Short Paper

Measuring Fuzziness in Rough Sets*

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This paper presents a new fuzziness measure for rough sets. Fuzziness measures for rough sets may be employed to describe the inconsistency of a decision table. The definition of the fuzziness of a rough set proposed by Chakrabarty *et al*. has two drawbacks. The first is that the fuzziness of a rough set may not be unique. The second is that the fuzziness of a rough set with a large boundary may be very small. The aim of this paper is to present a new definition of the fuzziness measure for rough sets. The proposed fuzziness measure overcomes the drawbacks of the measure proposed by Chakrabarty *et al*. That is, with the new measure, each rough set has a unique fuzziness, and the fuzziness is in proportional to the size of the boundary. Moreover, the fuzziness of a rough set can be easily computed with the boundary of the rough set.

Keywords: fuzziness measure, fuzziness of a rough set, fuzzy sets, rough sets, roughness of a fuzzy set

1. INTRODUCTION

Theories of fuzzy sets [26] and rough sets [12, 13, 15] are generalizations of classical set theory for modelling vagueness and uncertainty. Rough set theory belongs to the family of concepts concerning the modeling and representing of incomplete knowledge [14, 18]. There have been many studies on their connections and differences [2, 3, 9, 10, 17, 24, 25].

Various extensions of rough sets have been proposed [4, 7, 11, 21-23]. Hu *et al*. [7] presented a new rough set model based on database systems. Okuzaki *et al*. [11] presented a rough sets-based method for clustering nominal and numerical data. The probabilistic (stochastic) rough set model was first introduced by Wong and Ziarko [20]. Pawlak *et al.* [16] reviewed and compared the fundamental results for probabilistic and deterministic models of rough sets. Wei *et al.* [21] studied fuzziness in probabilistic rough sets using fuzzy sets.

Dubois and Prade [6] proposed the notion of a two-fold fuzzy set which is made up

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of a nested pair of fuzzy sets. In this notion, the differences and similarities between rough sets and twofold fuzzy sets are stressed. Banerjee and Pal [1] defined the roughness of a crisp set in terms of the ratio of its lower approximation to its upper approximation, and the measure of roughness was extended to fuzzy sets. Fuzziness measures of rough sets were studied in [4, 21]. As noted in [21], fuzziness measures of rough sets may be employed to describe the inconsistency of a decision table. Rough sets and fuzzy sets were also studied in [2, 3, 9, 10, 24, 25].

The focus of this paper is to propose a new fuzziness measure for rough sets to remedy the drawbacks of the measure proposed by Chakrabarty *et al*. [4]. Two problems arise in their measure. The first is that the fuzziness of a rough set may not be unique. The second is that a rough set with a large boundary may have fuzziness approaching 0. To deal with these problems, we propose a new measure of fuzziness for rough sets such that each rough set has a unique fuzziness and the fuzziness is proportional to the size of the boundary.

There are two ways to characterize a rough set. We may characterize a rough set by means of either the lower approximation and upper approximation or by the family of all sets having the same lower approximation and upper approximation. For example, given approximation space (U, R) and concept $X = \{1, 2\}$, where the universe $U = \{1, 2, 3\}$ and the equivalence relation $R = \{\{1\}, \{2, 3\}\}$, the rough set $R(X)$ may be represented either as $\langle\{1\}, \{1, 2, 3\}\rangle$, the pair of composed the lower approximation and the upper approximation, or $\{\{1, 2\}, \{1, 3\}\}$, the family of all sets having the same lower and upper approximation. If a rough set is represented as a pair composed of the lower approximation and the upper approximation, then the fuzziness of the rough set should be related to the boundary of the rough set. Consider a rough set with an extremely large boundary. If we measure the fuzziness of the rough set with a concept X almost equal to the lower approximation, then the fuzziness will approach 0 according to the definition in [4]. This will contradict with our intuition since the boundary of the rough set is extremely large. If a rough set is represented by a family of all sets having the same lower approximation and the upper approximation, then the fuzziness should remain the same no matter which member of the family is chosen. However, according to the definition in [4], different fuzziness may be obtained with different members of the family. An element of a family having the same lower approximation and the upper approximation is a crisp set and is usually referred as a concept. The fuzziness measure in [4] is more appropriately referred as the roughness of the concept X under the approximation space (U, R) rather than the fuzziness of the rough set $R(X)$ although Banerjee and Pal [1] have given another definition of the roughness of a crisp set.

