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Fuzzy Inventory with Backorder Defuzzification by Signed Distance Method

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In this paper, we consider fuzzy inventory with backorder. First, we fuzzify the storing cost a, backorder cost b, cost of placing an order c, total demand r, order quantity q, and shortage quantity s as the triangular fuzzy numbers in the total cost. From these, we can obtain the fuzzy total cost. Using the signed distance method to defuzzify, we get the estimate of the total cost in the fuzzy sense. Two special cases of the optimal solutions on fuzzifying the storage quantity and order quantity as triangular fuzzy numbers will be treated numerically by the Nedler-Mead algorithm.

Keywords: fuzzy inventory, fuzzy sets, fuzzy total cost, signed distance, extension principle

1. INTRODUCTION

In crisp inventory models, all the parameters in the total cost are known and have definite values without ambiguity; in addition the real variable of the total cost is positive. But, in reality, it is not so certain. Hence there is a need to consider the fuzzy inventory models. Due to the various fuzzy cases, one may consider different fuzzy inventory models as follows. Yao *et al*. [2, 7, 13-16] discussed fuzzy inventory with and without backorder models. Papers [2, 13, 15, 16] related to this paper treated fuzzy inventory with backorder. In [2], they fuzzified the shortage quantity s as a triangular fuzzy number in the total cost of inventory with backorder and kept the order quantity q as a crisp real variable. In this way, they obtained a fuzzy total cost. In [13, 15], they fuzzified the order quantity q as triangular fuzzy numbers and trapezoid fuzzy numbers, and kept the shortage quantity s as a crisp real variable in the total cost of inventory with backorder. In [2, 13, 15] the authors used the extension principle to find the membership functions of the fuzzy total cost. Then they defuzzified by the centroid to find the estimate of the total cost in the fuzzy sense. Such methods are very difficult and complicated. In [16], they fuzzified the total demand quantity to an interval-valued fuzzy set in the total cost of inventory with backorder. Then they used the extension principle to find the membership

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function and defuzzified to get an estimate of the total cost in the fuzzy sense. Papers [7, 13, 14] discussed fuzzy inventory without backorder. In [7, 13], they fuzzified order quantity q in the total cost of inventory without backorder to a triangular fuzzy number and trapezoid fuzzy number to get the fuzzy total cost. Then they used the extension principle to find their membership function. In [14], they fuzzified order quantity q and total demand quantity r in the total cost of inventory without backorder to triangular fuzzy numbers. In this way, they could compute a fuzzy total cost. Similarly, they used the extension principle to find their membership function. All the articles [2, 7, 13-16] used the extension principle and centroid to find the estimate of the total cost in the fuzzy sense. This treatment is difficult and very complicated. Petrovic and Sweeney [9] fuzzified the demand, lead time and inventory level into triangular fuzzy numbers in an inventory control model, then decided the order quantity by the method of fuzzy proposition. Vujosevic *et al*. [12] fuzzified the ordering cost into a trapezoidal fuzzy number in the total cost of an inventory without backorder model and obtained the fuzzy total cost. They did the defuzzification by using centroid and obtained the total cost in the fuzzy sense. Chen *et al*. [3] fuzzified the order cost, inventory cost, and backorder cost into trapezoidal fuzzy numbers and used the functional principle to obtain the estimate of the total cost in the fuzzy sense. Roy and Maiti [11] rewrote the problem of classic economic order quantity into a form of nonlinear programming problem, and introduced fuzziness both in the objective function and storage area. It was solved by fuzzifying both nonlinear and geometric programming techniques for linear membership functions. Ishii and Konno [4] fuzzified the shortage cost L to a fuzzy number in a classical newsboy problem aimed at finding an optimal ordering quantity in the sense of fuzzy ordering.

In this article we do not use the extension principle, centroid or other methods in the above papers. Instead, we will use the signed distance method to defuzzify the fuzzy total cost and obtain an estimate of the total cost in the fuzzy sense. We will discuss signed distance in section 2. Section 3 consists of three subsections. In section 3.1, we will fuzzify q, s, r, a, b, and c in the total cost to triangular fuzzy numbers, and get the fuzzy total cost of inventory with backorder. Through defuzzification by signed distance, we have the estimate of the total cost in the fuzzy sense. We fuzzify s to a triangular fuzzy number in the total cost and defuzzify by signed distance in section 3.2. In section 3.3, we fuzzify q as a triangular fuzzy number and defuzzify by the signed distance method in the total cost. We give an example in section 5 and make comparisons with [2, 15].

2. PRELIMINARIES

For the fuzzy total cost in inventory with backorder based on the signed distance method, all pertinent definitions of fuzzy sets are given below.

Definition 2.1 *Fuzzy Point* (Definition 2.1 in Pu and Liu [10]): Let $\tilde{a}$ be a fuzzy set on $R = (-\infty, \infty)$. It is called a fuzzy point if its membership function is

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{if } x \neq a \end{cases}. \tag{1}$$

Definition 2.2 *Level α Fuzzy Interval*: Let $[a, b; \alpha]$ be a fuzzy set on $R = (-\infty, \infty)$. It is called a level α fuzzy interval, $0 \le \alpha \le 1$, $a < b$, if its membership function is

$$\mu_{[a,\, b;\, \alpha]}(x) = \begin{cases} \alpha, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

Definition 2.3 *Triangular Fuzzy Numbers*: Let $\tilde{A} = (p, q, r)$, $p < q < r$, be a fuzzy set on $R = (-\infty, \infty)$. It is called a triangular fuzzy number, if its membership function is

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-p}{q-p}, & \text{if } p \le x \le q \\ \dfrac{r-x}{r-q}, & \text{if } q \le x \le r \\ 0, & \text{otherwise} \end{cases}. \tag{3}$$

If $p = q = r$ then $\tilde{P} = (p, p, p)$.

Let $\tilde{D}$ be a fuzzy number on R. Denote by $D(\alpha) = [D_L(\alpha), D_R(\alpha)]$ the α-cut of $\tilde{D}$, where $0 \le \alpha \le 1$. $D_L(\alpha)$ and $D_R(\alpha)$ are the left and right hand side of $D(\alpha)$; they exist and are integrable for $\alpha \in [0, 1]$. In addition, we let F be the family of all these fuzzy numbers $\tilde{D}$ on R.

As in Yao and Wu [17], we consider the definition of the signed distance on F.

Definition 2.4 *The Signed Distance*: We define $d_0(a, 0) = a$, for $a, 0 \in R$.

Remark 2.1: The meaning of Definition 2.4 is as the follows, if $0 < a$ then the distance between a and 0 is $d_0(a, 0) = a$. If $a < 0$ then the distance between a and 0 is $-d_0(a, 0) = -a$. Therefore, we call $d_0(a, 0) = a$ is the signed distance between a and 0.

For $\tilde{D} \in F$, from Definition 2.4, we have that the signed distance of $D_L(\alpha)$ and $D_R(\alpha)$ measured from 0 are $d_0(D_L(\alpha), 0) = D_L(\alpha)$ and $d_0(D_R(\alpha), 0) = D_R(\alpha)$, respectively. Therefore, we may define the signed distance of the interval $[D_L(\alpha), D_R(\alpha)]$, which is measured from the origin 0, by $d_0[[D_L(\alpha), D_R(\alpha)], 0] = \frac{1}{2}[d_0(D_L(\alpha), 0) + d_0(D_R(\alpha), 0)] = \frac{1}{2}[D_L(\alpha) + D_R(\alpha)]$.

For each $\alpha \in [0, 1]$, the crisp interval $[D_L(\alpha), D_R(\alpha)]$ and the level α fuzzy interval $[D_L(\alpha), D_R(\alpha); \alpha]$ are in one to one correspondence. Therefore, we may define the signed distance from $[D_L(\alpha), D_R(\alpha); \alpha]$ to $\tilde{0}$ as $d([D_L(\alpha), D_R(\alpha); \alpha], \tilde{0}) = d_0([D_L(\alpha), D_R(\alpha)], 0) = \frac{1}{2}[D_L(\alpha) + D_R(\alpha)]$.

Since $\tilde{D} \in F$, $D_L(\alpha)$ and $D_R(\alpha)$ exist and are integrable for $\alpha \in [0, 1]$, we have the following definition.

Definition 2.5 Let $\tilde{D} \in F$. We define the signed distance of $\tilde{D}$ measured from $\tilde{0}$ as

$$d(\tilde{D},\ \tilde{0}) = \frac{1}{2}\int_0^1 [D_L(\alpha) + D_R(\alpha)]d\alpha.$$

We have the ordering definition of F as follows:

Definition 2.6 Let $\tilde{D}, \tilde{E} \in F$. The ordering of $\tilde{D}, \tilde{E}$ is as follows

$$\tilde{D} \prec \tilde{E} \text{ iff } d(\tilde{D},\ \tilde{0}) < d(\tilde{E},\ \tilde{0}),$$
$$\tilde{D} \approx \tilde{E} \text{ iff } d(\tilde{D},\ \tilde{0}) = d(\tilde{E},\ \tilde{0}).$$

Using Definitions 2.5 and 2.6, and the properties of ordering relation <, = on R, we have the following Proposition.

