-Broadcasting**(**the last neighbor of** S_i, M)
    end {parallel do}
end {Algorithm σH_n-Broadcasting}

Algorithm σH-Basic-Broadcasting-All(S_i, k, M)
begin
  if $k = 0$ **then** stop
  else
    begin
      for each neighboring node S_{nei} of S whose flag is not equal to the address of S **do**
        begin
          M is forwarded from S to S_{nei}
          The flag of S_{nei} is set to the address of S
        end {for}
      Return ($k = k - 1$)
    end {else}
end {Algorithm σH_n-Broadcasting-All}

The time complexity of the above algorithm is $O(n)$.

3.4 Recursively Expanded Heawood Networks II

The major disadvantage of the RE Heawood networks is that their reliability depends on the availability of root nodes. In order to improve fault tolerance, a variation of the RE Heawood networks is proposed and called *recursively expanded Heawood networks II* (*RE Heawood networks II*).

3.4.1 Definition and properties

Definition 5 (Recursively Expanded Heawood Networks II)

- Let $\sigma\mathcal{H}_1 = H$.
- An n-dimensional network $\sigma\mathcal{H}_n$ is formed by connecting the nodes with the addresses ($n - 2$) and ((($n - 2$) ± 3) mod 14) of the 14 subnetworks $\sigma\mathcal{H}_{n-1}^j$ ($0 \leq j < 14$) to form a Heawood network.

Fig. 6 shows a 2-dimensional RE Heawood II network $\sigma\mathcal{H}_2$. Compared with the RE Heawood network shown in Fig. 5, every subnetwork $\sigma\mathcal{H}_1^j$ ($0 \leq j < 14$) has three edges.

According to the above definition, each node of a basic Heawood network $\sigma\mathcal{H}_1$ will connect to at most three nodes of other dimensions. Fig. 7 depicts the possible connections of every node of a Heawood network to nodes of different dimensions. Each tuple (i, j, k) represents the possible dimensions to which a node can connect.

3.4.2 Routing and broadcasting algorithms

The routing algorithm for RE Heawood networks can be used to perform routing operations on RE Heawood networks II with the following modification:

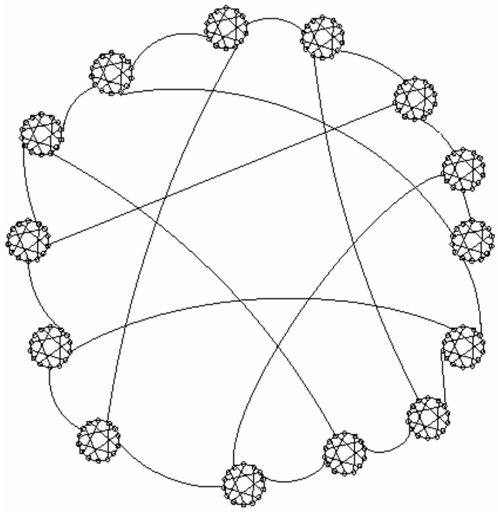


Fig. 6. A two-dimensional RE Heawood network II $\sigma\mathcal{H}_2$.

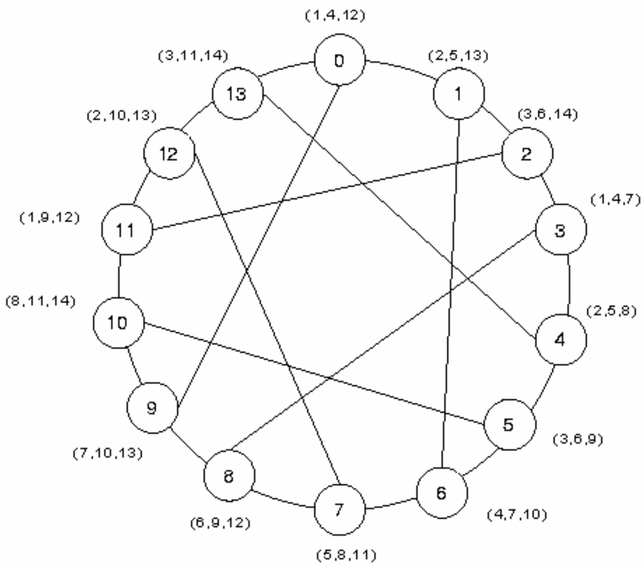


Fig. 7. Dimensions to which each node will connect in an RE Heawood network II.

Algorithm $\sigma\mathcal{H}_n$**-II-Basic-Routing**(S_i, D_i, $\mathcal{M}$)
begin
  while $D_i \neq S_i$ **do**
    Basic-Routing(S_i, $N_0(i)$, $\mathcal{M}$) [$N_0(i)$: one of the three port nodes with links to the nodes of $\sigma\mathcal{H}_n$]
end {Algorithm $\sigma\mathcal{H}_n$-II-Basic-Routing}

The same broadcasting algorithm for RE Heawood networks can be used to perform broadcasting operations on RE Heawood networks II as well.