The basics of rough sets and the fuzziness measure proposed in [4] are reviewed in section 2. The drawbacks of the fuzziness measure proposed by Chakrabarty *et al*. are also presented in section 2. In section 3, we present a new fuzziness measure of rough sets and show the uniqueness property and proportionality property of the measure. Finally, conclusions are drawn in section 4.

2. ROUGH SETS AND FUZZINESS

Let U be a nonempty set, and let R be an indiscernibility relation or equivalence relation on U. Then (U, R) is called a Pawlak approximation space. Let the concept X be a subset of U. Then the lower approximation of X in (U, R), denoted as $\underline{X}$, is defined as

$$\underline{X} = \{u \mid [u]_R \subseteq X\},$$

and the upper approximation of X in (U, R), denoted as $\overline{X}$, is defined as

$$\overline{X} = \{u \mid [u]_R \cap X \neq \phi\},$$

where $[u]_R$ is an equivalence class of R containing u. The equivalence classes of R and the empty set ϕ are called elementary or atomic sets in the approximation space (U, R). The union of one or more elementary sets is called a composed set. The family of all composed sets, including the empty set, is denoted by $Comp((U, R))$, which is a Boolean algebra and a subalgebra of Boolean algebra 2^U. Pawlak regards the group of subsets of U with the same upper and lower approximations in (U, R) as a rough set in (U, R). Using lower and upper approximations, we can define an equivalence relation $\approx_R$ on the powerset of U:

$$X \approx_R Y \Leftrightarrow \underline{X} = \underline{Y} \text{ and } \overline{X} = \overline{Y},$$

where $X, Y \in 2^U$ and R is an equivalence relation on U.

This equivalence relation induces a partition on the power set 2^U. An equivalence class of this partition is called a P-rough set. The set of all P-rough sets is denoted by $2^U/\approx_R$. More specifically, a P-rough set can be defined as follows:

Definition 2.1 Given the approximation space (U, R) and two sets $A_1, A_2 \in Comp((U, R))$ with $A_1 \subseteq A_2$, a P-rough set is the family of subsets of U described as follows:

$$\langle A_1, A_2\rangle = \{X \in 2^U \mid \underline{X} = A_1, \ \overline{X} = A_2\}.$$

Equivalently, a P-rough set containing $X \in 2^U$ can be defined as

$$R(X) = [X]\approx_R = \{Y \in 2^U \mid \underline{Y} = \underline{X}, \ \overline{Y} = \overline{X}\}.$$

In other words,

$$R(X) = \langle \underline{X}, \overline{X}\rangle.$$

A member of $[X]_R$ is also referred to as a generator of the P-rough set [5].

After Pawlak's initiative, Iwinski subsequently interpreted rough sets in an algebraic way [8]. Let $\mathbb{B}$ be a complete subalgebra of the Boolean algebra 2^U. The pair $(U, \mathbb{B})$ is called a rough universe. Iwinski defined rough sets as follows:

Definition 2.2 Given the rough universe $(U, \mathbb{B})$, the pair (A_1, A_2) is a rough set iff $A_1, A_2 \in \mathbb{B}$ and $A_1 \subseteq A_2$.

We shall call (A_1, A_2) an *I*-rough set. In fact, the views of Pawlak and Iwinski on rough sets are equivalent.

Theorem 2.1 There is a one-to-one correspondence between the approximation space (U, R) and rough universe $(U, \mathbb{B})$.

Proof: Given (U, R), algebra $\mathbb{B}$ can be constructed by letting the atoms of $\mathbb{B}$ be the equivalence classes of R. Likewise, given $(U, \mathbb{B})$, R can be constructed by letting the equivalence classes of R be the atoms of $\mathbb{B}$. ❑

It follows that $Comp((U, R)) = \mathbb{B}$. Hence there is a one-to-one correspondence between *P*-rough set $\langle A_1, A_2\rangle$ and *I*-rough set (A_1, A_2). For convenience, we shall let $R(X) = (\underline{X}, \overline{X}) = \langle \underline{X}, \overline{X}\rangle$ denote the rough set of X.