Proposition 1:
(a) $\tilde{D}, \tilde{E} \in F$, then one and only one of the following, $\tilde{D} \prec \tilde{E}$, $\tilde{D} \approx \tilde{E}$, or $\tilde{E} \prec \tilde{D}$, is true.
(b) $\tilde{C}, \tilde{D}, \tilde{E} \in F$, then the following three axioms of the ordering relations are true.
(i) $\tilde{C} \prec\approx \tilde{C}$.
(ii) $\tilde{C} \prec\approx \tilde{D}$ and $\tilde{D} \prec\approx \tilde{C}$, then $\tilde{C} \approx \tilde{D}$.
(iii) $\tilde{C} \prec\approx \tilde{D}$ and $\tilde{D} \prec\approx \tilde{E}$, then $\tilde{C} \prec\approx \tilde{E}$.

From Proposition 1, we have that $\prec$, $\approx$ are the linear order on F.

Remark 2.2: If $\tilde{C} = (p, q, r)$, then the left endpoint and the right endpoint of the α-cut of $\tilde{C}$ are $C_L(\alpha) = p + (q - p)\alpha$ and $C_R(\alpha) = r - (r - q)\alpha$, respectively. The centroid of $\tilde{C}$ is $C(\tilde{C}) = \frac{1}{3}(p + q + r)$, and the signed distance of $\tilde{C}$ is $d(\tilde{C},\ \tilde{0}) = \frac{1}{4}(2q + p + r)$. The mid-point of the interval $[p, r]$ is $M' = \frac{1}{2}(p + r)$.

$$C(\tilde{C}) - d(\tilde{C},\ \tilde{0}) = \frac{1}{6}(M' - q).$$

$$d(\tilde{C},\ \tilde{0}) - q = \frac{1}{2}(M' - q).$$

$$M' - C(\tilde{C}) = \frac{1}{3}(M' - q).$$

(a) If $M' > q$, then $p < q < d(\tilde{C},\ \tilde{0}) < C(\tilde{C}) < M' < r$.
(b) If $M' < q$, then $p < M' < C(\tilde{C}) < d(\tilde{C},\ \tilde{0}) < q < r$.
(c) If $M' = q$, then $p < M' = C(\tilde{C}) = d(\tilde{C},\ \tilde{0}) = q < r$.

From (a) and (b), we know that $d(\tilde{C},\ \tilde{0})$ is near q, and $C(\tilde{C})$ is near M'. From Figs. 1 and 2, we know that the membership grade of $\tilde{C}$ at q is 1. The membership grade of $\tilde{C}$ at $d(\tilde{C},\ \tilde{0})$ is greater than that at $C(\tilde{C})$. Then, we have $\mu_{\tilde{C}}(d(\tilde{C},\ \tilde{0})) > \mu_{\tilde{C}}(C(\tilde{C}))$. Therefore, from the membership grade viewpoint, it is better for us to defuzzify the fuzzy number $\tilde{C}$ by $d(\tilde{C},\ \tilde{0})$ than by $C(\tilde{C})$.

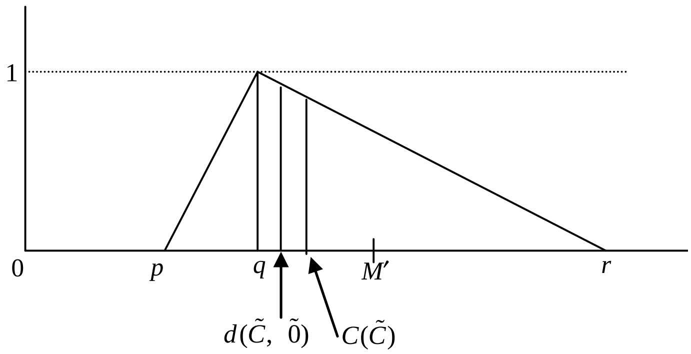

Fig. 1. Case $M' > q$.

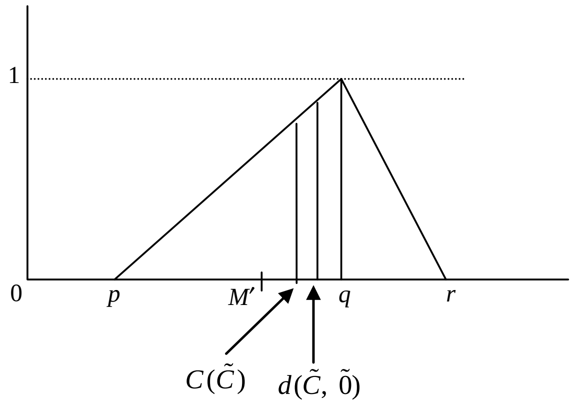

Fig. 2. Case $M' < q$.

Let $\tilde{D}, \tilde{E} \in F$. For $\alpha \in [0, 1]$, we have four operations $\oplus, \ominus, \otimes, \oslash$ for the level α fuzzy intervals. For more details, please refer to [6].

3. DEFUZZIFICATION BY SIGNED DISTANCE METHOD

For a crisp inventory model with backorder, we use the following notations and related parameters.

Fig. 3 illustrates the role of all of the parameters where

T is the length of plan (days).
a is the storing cost for one unit per day.
b is the backorder cost for one unit per day.
c is the cost of placing an order.
r is the total demand over the planning time period $[0, T]$.
t_q is length of a cycle.
q is the order quantity per cycle.

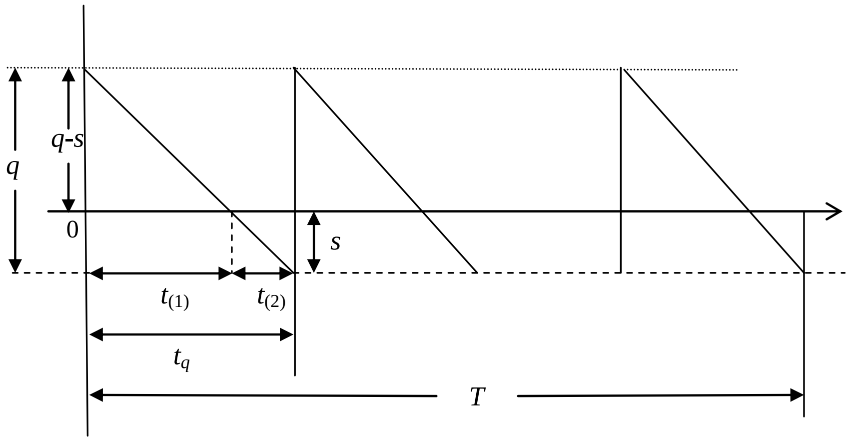

Fig. 3. Inventory with backorder.

s is the shortage quantity per cycle.

Then, we have $\frac{q-s}{t_{(1)}} = \frac{q}{t_q} = \frac{s}{t_{(2)}} = \frac{r}{T}$.

The crisp total cost on the planning period [0, T] is given by

$$F(q,\ s) = [at_{(1)}\frac{q-s}{2} + bt_{(2)}\frac{s}{2} + c]\frac{r}{q}$$

$$= \frac{a(q-s)^2T}{2q} + \frac{bs^2T}{2q} + \frac{cr}{q},\ (0 < s < q). \tag{4}$$

The crisp optimal solutions are

optimal order quantity $q_* = \sqrt{\frac{2(a+b)cr}{abT}}$.

optimal backorder quantity $s_* = \sqrt{\frac{2acr}{b(a+b)T}}$.

minimal total cost $F(q_*,\ s_*) = \sqrt{\frac{2abcrT}{a+b}}$.

3.1 Fuzzifying *q*, *s*, *r*, *a*, *b*, and *c* to Triangular Fuzzy Numbers in the Total Cost

Under the condition that the period of the ordering and arriving of the commodities per cycle is the same, we can find the total cost Eq. (4). In practical situations, it will fluctuate a little. It will influence the ordering quantity q and shortage quantity s. Therefore, we consider fuzzifying q and s to the following fuzzy numbers

$$\tilde{q} = (q_1,\ q,\ q_2), \tag{5}$$

$$\tilde{s} = (s_1,\ s,\ s_2), \tag{6}$$

where $0 < s_1 < s < s_2 < q_1 < q < q_2$ and s_1, s, s_2, q_1, q, and q_2 are unknown positive numbers.

Remark 3.1: Under the condition $0 < s < q$ in Eq. (4), we may fuzzify as follows: Since $0 < s_1 < s < s_2 < q_1 < q < q_2$, we have $d(\tilde{0},\ \tilde{0}) = 0 < d(\tilde{s},\ \tilde{0}) = \frac{1}{3}\ (s_1 + s + s_2) < d(\tilde{q},\ \tilde{0}) = \frac{1}{3}(q_1 + q + q_2)$. Then, we have $d(\tilde{0},\ \tilde{0}) < d(\tilde{s},\ \tilde{0}) < d(\tilde{q},\ \tilde{0})$. By definition 2.6, we have $\tilde{0} \prec \tilde{s} \prec \tilde{q}$.