3.5 Flooded Heawood Networks

A new form of hierarchical Heawood networks, called the *flooded Heawood network*, will be presented in this section. Similar to RE Heawood networks, it is obtained by recursively expanding the basic Heawood network.

3.5.1 Definition and properties

Definition 6 Flooded Heawood Networks

- Let $\sigma FH_1 = H$.
- Each node of σFH_1 is connected with 13 nodes to form a basic Heawood network, and the resulting network is a σFH_2 network.
- Each of the $14 \times 13^{n-1}$ nodes of σFH_n is connected to 13 nodes to form a basic Heawood network, and the resulting entire network is a σFH_{n+1} network.

Fig. 8 depicts a 3-dimensional flooded Heawood network σFH_3. When $n = 1$, σFH_1 is exactly a 1-dimensional RE Heawood network σH_1. However, flooded Heawood networks can be expanded to infinite dimensions.

An n-dimensional flooded Heawood network σFH_n can accommodate a total of 14^n nodes. Every leave node of σFH_n (i.e., nodes at dimension n) is connected to 3 neighboring nodes, while the internal nodes have 6 neighbors each. Therefore, the average degree of an n-dimensional flooded Heawood network will be

$$(3 \times 14^{n-1} + 3 \times 14^n) / 14^n \cong 3.2.$$

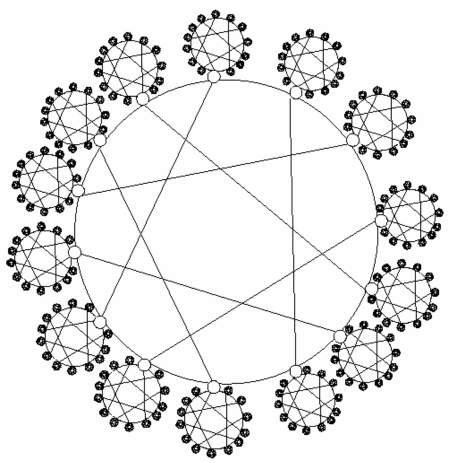


Fig. 8. A three-dimensional flooded Heawood network σFH_3.

3.5.2 Routing and broadcasting algorithms

Algorithm σFH_n**-Routing**(S, D, M)
{The addresses of source and destination nodes S and D are represented as $S \equiv (S_{n-1}S_{n-2} \dots S_0)$ and $D \equiv (D_{n-1}D_{n-2} \dots D_0)$, respectively.}
begin
    for $i = n$ **to** j **do** σFH**-Basic-Routing**(S_i, D_i, M) {where $D_j = S_j$}
    for $i = j$ **to** n **do** σFH**-Basic-Routing**(S_i, D_i, M)
end {Algorithm σFH_n-Routing}

Algorithm σFH**-Basic-Routing**(S_i, D_i, M)
begin
    while $D_i \neq S_i$ do
        begin
            σH**-Basic-Routing**(S_i, $N_0(i)$, M) [$N_0(i)$: the port node from level i to level $i + 1$]
            if adjacent(S_i, D_i) **then** forward M to D_i; Set $S_i = D_i$
            else if adjacent(D_i, $N_{plus}(S_i)$) **then** forward M to $N_{plus}(S_i)$; Set $S_i = N_{plus}(S_i)$
            else if adjacent(D_i, $N_{minus}(S_i)$) **then** forward M to $N_{minus}(S_i)$; Set $S_i = N_{minus}(S_i)$
            else if adjacent(D_i, $N_{distance}(S_i)$) **then** forward M to $N_{distance}(S_i)$; Set $S_i = N_{distance}(S_i)$
            else Set $S_i = N_{plus}(S_i)$.
        end {While}
end {Algorithm σFH_n-Basic-Routing}

Algorithm σFH_n**-Broadcasting**(S_k, M)
begin
    for $i = k$ **to** $n - 1$ **do Basic-Routing**(S_i, 0_i, M)
    for $i = n - 1$ **to** 0 **do Basic-Broadcasting**(S_i, M)
end {Algorithm σFH_n-Broadcasting}

Both σFH_n**-Routing** and σFH_n**-Broadcasting** have time complexity $O(n)$.