Example 2.1: Let (U, R) be the approximation space such that $U = \{1, 2, 3, 4\}$ and $R = \{\{1, 2\}, \{3, 4\}\}$. Assume that $X = \{1, 2, 3\}$. Let $R(X)$ be the rough set of X. Then $R(X) = (\{1, 2\}, \{1, 2, 3, 4\}) = \{\{1, 2, 3\}, \{1, 2, 4\}\}$, where $\{1, 2\}$, is the lower approximation of X, $\{1, 2, 3, 4\}$ is the upper approximation of X, and $\{\{1, 2, 3\}, \{1, 2, 4\}\}$ is the family of all sets having $\{1, 2\}$ and $\{1, 2, 3, 4\}$ as their lower and upper approximations.

The following definitions and preliminaries are required for the rest of this work:

Definition 2.3 The α-level set or α-cut, denoted by A_α of a fuzzy set A in U, comprises all the elements of U whose degrees of membership in A are all greater than or equal to α, where $0 < \alpha \le 1$. In other words,

$$A_\alpha = \{x \in U : u_A(x) \ge \alpha\}$$

is a crisp set. The membership function of a fuzzy set A can be expressed in terms of the characteristic function $\mu_{A_\alpha}(x)$ of its α-level sets using

$$\mu_A(x) = \sup_\alpha \min(\alpha, \mu_{A_\alpha}(x)), \text{ where } 0 < \alpha \le 1$$

and

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & \text{iff } x \in A_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

The strong α-cut A_{α^+} is given by

$$A_{\alpha^+} = \{x \in U : \mu_A(x) > \alpha\}.$$

Definition 2.4 Let A be a fuzzy set. Then the nearest ordinary set to A is denoted by $\underline{A}$ and is given by

$$\mu_{\underline{A}}(X) = \begin{cases} 0 & \text{if } \mu_A(x) < 0.5, \\ 1 & \text{if } \mu_A(x) > 0.5, \\ 0 \text{ or } 1 & \text{if } \mu_A(x) = 0.5. \end{cases}$$

By convention we take $\mu_{\underline{A}}(x) = 0$ for the last case. Thus, $\underline{A} = A_{0.5}$, where $A_{0.5}$ is the 0.5-cut of A.

Definition 2.5 [4] The index of fuzziness of a set A having n supporting points is defined as

$$v(A) = (2/n^k)d(A, \underline{A}),$$

where $d(A, \underline{A})$ denotes the distance between the fuzzy set A and its nearest ordinary set $\underline{A}$. The number 2 and the positive constant k appear in order to make $v(A)$ lie between 0 and 1. The value of k depends on the type of distance function used, e.g., $k = 1$ for a generalized Hamming distance, whereas $k = 0.5$ for an Euclidean distance. The corresponding indices of fuzziness are, respectively, called the 'linear index of fuzziness,' $v_l(A)$, and the 'quadratic index of fuzziness', $v_q(A)$.

For example, if

$$A = \{0 * 0, 1 * 1, 2 * 1, 3 * 0.2, 4 * 0.7, 5 * 0, 6 * 1, 7 * 1, 8 * 0.5\},$$

then the linear index of fuzziness of A is

$$v_l(A) = 2/9,$$

and the quadratic index of fuzziness of A is

$$v_q(A) = \sqrt{\frac{38}{225}}.$$

Let (U, R) be an approximation space, and let $X \subseteq U$. The rough set of X in (U, R) is $R(X) = (\underline{X}, \overline{X})$. Given a subset X of U, Chakrabarty *et al*. [4] defined an induced fuzzy set $\tilde{F}_X^R$ based on the rough membership function which measures the degree of rough belongingness of u in X [19]:

$$\frac{|[u]_R \cap X|}{|[u]_R|}, \tag{1}$$

where $|Y|$ denotes the cardinality of Y. The fuzzy set F_X^R is given by

$$F_X^R = \{(u, \mu_{F_X^R}(u)) : u \in U, \mu_{F_X^R}(u) = \frac{|[u]_R \cap X|}{|[u]_R|}\}.$$

Definition 2.6 [4] The fuzziness in the rough set $R(X)$ of X is denoted by f_X^R and is defined as the amount of fuzziness present in the fuzzy set F_X^R. The amount of fuzziness can be measured by means of a suitable index of fuzziness (linear or quadratic). The linear and the quadratic indices of fuzziness of the fuzzy set F_X^R are, respectively, called the linear fuzziness and the quadratic fuzziness of the rough set $R(X)$. They are denoted by $(f_X^R)_l$ and $(f_X^R)_q$, respectively.