It is difficult to determine a fixed value r for the total demand in the plan period. On the contrary, it is easier to set the total demand in the interval $[r - \Delta_1, r + \Delta_2]$, where $0 < \Delta_1 < r$, $0 < \Delta_2$ and Δ_1, Δ_2 are determined by the decision maker. Let r be a known number. The decision maker wants to choose a suitable value in the interval $[r - \Delta_1, r + \Delta_2]$ as an appropriate estimate of the total demand. If it happens that the value coincides with r, then the error is 0. If the value deviates from r farther from both sides of r, the error is bigger. The error will attain a maximum at the two endpoints $r - \Delta_1$, and $r + \Delta_2$. From the fuzzy point of view, we can transform the error to a confidence level. If the error is zero, then the confidence level is 1. The farther the value is from both sides of r, the less the confidence level is. At the two endpoints $r - \Delta_1$, and $r + \Delta_2$, the confidence level will be the minimum. We set them to zero.

From the arguments above and below, corresponding to the interval $[r - \Delta_1, r + \Delta_2]$, we set the following fuzzy numbers

$$\tilde{r} = (r - \Delta_1, r, r + \Delta_2),\ 0 < \Delta_1 < r,\ 0 < \Delta_2. \tag{7}$$

The membership grade of r in $\tilde{r}$ is 1. The farther the point in $[r - \Delta_1, r + \Delta_2]$ is from both sides of r, the less the membership grade is. The membership grade shares the same property with the confidence level. If we make a correspondence between membership grade and confidence level, it is reasonable to set a fuzzy number in Eq. (7) corresponding to the interval $[r - \Delta_1, r + \Delta_2]$. In a perfectly competitive market, it will be perturbed a little for the inventory cost per unit per day in the plan period. We can set the inventory cost per unit to lie in the interval $[a - \Delta_3, a + \Delta_4]$ where $0 < \Delta_3 < a$, and $0 < \Delta_4$. With the same arguments as above, corresponding to the interval $[a - \Delta_3, a + \Delta_4]$, we set the following fuzzy number

$$\tilde{a} = (a - \Delta_3, a, a + \Delta_4),\ 0 < \Delta_3 < a,\ 0 < \Delta_4,\ a \text{ is known}. \tag{8}$$

Similarly, let the backorder cost per unit per day lie in the interval $[b - \Delta_5, b + \Delta_6]$. Corresponding to the interval $[b - \Delta_5, b + \Delta_6]$, we set the following fuzzy number

$$\tilde{b} = (b - \Delta_5, b, b + \Delta_6),\ 0 < \Delta_5 < b,\ 0 < \Delta_6, \text{ and } b \text{ is known}. \tag{9}$$

Let the ordering cost each time lie in the interval $[c - \Delta_7, c + \Delta_8]$. Corresponding to the interval $[c - \Delta_7, c + \Delta_8]$, we set the following fuzzy number

$$\tilde{c} = (c - \Delta_7, c, c + \Delta_8),\ 0 < \Delta_7 < c,\ 0 < \Delta_8, \text{ and } c \text{ is known.} \tag{10}$$

Δ_j, $j = 1, 2, \ldots, 8$ will be decided appropriately by the decision maker.

From Eq. (4), let

$$K(q, s, r, a, b, c) = F(q, s) = \frac{T \cdot a \cdot q}{2} - T \cdot a \cdot s + \frac{T \cdot a \cdot s^2}{2q} + \frac{T \cdot b \cdot s^2}{2q} + \frac{c \cdot r}{q}. \tag{11}$$

We fuzzify q, s, r, a, b, and c in Eq. (11) to Eqs. (5) - (10), then we have the fuzzy total cost Eq. (12).

Let $\tilde{T} = (T, T, T)$, $\left(\frac{\tilde{T}}{2}\right) = \left(\frac{T}{2}, \frac{T}{2}, \frac{T}{2}\right)$ be the fuzzy points, and $\tilde{Q}_1 = \left(\frac{\tilde{T}}{2}\right) \otimes \tilde{a} \otimes \tilde{q}$, $\tilde{Q}_2 = \tilde{T} \otimes \tilde{a} \otimes \tilde{s}$, $\tilde{Q}_3 = \left(\frac{\tilde{T}}{2}\right) \otimes \tilde{a} \otimes (\tilde{s} \otimes \tilde{s}) \oplus\!\!\!\!\!\!\,\div\, \tilde{q}$, $\tilde{Q}_4 = \left(\frac{\tilde{T}}{2}\right) \otimes \tilde{b} \otimes (\tilde{s} \otimes \tilde{s}) \,⨸\, \tilde{q}$, $\tilde{Q}_5 = \tilde{c} \otimes \tilde{r} \otimes \tilde{s} \,⨸\, \tilde{q}$. Then we have the fuzzy total cost

$$K(\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c}) = \tilde{Q}_1 \ominus \tilde{Q}_2 \oplus \tilde{Q}_3 \oplus \tilde{Q}_4 \oplus \tilde{Q}_5. \tag{12}$$

If we use the extension principle to find the membership functions of the fuzzy total cost (Eq. (12)), and defuzzify by the centroid to find the estimate of the total cost in the fuzzy sense, then it will be difficult and complicated to derive the membership functions and centroid. In this article we don't use the extension principle or centroid. Instead, we will use the signed distance method (by Definition 2.5) to defuzzify the fuzzy total cost and obtain an estimate of the total cost in the fuzzy sense. The procedure is as follows:

The left and right hand side of the α-cut, $(0 \le \alpha \le 1)$, of $\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c}$ are

$$\left.\begin{aligned}
&q_L(\alpha) = q_1 + (q - q_1)\alpha > 0,\ q_R(\alpha) = q_2 - (q_2 - q)\alpha = q_2(1 - \alpha) + q\alpha > 0 \\
&s_L(\alpha) = s_1 + (s - s_1)\alpha > 0,\ s_R(\alpha) = s_2 - (s_2 - s)\alpha > 0 \\
&r_L(\alpha) = r - \Delta_1 + \alpha\Delta_1 > 0,\ r_R(\alpha) = r + \Delta_2 - \alpha\Delta_2 > 0 \\
&a_L(\alpha) = a - \Delta_3 + \alpha\Delta_3 > 0,\ a_R(\alpha) = a + \Delta_4 - \alpha\Delta_4 > 0 \\
&b_L(\alpha) = b - \Delta_5 + \alpha\Delta_5 > 0,\ b_R(\alpha) = b + \Delta_6 - \alpha\Delta_6 > 0 \\
&c_L(\alpha) = c - \Delta_7 + \alpha\Delta_7 > 0,\ c_R(\alpha) = c + \Delta_8 - \alpha\Delta_8 > 0
\end{aligned}\right\}. \tag{13}$$

From Eq. (13), we have the left and right hand side of the α-cut, $(0 \le \alpha \le 1)$, of $\tilde{Q}_j$, $j = 1, 2, \ldots, 5$ are

$$\left.\begin{aligned}
&Q_{1L}(\alpha) = \frac{T}{2} a_L(\alpha) q_L(\alpha),\ Q_{1R}(\alpha) = \frac{T}{2} a_R(\alpha) q_R(\alpha) \\
&Q_{2L}(\alpha) = T a_L(\alpha) s_L(\alpha),\ Q_{2R}(\alpha) = T a_R(\alpha) s_R(\alpha) \\
&Q_{3L}(\alpha) = \frac{T}{2} a_L(\alpha) s_L(\alpha)^2 / q_R(\alpha),\ Q_{3R}(\alpha) = \frac{T}{2} a_R(\alpha) s_R(\alpha)^2 / q_L(\alpha) \\
&Q_{4L}(\alpha) = \frac{T}{2} b_L(\alpha) s_L(\alpha)^2 / q_R(\alpha),\ Q_{4R}(\alpha) = \frac{T}{2} b_R(\alpha) s_R(\alpha)^2 / q_L(\alpha) \\
&Q_{5L}(\alpha) = c_L(\alpha) r_L(\alpha) / q_R(\alpha),\ Q_{5R}(\alpha) = c_R(\alpha) r_R(\alpha) / q_L(\alpha)
\end{aligned}\right\}. \tag{14}$$

The left and right hand side of the α-cut, $(0 \le \alpha \le 1)$, of $K(\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c})$ are

$$K(\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c})_L(\alpha) = Q_{1L}(\alpha) - Q_{2R}(\alpha) + Q_{3L}(\alpha) + Q_{4L}(\alpha) + Q_{5L}(\alpha),$$

$$K(\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c})_R(\alpha) = Q_{1R}(\alpha) - Q_{2L}(\alpha) + Q_{3R}(\alpha) + Q_{4R}(\alpha) + Q_{5R}(\alpha). \tag{15}$$