4. EVALUATION

This paper uses the following topological properties to compare the hierarchical interconnection networks presented in the previous sections with some popular hierarchical networks: diameter, degree, connectivity, and cost. Fig. 9 lists the topological properties of various hierarchical networks. The maximum number of links on the nodes of Petersen-based and Heawood-based hierarchical networks that are expanded using the same method are the same, but Heawood-based hierarchical networks can have more nodes to accommodate processors for multiprocessor computers. The cost of hypercube, folded Petersen, and folded Heawood networks is $O(n^2)$, which makes them unsuitable to use as interconnection networks underlying large multiprocessor systems, while the remaining hierarchical networks shown in Fig. 9 seem to be good candidates. In addition, the flooded Heawood network is suitable for integrated-circuit implementation due to its

	# of Nodes	Degree	Diameter	Cost
Hypercube	2^n	n	n	n^2
Folded Petersen	$5^n \times 2^n$	$3n$	$2n$	$6n^2$
Root-folded Petersen	$5^n \times 2^n$	3.3	$4n - 2$	$13.2n$
Recursively Expanded Petersen	$5^n \times 2^n$	6	$4n - 2$	$24n$
Folded Heawood	$7^n \times 2^n$	$3n$	$3n$	$9n^2$
Root-Folded Heawood	$7^n \times 2^n$	3.2	$6n - 3$	$19.2n$
Recursively Expanded Heawood	$7^n \times 2^n$	6	$6n - 3$	$36n$
Flooded Heawood	$7^n \times 2^n$	3.2	$6n - 3$	$19.2n$

Fig. 9. Topological properties of various hierarchical networks.

superior topological properties compared to the other networks, that is, low cost, small diameter, regular topology, high scalability, and small number of crossing edges. It is worth noting that the topological structure of flooded Heawood networks can be applied to hierarchical Petersen networks to achieve lower cost, a smaller diameter, and an even smaller number of crossing edges. Without loss of generality, it is assumed that the numbers of nodes in these hierarchical Heawood networks are powers of 14. The larger power can be considered if the network size is in between two powers of 14 without losing these properties.

5. CONCLUSIONS

This paper has presented several hierarchical interconnection networks, derived from the Heawood graph, for high-performance multicomputer systems. All the Heawood-based hierarchical networks presented in this paper have the following nice properties: regular topology, high scalability, and small diameter. Furthermore, this paper has also demonstrated that the routing and broadcasting algorithms for these hierarchical networks are as elegant as those for Petersen-based networks and hypercubes.

REFERENCES

1. S. Abraham and K. Padmanabhan, "An analysis of the twisted cube topology," in *Proceedings of International Conference on Parallel Processing*, Vol. 1, 1989, pp. 116-120.
2. S. K. Das, "Designing recursive networks combinatorially," in *Proceedings of International Conference on Graph Theory, Combinatorics, and Computing*, 1991.
3. S. K. Das and A. Mao, "Embeddings in recursive combinatorial networks," in *Proceedings of Workshop on Graph-Theoretic Concepts in Computer Science*, 1992, pp. 184-204.
4. S. K. Das, S. Öhring, and A. K. Banerjee, "Embeddings into hyper Petersen networks: Yet another hypercube-like interconnection topology," *Journal of VLSI, Special Issue on Interconnection Networks*, Vol. 2, 1995, pp. 335-351.
5. A. H. Esfahanian, L. M. Ni, and B. E. Sagan, "On enhancing hypercube multiproc-

essors," in *Proceedings of International Conference on Parallel Processing*, 1988, pp. 86-89.
6. P. A. J. Hilbers, M. R. J. Koopman, and J. L. A. van de Snepacheut, "The twisted cube," in *Proceedings of Parallel Architectures and Algorithms Europe*, 1987, pp. 152-159.
7. K. Hwang, *Advanced Computer Architecture: Parallelism, Scalability, Programmability*, McGraw-Hill, New York, 1993.
8. V. Kumar, A. Grama, A. Gupta, and G. Karypis, *Introduction to Parallel Computing: Design and Analysis of Algorithms*, The Benjamin/Cummings Publishing Company, Inc., Redwood City, California, 1994.
9. M. B. Lin and G. E. Jan, "Routing and broadcasting algorithms for the root-folded Petersen networks," *Journal of Marine Science and Technology*, Vol. 6, 1998, pp. 65-70.
10. J. A. McHugh, *Algorithmic Graph Theory*, Englewood Cliffs: Prentice-Hall, New Jersey, 1990.
11. S. Öhring and S. K. Das, "The folded Petersen network: a new communication-efficient multiprocessor topology," in *Proceedings of 1993 International Conference on Parallel Processing*, Vol. I, 1993, pp. 311-314.
12. S. Öhring and S. K. Das, "Folded Petersen cube networks: new competitors for the hypercubes," *IEEE Transactions on Parallel and Distributed Systems,* Vol. 7, 1996, pp. 151-168.
13. H. Shen, "A high performance interconnection network for multiprocessor," *Parallel Computing*, 1992, pp. 993-1001.
14. L. Uhr, *Multicomputer Architecture for Artificial Intelligence*, Wiley Interscience, New York, 1987.

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