Example 2.2: Let (U, R) be an approximation space, where $U = \{1, 2, 3, 4, 5, 6\}$ and $R = \{\{1, 2\}, \{3, 4, 5, 6\}\}$. Let us consider a subset $X = \{1, 2, 3\}$ of U. Then the rough set of X is $R(X) = (\underline{X}, \overline{X})$, where $\underline{X} = \{1, 2\}$ and $\overline{X} = \{1, 2, 3, 4, 5, 6\}$. Therefore, $\mu_{F_X^R}(1) = \frac{|[1]_R \cap X|}{|[1]_R|} = 1$. Similarly, we get, $\mu_{F_X^R}(2) = 1$, $\mu_{F_X^R}(3) = \frac{1}{4}$, $\mu_{F_X^R}(4) = \frac{1}{4}$, $\mu_{F_X^R}(5) = \frac{1}{4}$, and $\mu_{F_X^R}(6) = \frac{1}{4}$. Hence,

$$F_X^R = \{1*1,\ 2*1,\ 3*\frac{1}{4},\ 4*\frac{1}{4},\ 5*\frac{1}{4},\ 6*\frac{1}{4}\}.$$

The linear fuzziness in the rough set $R(X)$ is $(f_X^R)_l = (2/6)d(F_X^R, \underline{F_X^R})$, where $\underline{F_X^R}$ denotes the nearest ordinary set of F_X^R and $d(F_X^R, \underline{F_X^R})$ denotes the Hamming distance between F_X^R and $\underline{F_X^R}$. Thus, $(f_X^R)_l = (2/6)\sum_{u_i \in U} |\mu_{F_X^R}(u_i) - \mu_{\underline{F_X^R}}(u_i)| = 0.33333$. The quadratic fuzziness in the rough set $R(X)$ is $(f_X^R)_q = (2/\sqrt{6})d'(F_X^R, \underline{F_X^R})$, where $\underline{F_X^R}$ denotes the nearest ordinary set of F_X^R and $d'(F_X^R, \underline{F_X^R})$ denotes the Euclidean distance between F_X^R and $\underline{F_X^R}$. Therefore,

$$(f_X^R)_q = (2/\sqrt{6})\sqrt{\sum_{u_i \in U} (\mu_{F_X^R}(u_i) - \mu_{\underline{F_X^R}}(u_i))^2} = 0.408248.$$

Example 2.3: Let (U, R) be an approximation space, where $U = \{1, 2, 3, 4, 5, 6\}$ and $R = \{\{1, 2\}, \{3, 4, 5, 6\}\}$. Let us consider a subset $Y = \{1, 2, 3, 4\}$ of U. Then the rough set of Y is $R(Y) = (\underline{Y}, \overline{Y})$, where $\underline{Y} = \{1, 2\}$ and $\overline{Y} = \{1, 2, 3, 4, 5, 6\}$. Therefore, $\mu_{F_Y^R}(1) = \frac{|[1]_R \cap Y|}{|[1]_R|} = 1$. Similarly we get, $\mu_{F_Y^R}(2) = 1$, $\mu_{F_Y^R}(3) = \frac{1}{2}$, $\mu_{F_Y^R}(4) = \frac{1}{2}$, $\mu_{F_Y^R}(5) = \frac{1}{2}$, and $\mu_{F_Y^R}(6) = \frac{1}{2}$. Hence,

$$F_Y^R = \{1*1,\ 2*1,\ 3*\frac{1}{2},\ 4*\frac{1}{2},\ 5*\frac{1}{2},\ 6*\frac{1}{2}\}.$$

The linear fuzziness in the rough set $R(Y)$ is $(f_Y^R)_l = (2/6)d(F_Y^R,\ \underline{F_Y^R})$, where $\underline{F_Y^R}$ denotes the nearest ordinary set of F_Y^R and $d(F_Y^R,\ \underline{F_Y^R})$ denotes the Hamming distance between F_Y^R and $\underline{F_Y^R}$. Thus, $(f_Y^R)_l = (2/6)\sum_{u_i \in U} |\mu_{F_Y^R}(u_i) - \mu_{\underline{F_Y^R}}(u_i)| = 0.666667$. The quadratic fuzziness in the rough set $R(Y)$ is $(f_Y^R)_q = (2/\sqrt{6})d'(F_Y^R,\ \underline{F_Y^R})$, where $\underline{F_Y^R}$ denotes the nearest ordinary set of F_Y^R and $d'(F_Y^R,\ \underline{F_Y^R})$ denotes the Euclidean distance between F_Y^R and $\underline{F_Y^R}$. Therefore,

$$(f_Y^R)_q = (2/\sqrt{6})\sqrt{\sum_{u_i \in U} (\mu_{F_Y^R}(u_i) - \mu_{\underline{F_Y^R}}(u_i))^2} = 0.816497.$$

From Examples 2.2 and 2.3, we find that

$R(X) = R(Y) = (\{1, 2\}, \{1, 2, 3, 4, 5, 6\})$.