Trivially, we have $K(\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c}) \in F$. By Definition 2.5

$$d(K(\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c}), \tilde{0}) = \frac{1}{2}\int_0^1 [K(\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c})_L(\alpha) + K(\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c})_R(\alpha)]d\alpha$$

$$\equiv FC(q_1, q, q_2, s_1, s, s_2; \Delta_j, j = 1, 2, \ldots, 8). \tag{16}$$

Let

$$G(e, f, g, h, k, w) = \int_0^1 \frac{e\alpha^3 + f\alpha^2 + g\alpha + h}{k\alpha + w} d\alpha$$

$$= \frac{e}{3k} + \frac{fk - ew}{2k^2} + \frac{gk^2 - fkw + ew^2}{k^3} + \frac{hk^3 - gk^2w + fkw^2 - ew^3}{k^4} \ln\left|\frac{k+w}{w}\right|. \tag{17}$$

$$H(w, v, u, t, p) = \int_0^1 \frac{w\alpha^2 + v\alpha + u}{t\alpha + p} d\alpha$$

$$= \frac{w}{2t} + \frac{vt - wp}{t^2} + \frac{ut^2 - vtp + wp^2}{t^3} \ln\left|\frac{t+p}{p}\right|. \tag{18}$$

From Eqs. (5) to (18), we have the following Proposition.

Proposition 2: If we fuzzify the q, s, r, a, b, and c in the crisp inventory with backorder in Eq. (11) as the fuzzy numbers $\tilde{q}$ (in Eq. (5)), $\tilde{s}$ (in Eq. (6)), $\tilde{r}$ (in Eq. (7)), $\tilde{a}$ (in Eq. (8)), $\tilde{b}$ (in Eq. (9)), and $\tilde{c}$ (in Eq. (10)), then we can obtain the estimate of the total cost in the fuzzy sense as

$$FC(q_1, q, q_2, s_1, s, s_2; \Delta_j, j = 1, 2, \ldots, 8)$$

$$= \frac{T}{24}[3a(q_1 + 2q + q_2) + (\Delta_4 - \Delta_3)q + 2\Delta_4 q_2 - 2\Delta_3 q_1] - \frac{T}{12}[3a(s_1 + 2s + s_2) + (\Delta_4 - \Delta_3)s + 2\Delta_4 s_2 - 2\Delta_3 s_1] + \frac{T}{4}G(e_1, f_1, g_1, h_1, k_1, w_1) + \frac{T}{4}G(e_2, f_2, g_2, h_2, k_2, w_2) + \frac{T}{4}G(e_3, f_3, g_3, h_3, k_3, w_3) + \frac{T}{4}G(e_4, f_4, g_4, h_4, k_4, w_4) + \frac{1}{2}H(w_1, v_1, u_1, t_1, p_1) + \frac{1}{2}H(w_2, v_2, u_2, t_2, p_2),$$

where

$e_1 = \Delta_3(s - s_1)^2$, $f_1 = (a - \Delta_3)(s - s_1)^2 + 2\Delta_3 s_1(s - s_1)$,
$g_1 = \Delta_3 s_1^2 + 2(a - \Delta_3)s_1(s - s_1)$, $h_1 = (a - \Delta_3)s_1^2$, $k_1 = - q_2 + q$, $w_1 = q_2$,
$e_2 = - \Delta_4(s_2 - s)^2$, $f_2 = (a + \Delta_4)(s_2 - s)^2 + 2\Delta_4 s_2(s_2 - s)$,
$g_2 = - \Delta_4 s_2^2 - 2(a + \Delta_4)s_2(s_2 - s)$, $h_2 = (a + \Delta_4)s_2^2$, $k_2 = q - q_1$, $w_2 = q_1$,
$e_3 = \Delta_5(s - s_1)^2$, $f_3 = (b - \Delta_5)(s - s_1)^2 + 2\Delta_5 s_1(s - s_1)$,
$g_3 = \Delta_5 s_1^2 + 2(b - \Delta_5)s_1(s - s_1)$, $h_3 = (b - \Delta_5)s_1^2$, $k_3 = - q_2 + q$, $w_3 = q_2$,
$e_4 = - \Delta_6(s_2 - s)^2$, $f_4 = (b + \Delta_6)(s_2 - s)^2 + 2\Delta_6 s_2(s_2 - s)$,
$g_4 = - \Delta_6 s_2^2 - 2(b + \Delta_6)s_2(s_2 - s)$, $h_4 = (b + \Delta_6)s_2^2$, $k_4 = q - q_1$, $w_4 = q_1$,
$w_1 = \Delta_7\Delta_1$, $v_1 = \Delta_7(r - \Delta_1) + (c - \Delta_7)\Delta_1$, $u_1 = (c - \Delta_7)(r - \Delta_1)$, $t_1 = - q_2 + q$, $p_1 = q_2$,
$w_2 = \Delta_8\Delta_2$, $v_2 = - \Delta_8(r + \Delta_2) - (c + \Delta_8)\Delta_2$, $u_2 = (c + \Delta_8)(r + \Delta_2)$, $t_2 = q - q_1$, $p_2 = q_1$.

3.2 Fuzzify *s* in the Total Cost as Triangular Fuzzy Number

We replace the fuzzy numbers in Eqs. (5) to (10) by $\tilde{q} = (q, q, q)$, $\tilde{s} = (s_1, s_2, s_3)$, $\tilde{r} = (r, r, r)$, $\tilde{a} = (a, a, a)$, $\tilde{b} = (b, b, b)$, $\tilde{c} = (c, c, c)$, and $\tilde{T} = (T, T, T)$. Under this condition, $k_j = 0$, $j = 1, 2, 3, 4$, and $t_j = 0$, $j = 1, 2$, in Proposition 2, from Eqs. (17) and (18), we cannot solve $G(e_j, f_j, g_j, h_j, k_j, w_j)$ and $H(w_j, v_j, u_j, t_j, p_j)$. Therefore, from Eq. (13) we could solve it as follows.

The left and right hand side of the α-cut, $(0 \le \alpha \le 1)$, of $\tilde{q}, \tilde{s}, \tilde{r}, \tilde{a}, \tilde{b}, \tilde{c}$ are

$$\left.\begin{aligned}
&q_L(\alpha) = q_R(\alpha) = q\\
&s_L(\alpha) = s_1 + (s - s_1)\alpha > 0,\ s_R(\alpha) = s_2 - (s_2 - s)\alpha > 0\\
&r_L(\alpha) = r_R(\alpha) = r\\
&a_L(\alpha) = a_R(\alpha) = a\\
&b_L(\alpha) = b_R(\alpha) = b\\
&c_L(\alpha) = c_R(\alpha) = c
\end{aligned}\right\}.$$

From Eq. (14), we have

$$\left.\begin{aligned}
&Q_{1L}(\alpha) = Q_{1R}(\alpha) = \frac{Ta}{2}q\\
&Q_{2L}(\alpha) = Ta[s_1 + (s - s_1)\alpha],\ Q_{2R}(\alpha) = Ta[s_2 - (s_2 - s)\alpha)]\\
&Q_{3L}(\alpha) = \frac{Ta}{2q}[s_1 + (s - s_1)\alpha]^2,\ Q_{3R}(\alpha) = \frac{Ta}{2q}[s_2 - (s_2 - s)\alpha]^2\\
&Q_{4L}(\alpha) = \frac{Tb}{2q}[s_1 + (s - s_1)\alpha]^2,\ Q_{4R}(\alpha) = \frac{Tb}{2q}[s_2 - (s_2 - s)\alpha]^2\\
&Q_{5L}(\alpha) = Q_{5R}(\alpha) = \frac{cr}{q}
\end{aligned}\right\}.$$

The left and right hand side of the α-cut, $(0 \le \alpha \le 1)$, of $K(\tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})$ are

$$K(\tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})_L(\alpha) = \frac{Taq}{2} - Ta[s_2 - (s_2 - s)\alpha] + \frac{Ta}{2q}[s_1 + (s - s_1)\alpha]^2 + \frac{Tb}{2q}[s_1 + (s - s_1)\alpha]^2 + \frac{cr}{q},$$

$$K(\tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})_R(\alpha) = \frac{Taq}{2} - Ta[s_1 + (s - s_1)\alpha] + \frac{Ta}{2q}[s_2 - (s_2 - s)\alpha]^2 + \frac{Tb}{2q}[s_2 - (s_2 - s)\alpha]^2 + \frac{cr}{q}.$$

By Definition 2.5, we have

$$d(K(\tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c}),\ \tilde{0}) = \frac{1}{2}\int_0^1 [K(\tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})_L(\alpha) + K(\tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})_R(\alpha)]d\alpha$$
$$\equiv F_q(q,\ s_1,\ s,\ s_2). \tag{19}$$

We have the following proposition.

Proposition 3: If we fuzzify s in the crisp inventory with backorder in Eq. (11) as the fuzzy numbers $\tilde{s}$ (in Eq. (6)), then we obtain an estimate of the total cost in the fuzzy sense as

$$F_q(q,\ s_1,\ s,\ s_2) = \frac{Taq}{2} + \frac{cr}{q} - \frac{Ta}{4}[s_1 + 2s + s_2] + \frac{Ta}{12q}[s_1^2 + 2s^2 + s_2^2 + s_1 s + s_2 s] + \frac{Tb}{12q}[s_1^2 + 2s^2 + s_2^2 + s_1 s + s_2 s].$$

Remark 3.2: Since $d(\tilde{s},\ \tilde{0}) = \frac{1}{4}(s_1 + 2s + s_2)$, $d(\tilde{s}^2,\ \tilde{0}) = \frac{1}{6}(s_1^2 + 2s^2 + s_2^2 + s_1 s + s_2 s)$, therefore, we have $F_q(q,\ s_1,\ s,\ s_2) = \frac{Taq}{2} + \frac{cr}{q} - Tad(\tilde{s},\ \tilde{0}) + \frac{Ta}{2q}\, d(\tilde{s}^2,\ \tilde{0}) + \frac{Tb}{2q} d(\tilde{s}^2,\ \tilde{0})$. This formula is just the same as if we replaced s by $d(\tilde{s},\ \tilde{0})$ and s^2 by $d(\tilde{s}^2,\ \tilde{0})$ in Eq. (11).

3.3 Fuzzify q in the Total Cost as Triangular Fuzzy Number

We replace the fuzzy numbers in Eqs. (5) to (10) by $\tilde{q} = (q_1, q, q_2)$, $\tilde{s} = (s, s, s)$, $\tilde{r} = (r, r, r)$, $\tilde{a} = (a, a, a)$, $\tilde{b} = (b, b, b)$, $\tilde{c} = (c, c, c)$, and $\tilde{T} = (T, T, T)$. Let $s_1 = s_2 = s$ and $\Delta_j = 0$, $j = 1, 2, \ldots, 8$ in Proposition 2, then we have the following proposition.

Proposition 4: If we fuzzify q in the crisp inventory with backorder in Eq. (11) as the fuzzy numbers $\tilde{q}$ (in Eq. (5)), then we can obtain an estimate of the total cost in the fuzzy sense as

$$F_s(s,\ q_1,\ q,\ q_2) = \frac{Ta}{8}[q_1 + 2q + q_2] - Tas + \left(\frac{Tas^2}{4} + \frac{Tbs^2}{4} + \frac{cr}{2}\right)\left[\frac{1}{q_2 - q}\ln\frac{q_2}{q} + \frac{1}{q - q_1}\ln\frac{q}{q_1}\right].$$

Remark 3.3: Since $d(\tilde{q},\ \tilde{0}) = \frac{1}{4}(q_1 + 2q + q_2)$, $d\left(\frac{1}{\tilde{q}},\ \tilde{0}\right) = \frac{1}{2}\left[\frac{\ln q_2 - \ln q}{q_2 - q} + \frac{\ln q - \ln q_1}{q - q_1}\right]$, we have $F_s(s, q_1, q, q_2) = \frac{Ta}{2}d(\tilde{q},\ \tilde{0}) - Tas + \left(\frac{Tas^2}{2} + \frac{Tbs^2}{2} + cr\right)d\left(\frac{1}{\tilde{q}},\ \tilde{0}\right)$. This formula is just the same as if we replaced q by $d(\tilde{q},\ \tilde{0})$ and $\frac{1}{q}$ by $d\left(\frac{1}{\tilde{q}},\ \tilde{0}\right)$ in Eq. (11).

4. OPTIMAL SOLUTION

For $FC(q_1, q, q_2, s_1, s, s_2; \Delta_j, j = 1, 2, \ldots, 8)$, the estimated total cost in the fuzzy sense is shown in Proposition 2. For given Δ_j, $j = 1, 2, \ldots, 8$, there are six variables q_1, q, q_2, s_1, s, s_2 which satisfy $0 < s_1 < s < s_2 < q_1 < q < q_2$. We want to find $q_1^{(0)},\ q^{(0)},\ q_2^{(0)},\ s_1^{(0)},\ s^{(0)},\ s_2^{(0)}$ such that $FC(q_1^{(0)},\ q^{(0)},\ q_2^{(0)},\ s_1^{(0)},\ s^{(0)},\ s_2^{(0)};\ \Delta_j,\ j = 1, 2, \ldots, 8)$ is the minimum, with the optimal order quantity $q^{(00)} = \frac{1}{4}(q_1^{(0)} + 2q^{(0)} + q_2^{(0)})$ and the optimal backorder quantity $s^{(00)} = \frac{1}{4}(s_1^{(0)} + 2s^{(0)} + s_2^{(0)})$. For the estimated total cost in the fuzzy sense, $F_q(q, s_1, s, s_2)$ is shown in Proposition 3. There are four variables q, s_1, s, s_2 which satisfy $0 < s_1 < s < s_2 < q$. We want to find $q^{(0)},\ s_1^{(0)},\ s^{(0)},\ s_2^{(0)}$ such that $F_q(q^{(0)},\ s_1^{(0)},\ s^{(0)},\ s_2^{(0)})$ is minimum, the optimal order quantity $q^{(00)} = q^{(0)}$, and the optimal backorder quantity $s^{(00)} = \frac{1}{4}(s_1^{(0)} + 2s^{(0)} + s_2^{(0)})$. For the estimated total cost in the fuzzy sense, $F_s(s, q_1, q, q_2)$ is shown in Proposition 4. There are four variables s, q_1, q, q_2 which satisfy $0 < s < q_1 < q < q_2$. We want to find $s^{(0)},\ q_1^{(0)},\ q^{(0)},\ q_2^{(0)}$ such that $F_s(s^{(0)},\ q_1^{(0)},\ q^{(0)},\ q_2^{(0)})$ is the minimum, the optimal order quantity $q^{(00)} = \frac{1}{4}(q_1^{(0)} + 2q^{(0)} + q_2^{(0)})$, and the optimal backorder quantity $s^{(00)} = s^{(0)}$. We apply the Nelder-Mead method [8]. But, in our paper, the variables should have the ordering relation as shown above. For example, in Proposition 2, q_1, q, q_2, s_1, s, s_2 should satisfy $0 < s_1 < s < s_2 < q_1 < q < q_2$. Therefore, when we apply the Nelder-Mead simplex algorithm [1], the two transformations Eqs. (20) and (21) we use are shown in Figs. 4 and 5 instead of the two transformations of Algorithm 6.5 of the Nelder-Mead method [8].

$$R = X + e(X - G) = (1 + e)X - eG, \text{ where } 0 < e \leq 1, \tag{20}$$

$$E = X + d(R - X) = (1 - d)X + dR, \text{ where } d > 1. \tag{21}$$

For the Proposition 4 (similar arguments as Propositions 2 and 3), we denote q_1 for $R(1)$, $X(1)$, $G(1)$, and $E(1)$, q for $R(2)$, $X(2)$, $G(2)$, and $E(2)$, and q_2 for $R(3)$, $X(3)$, $G(3)$, and $E(3)$, s for $R(4)$, $X(4)$, $G(4)$, and $E(4)$ instead of the symbols in Algorithm 6.5 of the Nelder-Mead method [8].

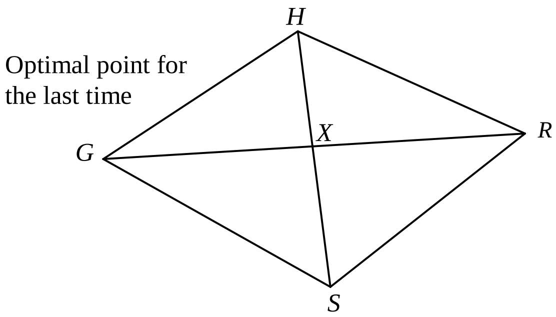

Fig. 4. Contraction step.

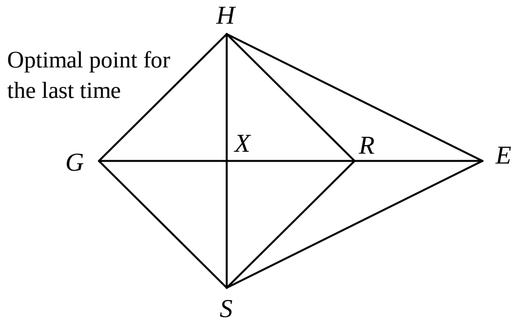

Fig. 5. Expansion step.

Suppose $X(1) < X(2) < X(3) < X(4)$ and $G(1) < G(2) < G(3) < G(4)$.