That is, the rough sets $R(X)$ and $R(Y)$ are the same but their fuzziness is different.

Example 2.4: Assume that $n > 3$. Let (U, R) be an approximation space, where $U = \{i \mid 1 \le i \le n,\ i \in N\}$ and $R = \{\{1, 2\}, \{3, \ldots, n\}\}$. Let us consider a subset $X = \{1, 2, 3\}$ of U. Then the rough set of X is $R(X) = (\underline{X},\ \overline{X})$, where $\underline{X} = \{1, 2\}$ and $\overline{X} = U$. Therefore, $\mu_{F_X^R}(1) = \frac{|[1]_R \cap X|}{|[1]_R|} = 1$. Similarly we get, $\mu_{F_X^R}(2) = 1$, $\mu_{F_X^R}(i) = \frac{1}{n-2}$ for $3 \le i \le n$ and $i \in U$. Hence,

$$F_X^R = \{1*1,\ 2*1,\ 3*\frac{1}{n-2},\ \ldots,\ n*\frac{1}{n-2}\}.$$

The linear fuzziness in the rough set $R(X)$ is $(f_X^R)_l = (2/n)d(F_X^R,\ \underline{F_X^R})$, where $\underline{F_X^R}$ denotes the nearest ordinary set of F_X^R and $d(F_X^R,\ \underline{F_X^R})$ denotes the Hamming distance between F_X^R and $\underline{F_X^R}$. Thus, $(f_X^R)_l = (2/n)\sum_{u_i \in U} |\mu_{F_X^R}(u_i) - \mu_{\underline{F_X^R}}(u_i)| = 2/n$. The quadratic fuzziness in the rough set $R(X)$ is $(f_X^R)_q = (2/\sqrt{n})d'(F_X^R,\ \underline{F_X^R})$, where $\underline{F_X^R}$ denotes the nearest ordinary set of F_X^R and $d'(F_X^R,\ \underline{F_X^R})$ denotes the Euclidean distance between F_X^R and $\underline{F_X^R}$. Therefore,

$$(f_X^R)_q = (2/\sqrt{n})\sqrt{\sum_{u_i \in U} (\mu_{F_X^R}(u_i) - \mu_{\underline{F_X^R}}(u_i))^2} = \frac{2}{\sqrt{n^2 - 2n}}.$$

From Example 2.4, we find that the boundary of $R(X)$ is

$\{i \mid 3 \le i \le n, i \in U\}$.

As n approaches infinity, the fuzziness of $R(X)$ approaches 0, whereas its boundary becomes extremely large.

In short, Example 2.3 shows that with the fuzziness measure defined by Chakrabarty *et al*., the fuzziness of a rough set may not be unique, and Example 2.4 shows that the fuzziness of a rough set with an extremely large boundary may be very small.

3. NEW FUZZINESS MEASURE

In the previous section, we found that there are some drawbacks of the fuzziness measure proposed by Chakrabarty *et al*. To remedy these drawbacks, we propose a new fuzziness measure for a rough set $R(X)$. For an element $u \in U$, the degree of rough belongingness of u in X is given by

$$F_X^R(u) = \frac{|[u]_R \cap X|}{|[u]_R|}.$$

Let $B_{R(X)}^u = \{F_Y^R(u) \mid Y \in R(X)\}$. That is, $B_{R(X)}^u$ is the set of all possible degrees of rough belongingness of u in the rough set $R(X)$. We take the average value of all possible values in $B_{R(X)}^u$, $\frac{\sum_{v \in B_{R(X)}^u} v}{|B_{R(X)}^u|}$, as the degree of rough belongingness of u in $R(X)$. This immediately induces a fuzzy set G_X^R of U given by

$$G_X^R = \{(u, \mu_{G_X^R}(u)) : u \in U, \mu_{G_X^R}(u) = \frac{\sum_{v \in B_{R(X)}^u} v}{|B_{R(X)}^u|}\}.$$

Definition 3.1 The fuzziness in the rough set $R(X)$ of X is denoted by g_X^R and is defined as the amount of fuzziness present in the fuzzy set G_X^R. The amount of fuzziness can be measured by means of a suitable index of fuzziness (linear or quadratic). The linear and the quadratic indices of fuzziness of the fuzzy set G_X^R are, respectively, called the linear fuzziness and the quadratic fuzziness of the rough set $R(X)$. They are denoted by $(g_X^R)_l$ and $(g_X^R)_q$, respectively.

Example 3.1: Let (U, R) be an approximation space, where $U = \{1, 2, 3, 4, 5, 6\}$ and $R = \{\{1, 2\}, \{3, 4, 5, 6\}\}$. Let us consider a subset $X = \{1, 2, 3\}$ of U. Then the rough set of X is $R(X) = (\underline{X}, \overline{X})$, where $\underline{X} = \{1, 2\}$ and $\overline{X} = \{1, 2, 3, 4, 5, 6\}$. The rough set is

$$\begin{aligned} R(X) = \{&\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \\ &\{1, 2, 3, 6\}, \{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 2, 5, 6\}, \{1, 2, 3, 4, 5\}, \\ &\{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}\}. \end{aligned}$$

Let

$X_1 = \{1, 2, 3\}$, $X_2 = \{1, 2, 4\}$, $X_3 = \{1, 2, 5\}$, $X_4 = \{1, 2, 6\}$,
$X_5 = \{1, 2, 3, 4\}$, $X_6 = \{1, 2, 3, 5\}$, $X_7 = \{1, 2, 3, 6\}$,
$X_8 = \{1, 2, 4, 5\}$, $X_9 = \{1, 2, 4, 6\}$, $X_{10} = \{1, 2, 5, 6\}$,
$X_{11} = \{1, 2, 3, 4, 5\}$, $X_{12} = \{1, 2, 3, 4, 6\}$, $X_{13} = \{1, 2, 3, 5, 6\}$, $X_{14} = \{1, 2, 4, 5, 6\}$.

Then

$$F_{X_1}^R(1) = F_{X_2}^R(1) = \cdots = F_{X_{12}}^R(1) = 1,$$
$$F_{X_1}^R(2) = F_{X_2}^R(2) = \cdots = F_{X_{12}}^R(2) = 1,$$
$$F_{X_1}^R(3) = F_{X_2}^R(3) = F_{X_3}^R(3) = F_{X_4}^R(3) = \frac{1}{4},$$
$$F_{X_5}^R(3) = F_{X_6}^R(3) = F_{X_7}^R(3) = F_{X_8}^R(3) = F_{X_9}^R(3) = F_{X_{10}}^R(3) = \frac{1}{2},$$
$$F_{X_{11}}^R(3) = F_{X_{12}}^R(3) = F_{X_{13}}^R(3) = F_{X_{14}}^R(3) = \frac{3}{4},$$
$$F_{X_1}^R(4) = F_{X_2}^R(4) = F_{X_3}^R(4) = F_{X_4}^R(4) = \frac{1}{4},$$
$$F_{X_5}^R(4) = F_{X_6}^R(4) = F_{X_7}^R(4) = F_{X_8}^R(4) = F_{X_9}^R(4) = F_{X_{10}}^R(4) = \frac{1}{2},$$
$$F_{X_{11}}^R(4) = F_{X_{12}}^R(4) = F_{X_{13}}^R(4) = F_{X_{14}}^R(4) = \frac{3}{4},$$
$$F_{X_1}^R(5) = F_{X_2}^R(5) = F_{X_3}^R(5) = F_{X_4}^R(5) = \frac{1}{4},$$
$$F_{X_5}^R(5) = F_{X_6}^R(5) = F_{X_7}^R(5) = F_{X_8}^R(5) = F_{X_9}^R(5) = F_{X_{10}}^R(5) = \frac{1}{2},$$
$$F_{X_{11}}^R(5) = F_{X_{12}}^R(5) = F_{X_{13}}^R(5) = F_{X_{14}}^R(5) = \frac{3}{4},$$
$$F_{X_1}^R(6) = F_{X_2}^R(6) = F_{X_3}^R(6) = F_{X_4}^R(6) = \frac{1}{4},$$
$$F_{X_5}^R(6) = F_{X_6}^R(6) = F_{X_7}^R(6) = F_{X_8}^R(6) = F_{X_9}^R(6) = F_{X_{10}}^R(6) = \frac{1}{2},$$
$$F_{X_{11}}^R(6) = F_{X_{12}}^R(6) = F_{X_{13}}^R(6) = F_{X_{14}}^R(6) = \frac{3}{4}.$$