Step 1: Let $H(k + 1, k) = X(k) - X(k + 1) - G(k) + G(k + 1)$, for $k = 1, 2, 3$ and

$$I(H(k+1,\ k)) = \begin{cases} 1, & \text{if } H(k+1,\ k) > 0 \\ 0, & \text{if } H(k+1, k) \leq 0 \end{cases},$$

$$H_* = \min\begin{bmatrix} \frac{X(2)-X(1)}{H(2,\ 1)} I(H(2,\ 1)), & \frac{X(3)-X(2)}{H(3,\ 2)} I(H(3,\ 2)), \\ \frac{X(4)-X(3)}{H(4,\ 3)} I(H(4,\ 3)),\ 1 & \end{bmatrix},\ \text{for } k = 1, 2, 3.$$

If we take e in Eq. (20) satisfying

$$0 < e < H_*, \tag{22}$$

then it is easy to show that $R(1) < R(2) < R(3) < R(4)$.

Step 2: Let $L(k + 1, k) = X(k + 1) - X(k) + R(k) - R(k + 1)$, for $k = 1, 2, 3$ and

$$L_* = \begin{cases} \infty, & \text{if } L(k+1,\ k) \le 0,\ \forall k = 1,\ 2,\ 3 \\ \min\left[\dfrac{X(k+1) - X(k)}{L(k+1,\ k)} I(L(k+1,\ k)) \right], & \text{otherwise} \end{cases}.$$

If we take d in Eq. (21) satisfying

$$1 < d < L_*, \tag{23}$$

then it is easy to show that $E(1) < E(2) < E(3) < E(4)$.

We modify $R(k) = 2M(k) - V(H_i, k)$ in subroutine Newpoints of Algorithm 6.5 [8] to be $R(k) = (1 + e)M(k) - eV(H_i, k)$ where e satisfies Eq. (22). Also, we modify $E(k) = 2R(k) - M(k)$ to be $E(k) = dR(k) + (1 - d)M(k)$ where d satisfies Eq. (23). Applying the modified Algorithm 6.5 [8], we can find $s^{(0)}$, $q_1^{(0)}$, $q^{(0)}$, $q_2^{(0)}$ such that $F_s(s^{(0)}, q_1^{(0)}, q^{(0)}, q_2^{(0)})$ (in Proposition 4) is the local minimal value. Then the optimal order quantity is $q^{(00)} = \frac{1}{4}(q_1^{(0)} + 2q^{(0)} + q_2^{(0)})$, and the optimal backorder quantity $s^{(00)} = s^{(0)}$.

5. NUMERICAL EXAMPLE

Example 5.1: In [2], Chang *et al*. fuzzified s as triangular fuzzy number $\tilde{s} = (s_1, s, s_2)$ and kept the q as crisp variable in Eq. (4), they obtained the fuzzy total cost $G_q(\tilde{s})$. They derived the membership function by the extension principle, and then they obtained the estimate of the total cost in the fuzzy sense $M = (s_1, s, s_2, q)$ by the centroid method. But, it is very complex and difficult to derive [2].

(1) Example 4.1 in [2]: given $a = 10$, $b = 20$, $c = 200$, $r = 2000$, $T = 12$, we get the optimal solution $q_* = 100$, $s_* = 33.3333$ and the minimal total cost $F(q_*, s_*) = 8000$ in the crisp case. From [2], we have the results shown in Table 1 for each case of given sets of initial points s_1, s_0, s_2, q, where $R_q = \frac{q^{**} - q_*}{q_*} \times 100\%$, $R_s = \frac{s^{**} - s_*}{s_*} \times 100\%$, $M = M(s_1^*, s_0^*, s_2^*, q^{**})$, $R_c = \frac{M - F(q_*, s_*)}{F(q_*, s_*)} \times 100\%$.

(2) Using the data of the initial points s_1, s_0, s_2, q as shown in Table 1 for each case, and Proposition 2 in this article, we have the optimal quantities $s_1^{(0)}$, $s_0^{(0)}$, $s_2^{(0)}$, q^0, and $s^{(00)} = \frac{1}{4}(s_1^{(0)} + 2s_0^{(0)} + s_2^{(0)})$, $q^{(00)} = q^{(0)}$, as shown in Table 2, where $r_q = \frac{q^{(00)} - q_*}{q_*} \times 100\%$, $r_s = \frac{s^{(00)} - s_*}{s_*} \times 100\%$, $r_c = \frac{F_q - F(q_*, s_*)}{F(q_*, s_*)} \times 100\%$, $t_q = \frac{q^{(00)} - q_* *}{q_* *} \times 100\%$, $t_s = \frac{s^{(00)} - s_{**}}{s_{**}} \times 100\%$, $t_c = \frac{F_q - M}{M} \times 100\%$, and $F_q = F_q(q^{(00)}, s_1^{(0)}, s_0^{(0)}, s_2^{(0)})$.

Example 5.2: The formula form of total cost in [15] and in this paper are not the same. If we interchange s with $q - s$ in Eq. (4) within this paper, then we obtain the same total cost form as in [15]: $F^*(q, s) = \frac{a \cdot s^2 \cdot T}{2q} + \frac{b \cdot (q - s)^2 \cdot T}{2q} + \frac{c \cdot r}{q}$. Let $G_s(q) = F^*(q, s)$. For s

Table 1. Results of Example 4.1 shown in [2] and comparison with the crisp case.

Case	Initial points (s_1, s_0, s_2, q)	Optimal quantities	M	R_q(%)	R_s(%)	R_c(%)
1	(32, 33, 35, 100) (32, 34, 35, 100) (33, 35, 36, 100) (34, 35, 36, 101) (33, 34, 36, 101)	$s_1^* = 32.000002$ $s_0^* = 33.000197$ $s_2^* = 35.000003$ $s^{**} = 33.333401$ $q^{**} = 100.000003$	8001.456814	0.000	0.000	0.018
2	(32, 33, 36, 101) (31, 33, 35, 100) (32, 35, 36, 101) (32, 33, 34, 102) (33, 34, 36, 101)	$s_1^* = 31.999998$ $s_0^* = 32.934025$ $s_2^* = 35.706559$ $s^{**} = 33.546861$ $q^{**} = 100.614982$	8002.503489	0.615	0.641	0.031
3	(32, 33, 35, 100) (31, 32, 35, 100) (33, 35, 36, 101) (34, 35, 36, 101) (33, 34, 36, 101)	$s_1^* = 31.999999$ $s_0^* = 32.999946$ $s_2^* = 35.000293$ $s^{**} = 33.33383$ $q^{**} = 100.000029$	8001.457104	0.000	0.000	0.018
4	(32, 33, 37, 101) (31, 32, 35, 100) (32, 35, 36, 101) (32, 33, 34, 102) (33, 34, 36, 101)	$s_1^* = 32.002370$ $s_0^* = 33.439507$ $s_2^* = 35.167474$ $s^{**} = 33.353645$ $q^{**} = 100.000003$	8001.600941	0.878	0.609	

Table 2. Computed result of Example 5.1 by Proposition 2 using the initial points shown in Table 1, comparison with the crisp case and [2].

Case	Initial points (s_1, s_0, s_2, q)	Optimal quantities	F_q	r_q(%) / t_q(%)	r_s(%) / t_s(%)	r_c(%) / t_c(%)
1	(32, 33, 35, 100) (32, 34, 35, 100) (33, 35, 36, 100) (34, 35, 36, 101) (33, 34, 36, 101)	$s_1^{(0)} = 32.703877$ $s_0^{(0)} = 33.863650$ $s_2^{(0)} = 34.792244$ $s^{**} = 33.805855$ $q^{**} = 100.392804$	8000.924310	0.393 / 0.393	1.418 / 1.417	0.012 / -0.007
2	(32, 33, 36, 101) (31, 33, 35, 100) (32, 35, 36, 101) (32, 33, 34, 102) (33, 34, 36, 101)	$s_1^{(0)} = 32.088259$ $s_0^{(0)} = 33.361536$ $s_2^{(0)} = 35.429495$ $s^{**} = 33.560206$ $q^{**} = 100.724941$	8001.895131	0.725 / 0.725	0.681 / 0.680	0.024 / -0.008
3	(32, 33, 35, 100) (31, 32, 35, 100) (33, 35, 36, 101) (34, 35, 36, 101) (33, 34, 36, 101)	$s_1^{(0)} = 32.000000$ $s_0^{(0)} = 33.000000$ $s_2^{(0)} = 35.000000$ $s^{**} = 33.250000$ $q^{**} = 100.000000$	8001.40000	0.000 / 0.000	-0.250 / -0.250	0.018 / -0.001
4	(32, 33, 37, 101) (31, 32, 35, 100) (32, 35, 36, 101) (32, 33, 34, 102) (33, 34, 36, 101)	$s_1^{(0)} = 32.366281$ $s_0^{(0)} = 33.554513$ $s_2^{(0)} = 35.357182$ $s^{**} = 33.708122$ $q^{**} = 100.998807$	8001.740793	0.999 / 0.999	1.124 / 1.063	0.022 / 0.002

> 0, Yao and Lee [15] fuzzified q as triangular fuzzy number $\tilde{q} = (q_1, q, q_2)$ and kept s as a crisp variable. They obtained the fuzzy total cost $G_s(\tilde{q})$. They derived the membership function by the extension principle, and then they obtained the estimate of the total cost in the fuzzy sense $M = (q_1, q, q_2, s)$ by the centroid method. But, it is very complex and difficult to derive (see [14]). In order to compare $F^*(q, s)$ in [14] with $F(q, s)$ in Eq. (15), we interchange a and b.