Hence,

$$B_{R(X)}^1 = B_{R(X)}^2 = \{1\}$$

and

$$B_{R(X)}^3 = B_{R(X)}^4 = B_{R(X)}^5 = B_{R(X)}^6 = \{\frac{1}{4}, \ \frac{1}{2}, \ \frac{3}{4}\}.$$

Therefore, $\mu_{G_X^R}(1) = 1$. Similarly we get, $\mu_{G_X^R}(2) = 1$, $\mu_{G_X^R}(3) = 0.5$, $\mu_{G_X^R}(4) = 0.5$, $\mu_{G_X^R}(5) = 0.5$, and $\mu_{G_X^R}(6) = 0.5$. Hence,

$$G_X^R = \{1 * 1, 2 * 1, 3 * 0.5, 4 * 0.5, 5 * 0.5, 6 * 0.5\}.$$

The linear fuzziness in the rough set $R(X)$ is $(g_X^R)_l = (2/6)d(G_X^R, \underline{G_X^R})$, where $\underline{G_X^R}$ denotes the nearest ordinary set of G_X^R and $d(G_X^R, \underline{G_X^R})$ denotes the Hamming distance between G_X^R and $\underline{G_X^R}$. Thus, $(g_X^R)_l = (2/6)\sum_{u_i \in U} |\mu_{G_X^R}(u_i) - \mu_{\underline{G_X^R}}(u_i)| = 0.666667$. The quadratic fuzziness in the rough set $R(X)$ is $(g_X^R)_q = (2/\sqrt{6})d'(G_X^R, \underline{G_X^R})$, where $\underline{G_X^R}$ denotes the nearest ordinary set of G_X^R and $d'(G_X^R, \underline{G_X^R})$ denotes the Euclidean distance between G_X^R and $\underline{G_X^R}$. Therefore,

$$(g_X^R)_q = (2/\sqrt{6})\sqrt{\sum_{u_i \in U} (\mu_{G_X^R}(u_i) - \mu_{\underline{G_X^R}}(u_i))^2} = 0.816497.$$

Example 3.2: Let (U, R) be an approximation space, where $U = \{1, 2, 3, 4, 5, 6\}$ and $R = \{\{1, 2\}, \{3, 4, 5, 6\}\}$. Let us consider a subset $Y = \{1, 2, 3, 4\}$ of U. Then the rough set of Y is $R(Y) = (\underline{Y}, \overline{Y})$, where $\underline{Y} = \{1, 2\}$ and $\overline{Y} = \{1, 2, 3, 4, 5, 6\}$. The rough set is

$$\begin{aligned} R(Y) = \{&\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \\ &\{1, 2, 3, 6\}, \{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 2, 5, 6\}, \{1, 2, 3, 4, 5\}, \\ &\{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}\}. \end{aligned}$$

Let

$$\begin{aligned} &Y_1 = \{1, 2, 3\}, Y_2 = \{1, 2, 4\}, Y_3 = \{1, 2, 5\}, Y_4 = \{1, 2, 6\}, \\ &Y_5 = \{1, 2, 3, 4\}, Y_6 = \{1, 2, 3, 5\}, Y_7 = \{1, 2, 3, 6\}, \\ &Y_8 = \{1, 2, 4, 5\}, Y_9 = \{1, 2, 4, 6\}, Y_{10} = \{1, 2, 5, 6\}, \\ &Y_{11} = \{1, 2, 3, 4, 5\}, Y_{12} = \{1, 2, 3, 4, 6\}, Y_{13} = \{1, 2, 3, 5, 6\}, Y_{14} = \{1, 2, 4, 5, 6\}. \end{aligned}$$