(1) Example 4.1 in [14], given $a = 10$, $b = 20$, $c = 200$, $r = 2000$, $T = 12$, we get the optimal order quantity $q_* = 100$, optimal backorder quantity $s_* = 66.666667$, and the minimal total cost $F(q_*, s_*) = 8000$ in the crisp case. From [15], we have the results shown in Table 3 for each case of given sets of initial points q_1, q, q_2, s, where $R_q = \dfrac{q^{**} - q_*}{q_*} \times 100\%$, $q^{**} = \dfrac{1}{3}(q_1^* + q_0^* + q_2^*)$, $M = M(q_1^*, q_0^*, q_2^*, s^{**})$, and $R_c = \dfrac{M - F(q_*, s_*)}{F(q_*, s_*)} \times 100\%$.

Table 3. Result of Example 4.1 shown in [8] and comparison with the crisp case.

Case	Initial points (q_1, q_0, q_2, s)	Optimal quantities	M	R_q(%)	R_c(%)
1	(101, 102, 103, 66.666667) (101, 103, 104, 66.666667) (101, 102, 104, 66.666667) (100, 101, 102, 66.666667) (101, 102, 104, 66.666667)	$q_1^* = 100.003364$ $q_0^* = 101.003085$ $q_2^* = 102.004617$ $q^{**} = 101.003689$ $s^{**} = 66.666667$	8001.724438	1.004	0.022
2	(98, 99, 100, 55) (97, 98, 99, 50) (96, 98, 99, 50) (101, 102, 103, 50) (96, 97, 98, 50)	$q_1^* = 100.999926$ $q_0^* = 101.999926$ $q_2^* = 102.999926$ $q^{**} = 101.999926$ $s^{**} = 59.99987$	8114.999891	2.0	1.437
3	(101, 102, 103, 67) (101, 103, 104, 67) (101, 102, 104, 67) (101, 103, 105, 67) (101, 102, 105, 67)	$q_1^* = 101.000000$ $q_0^* = 102.00110$ $q_2^* = 103.00430$ $q^{**} = 102.01801$ $s^{**} = 66.666667$	8003.881534	2.0	0.046
4	(98, 99, 100, 90) (97, 98, 99, 85) (96, 98, 99, 85) (101, 102, 103, 88) (96, 97, 98, 85)	$q_1^* = 97.445160$ $q_0^* = 98.722647$ $q_2^* = 99.722647$ $q^{**} = 98.630152$ $s^{**} = 85.282626$	8697.957806	– 1.37	8.724

(2) As the discussion above, if we apply the Proposition 3 we should change the value of a and b in Example 4.1 in [15]. Given $a = 20$, $b = 10$, $c = 200$, $r = 2000$, $T = 12$, we get the optimal order quantity $q_{**} = 100$, optimal backorder quantity $s_{**} = 66.666667$, and the minimal total cost $F(q_{**}, s_{**}) = 8000$ in the crisp case. Using the data of the

initial points q_1, q, q_2, s as shown in Table 3 for each case, and Proposition 3 in this article, we obtain the optimal quantities $(q_1^{(0)},\ q_0^{(0)},\ q_2^{(0)},\ s^{(0)})$, and $q^{(00)} = \frac{1}{4}(q_1^{(0)} + 2q_0^{(0)} + q_2^{(0)})$, $s^{(00)} = s^{(0)}$, as shown in Table 4. Let $r_q = \frac{q^{(00)} - q_{**}}{q_{**}} \times 100\%$, $F_s = F_s(q_1^{(0)},\ q_0^{(0)},\ q_2^{(0)},\ s^{(00)})$, $r_c = \frac{F_s - F(q_{**},\ s_{**})}{F(q_{**},\ s_{**})} \times 100\%$, $t_q = \frac{q^{(00)} - q_{**}}{q^{**}} \times 100\%$, $M = M(q_1^*,\ q_0^*,\ q_2^*,\ s^{**})$, $t_c = \frac{F_s - M}{M} \times 100\%$.

Table 4. Computed result of Example 5.2 using the initial points shown in Table 3 and comparison with the crisp case.

Case	Initial points (q_1, q_0, q_2, s)	Optimal quantities	M	r_q(%) / t_q(%)	r_c(%) / t_c(%)
1	(101, 102, 103, 66.666667) (101, 103, 104, 66.666667) (101, 102, 104, 66.666667) (100, 101, 102, 66.666667) (101, 102, 104, 66.666667)	$q_1^{(0)}$ = 100.000000 $q_0^{(0)}$ = 101.000000 $q_2^{(0)}$ = 102.000000 q^{**} = 101.000000 s^{**} = 66.666667	8001.576379	1.000 / – 0.0036	0.020 / 0.0019
2	(98, 99, 100, 55) (97, 98, 99, 50) (96, 98, 99, 50) (101, 102, 103, 50) (96, 97, 98, 50)	$q_1^{(0)}$ = 101.000000 $q_0^{(0)}$ = 102.000000 $q_2^{(0)}$ = 103.000000 q^{**} = 102.000000 s^{**} = 60.000000	8114.839009	2.0 / 0.0001	1.435 / -0.002
3	(101, 102, 103, 67) (101, 103, 104, 67) (101, 102, 104, 67) (101, 103, 105, 67) (101, 102, 105, 67)	$q_1^{(0)}$ = 101.000000 $q_0^{(0)}$ = 102.000000 $q_2^{(0)}$ = 103.000000 q^{**} = 102.000000 s^{**} = 67.000000	8003.712805	2.0 / 0.0177	0.046 / 0.002
4	(98, 99, 100, 90) (97, 98, 99, 85) (96, 98, 99, 85) (101, 102, 103, 88) (96, 97, 98, 85)	$q_1^{(0)}$ = 97.445001 $q_0^{(0)}$ = 98.72250 $q_2^{(0)}$ = 99.72250 q^{**} = 98.653125 s^{**} = 85.282500	8696.252046	– 1.347 / 0.0233	8.704 / 0.0196

Example 5.3: We apply Proposition 1 of this article to solve the optimal solution $q_1^{(0)}$, $q^{(0)}$, $q_2^{(0)}$, $s_1^{(0)}$, $s^{(0)}$, $s_2^{(0)}$, the optimal order quantity $q^{(00)} = \frac{1}{4}(q_1^{(0)} + 2q^{(0)} + q_2^{(0)})$, the optimal backorder quantity $s^{(00)} = \frac{1}{4}(s_1^{(0)} + 2s^{(0)} + s_2^{(0)})$, and the minimal total cost $F_c(q_1^{(0)},\ q^{(0)},\ q_2^{(0)},\ s_1^{(0)},\ s^{(0)},\ s_2^{(0)};\ \Delta_j, j = 1, 2, \ldots, 8) \equiv F_c$. Let $r_q = \frac{q^{(00)} - q_*}{q_*} \times 100\%$, $r_s = \frac{s^{(00)} - s_*}{s_*} \times 100\%$, $r_c = \frac{F_c - F(q_*,\ s_*)}{F(q_*,\ s_*)} \times 100\%$.

Given a = 10, b = 20, c = 200, r = 2000, T = 12, we get the crisp optimal order quantity q_* = 100, optimal backorder quantity s_* = 33.333333, and the minimal total cost $F(q_*, s_*)$ = 8000 in the crisp case. Since there are six variables $0 < s_1 < s < s_2 < q_1 < q < q_2$, we should give a set of seven initial points. If we assign two sets of initial points as shown in Tables 5 and 6, and the Δ_j, j = 1, 2, …, 8, then we have the computation results shown in Tables 7 and 8.

Table 5. A set of initial points for Proposition 1.

s_1	s	s_2	q_1	q	q_2
32.5	33	34	99.1	100	101
32.5	34	35	99.5	101	102
31.5	33.5	34	99	100.5	101.5
31.5	33.7	34.5	98.9	101.2	102
32.8	33.8	34.6	98.5	100.8	101
32.6	33.4	34.7	98.6	101.5	102
31.8	33.2	34.8	99	100.9	101.5

Table 6. A set of initial points for Proposition 1.

s_1	s	s_2	q_1	q	q_2
32	33	33.5	99.1	100	101
32.5	34	34.5	99.5	101	102
32.5	33.5	34	99	100.5	101.5
31.5	33.7	34.5	98.9	101.2	102
32.8	33.8	34.6	98.5	100.8	101
32.6	33.4	34.7	98.6	101.5	102
31.8	33.2	34.8	99	100.9	101.5

6. DISCUSSIONS

(A) The comparison of this article with [2] and [15].

In this article, we use the signed distance to find the estimated total cost in the fuzzy sense in an easy way. The articles [2] and [15] used the extension principle and the centroid method. It was difficult for them to get the estimated total cost in the fuzzy sense. From Table 2, we know that the computation results from Proposition 2 in this paper are very close to those of [2]. From Table 4, we know that the computation results from Proposition 4 in this paper are very close to those of [15]. Moreover, from Tables 7 and 8, we find that the computation results from Proposition 2 (fuzzification of six variables) are very close to the crisp case if the fuzzy situation is insignificant.

Table 7. Example 5.3 (Computed result with Proposition 1).

Δ_1	0.1	0.1	0.2	0.1	1
Δ_2	0.2	0.1	0.1	0.1	3
Δ_3	0.1	0.1	0.1	0.2	2
Δ_4	0.2	0.1	0.2	0.2	3
Δ_5	0.1	0.1	0.1	0.1	1
Δ_6	0.2	0.1	0.2	0.2	2
Δ_7	0.1	0.1	0.2	0.2	3
Δ_8	0.2	0.1	0.3	0.1	2
$s_1^{(0)}$	32.500000	32.500000	32.500000	32.500000	1
$s^{(0)}$	33.000001	33.000001	33.000001	33.000001	32.499998
$s_2^{(0)}$	34.000001	33.500001	33.500001	33.500001	33.000001
$q_1^{(0)}$	99.100000	99.100000	99.100000	99.100000	33.500001
$q^{(0)}$	100.000001	100.000001	100.000001	100.000001	99.100000
$q_2^{(0)}$	101.000000	101.000000	101.000000	101.000000	100.000001
$s^{(00)}$	33.125000	33.000000	33.000001	33.000000	101.000001
$q^{(00)}$	100.025	100.025	100.025	00.025	100.025
FC	8010.006123	8000.897788	8009.811115	8002.154553	7881.604961
$r_q(\%)$	0.025	0.025	0.025	0.025	0.025
$r_s(\%)$	– 0.625	– 1.000	– 1.000	– 1.000	– 1.000
$r_c(\%)$	0.125	1.011	0.123	0.027	– 1.480

Table 8. Example 5.3 (Computed result with Proposition 1).

Δ_1	0.1	0.05	0.05	0.01	1
Δ_2	0.2	0.1	0.05	0.01	2
Δ_3	0.1	0.05	0.05	0.01	3
Δ_4	0.2	0.1	0.05	0.01	1
Δ_5	0.1	0.05	0.05	0.01	2
Δ_6	0.2	0.1	0.05	0.01	3
Δ_7	0.1	0.05	0.05	0.01	1
Δ_8	0.2	0.2	0.05	0.01	2
$s_1^{(0)}$	32.352267	32.336538	32.336535	32.336536	32.000000
$s^{(0)}$	33.375223	33.359652	33.359649	33.359650	33.000000
$s_2^{(0)}$	33.925999	33.915131	33.915127	33.915129	33.500000
$q_1^{(0)}$	98.999076	99.012653	99.012655	99.012655	99.100000
$q^{(0)}$	100.380270	100.370054	100.370049	100.370051	100.000000
$q_2^{(0)}$	101.261846	101.266787	101.266786	101.266787	101.000000
$s^{(00)}$	33.257178	33.242743	33.242740	33.242741	32.875000
$q^{(00)}$	100.257178	100.254887	100.254885	100.254886	100.025000
FC	8010.244311	8006.160907	8001.152777	8001.082155	7890.685151
$r_q(\%)$	0.255	0.255	0.255	0.225	0.025
$r_s(\%)$	– 0.228	– 0.272	– 0.272	– 0.272	– 1.375
$r_c(\%)$	0.128	0.077	0.014	0.014	– 1.366

(B) If we let $\Delta_j = 0$, $j = 1, 2, \ldots, 8$ in Proposition 2, then we have the total cost in the fuzzy sense

$$FC(q_1, q, q_2, s_1, s, s_2; \Delta_j = 0, j = 1, 2, \ldots, 8)$$
$$= \frac{Ta}{8}(q_1 + 2q + q_2) - \frac{Ta}{4}(s_1 + 2s + s_2) + \frac{T}{4}\left[\frac{(a+b)(s-s_1)^2}{2(q-q_2)} + \frac{2(a+b)s_1(s-s_1)}{q-q_2} + \frac{(a+b)s_1^2}{q-q_2}\ln\frac{q}{q_2} + \frac{(a+b)(s_2-s)^2}{2(q-q_1)} - \frac{2(a+b)s_2(s_2-s)}{q-q_1} + \frac{(a+b)s_2^2}{q-q_1}\ln\frac{q}{q_1}\right] + \frac{cr}{2}\left(\frac{1}{q-q_2}\ln\frac{q}{q_2} + \frac{1}{q-q_1}\ln\frac{q}{q_1}\right).$$

If we let $s_1 = s_2 = s$ then we obtain Proposition 4.

(C) In section 3.1, we fuzzify the length of plan T.

We fuzzify q, s, r, a, b in the crisp total cost in Eq. (4) as the triangular fuzzy numbers. In Eqs. (5) to (9), we do the same way and fuzzify the length of plan as the following triangular fuzzy number

$$\tilde{T} = (T - w_1,\ T,\ T + w_2),\ 0 < w_1 < T,\ 0 < w_2$$

where w_1, w_2 are properly determined by the decision maker, $T_L(\alpha) = T - (1 - \alpha)w_1 > 0$, $T_R(\alpha) = T + (1 - \alpha)w_2 > 0$, and fuzzify total cost in Eq. (12) with $\tilde{Q}_1 = \frac{1}{2}\tilde{T} \otimes \tilde{a} \otimes \tilde{q}$, $\tilde{Q}_2 = \tilde{T} \otimes \tilde{a} \otimes \tilde{s}$, $\tilde{Q}_3 = \frac{1}{2}\tilde{T} \otimes \tilde{a} \otimes (\tilde{s} \otimes \tilde{s}) \ominus \tilde{q}$, $\tilde{Q}_4 = \frac{1}{2} \otimes \tilde{b} \otimes (\tilde{s} \otimes \tilde{s}) \ominus \tilde{q}$, $\tilde{Q}_5 = \tilde{c} \otimes \tilde{r} \ominus \tilde{q}$. Eq. (14) becomes

$$\left.\begin{aligned}
&Q_{1L}(\alpha) = \frac{1}{2}T_L(\alpha)a_L(\alpha)q_L(\alpha),\ Q_{1R}(\alpha) = \frac{1}{2}T_R(\alpha)a_R(\alpha)q_R(\alpha)\\
&Q_{2L}(\alpha) = T_L(\alpha)a_L(\alpha)s_L(\alpha),\ Q_{2R}(\alpha) = T_R(\alpha)a_R(\alpha)s_R(\alpha)\\
&Q_{3L}(\alpha) = \frac{1}{2}T_L(\alpha)a_L(\alpha)s_L(\alpha)^2 / q_R(\alpha),\ Q_{3R}(\alpha) = \frac{1}{2}T_R(\alpha)a_R(\alpha)s_R(\alpha)^2 / q_L(\alpha)\\
&Q_{4L}(\alpha) = \frac{1}{2}T_L(\alpha)b_L(\alpha)s_L(\alpha)^2 / q_R(\alpha),\ Q_{4R}(\alpha) = \frac{1}{2}T_R(\alpha)b_R(\alpha)s_R(\alpha)^2 / q_L(\alpha)\\
&Q_{5L}(\alpha) = c_L(\alpha)r_L(\alpha) / q_R(\alpha),\ Q_{5R}(\alpha) = c_R(\alpha)r_R(\alpha) / q_L(\alpha).
\end{aligned}\right\}.$$

Eq. (15) becomes

$$K(\tilde{T},\ \tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})_L(\alpha) = Q_{1L}(\alpha) - Q_{2R}(\alpha) + Q_{3L}(\alpha) + Q_{4L}(\alpha) + Q_{5L}(\alpha),$$
$$K(\tilde{T},\ \tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})_R(\alpha) = Q_{1R}(\alpha) - Q_{2L}(\alpha) + Q_{3R}(\alpha) + Q_{4R}(\alpha) + Q_{5R}(\alpha).$$

By the same way as section 3.1, we have

$$d(K(\tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c}),\ \tilde{0}) = \frac{1}{2}\int_0^1 [K(\tilde{T},\ \tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})_L(\alpha) + K(\tilde{T},\ \tilde{q},\ \tilde{s},\ \tilde{r},\ \tilde{a},\ \tilde{b},\ \tilde{c})_R(\alpha)]d\alpha.$$

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