Then

$$\begin{aligned} &F_{Y_1}^R(1) = F_{Y_2}^R(1) = \cdots = F_{Y_{12}}^R(1) = 1, \\ &F_{Y_1}^R(2) = F_{Y_2}^R(2) = \cdots = F_{Y_{12}}^R(2) = 1, \\ &F_{Y_1}^R(3) = F_{Y_2}^R(3) = F_{Y_3}^R(3) = F_{Y_4}^R(3) = \frac{1}{4}, \\ &F_{Y_5}^R(3) = F_{Y_6}^R(3) = F_{Y_7}^R(3) = F_{Y_8}^R(3) = F_{Y_9}^R(3) = F_{Y_{10}}^R(3) = \frac{1}{2}, \\ &F_{Y_{11}}^R(3) = F_{Y_{12}}^R(3) = F_{Y_{13}}^R(3) = F_{Y_{14}}^R(3) = \frac{3}{4}, \end{aligned}$$

$$F_{Y_1}^R(4) = F_{Y_2}^R(4) = F_{Y_3}^R(4) = F_{Y_4}^R(4) = \frac{1}{4},$$
$$F_{Y_5}^R(4) = F_{Y_6}^R(4) = F_{Y_7}^R(4) = F_{Y_8}^R(4) = F_{Y_9}^R(4) = F_{Y_{10}}^R(4) = \frac{1}{2},$$
$$F_{Y_{11}}^R(4) = F_{Y_{12}}^R(4) = F_{Y_{13}}^R(4) = F_{Y_{14}}^R(4) = \frac{3}{4},$$
$$F_{Y_1}^R(5) = F_{Y_2}^R(5) = F_{Y_3}^R(5) = F_{Y_4}^R(5) = \frac{1}{4},$$
$$F_{Y_5}^R(5) = F_{Y_6}^R(5) = F_{Y_7}^R(5) = F_{Y_8}^R(5) = F_{Y_9}^R(5) = F_{Y_{10}}^R(5) = \frac{1}{2},$$
$$F_{Y_{11}}^R(5) = F_{Y_{12}}^R(5) = F_{Y_{13}}^R(5) = F_{Y_{14}}^R(5) = \frac{3}{4},$$
$$F_{Y_1}^R(6) = F_{Y_2}^R(6) = F_{Y_3}^R(6) = F_{Y_4}^R(6) = \frac{1}{4},$$
$$F_{Y_5}^R(6) = F_{Y_6}^R(6) = F_{Y_7}^R(6) = F_{Y_8}^R(6) = F_{Y_9}^R(6) = F_{Y_{10}}^R(6) = \frac{1}{2},$$
$$F_{Y_{11}}^R(6) = F_{Y_{12}}^R(6) = F_{Y_{13}}^R(6) = F_{Y_{14}}^R(6) = \frac{3}{4}.$$

Hence,

$$B_{R(Y)}^1 = B_{R(Y)}^2 = \{1\}$$

and

$$B_{R(Y)}^3 = B_{R(Y)}^4 = B_{R(Y)}^5 = B_{R(Y)}^6 = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}.$$

Therefore, $\mu_{G_Y^R}(1) = 1$. Similarly we get, $\mu_{G_Y^R}(2) = 1$, $\mu_{G_Y^R}(3) = 0.5$, $\mu_{G_Y^R}(4) = 0.5$, $\mu_{G_Y^R}(5) = 0.5$, and $\mu_{G_Y^R}(6) = 0.5$. Hence,

$$G_Y^R = \{1 * 1, 2 * 1, 3 * 0.5, 4 * 0.5, 5 * 0.5, 6 * 0.5\}.$$

The linear fuzziness in the rough set $R(Y)$ is $(g_Y^R)_l = (2/6)d(G_Y^R, \underline{G_Y^R})$, where $\underline{G_Y^R}$ denotes the nearest ordinary set of G_Y^R and $d(G_Y^R, \underline{G_Y^R})$ denotes the Hamming distance between G_Y^R and $\underline{G_Y^R}$. Thus, $(g_Y^R)_l = (2/6)\sum_{u_i \in U} |\mu_{G_Y^R}(u_i) - \mu_{\underline{G_Y^R}}(u_i)| = 0.666667$. The quadratic fuzziness in the rough set $R(Y)$ is $(g_Y^R)_q = (2/\sqrt{6})d'(G_Y^R, \underline{G_Y^R})$, where $\underline{G_Y^R}$ denotes the nearest ordinary set of G_Y^R and $d'(G_Y^R, \underline{G_Y^R})$ denotes the Euclidean distance between G_Y^R and $\underline{G_Y^R}$. Therefore,

$$(g_Y^R)_q = (2/\sqrt{6})\sqrt{\sum_{u_i \in U} (\mu_{G_Y^R}(u_i) - \mu_{\underline{G_Y^R}}(u_i))^2} = 0.816497.$$

From the above examples, we find that if $R(X) = R(Y)$, then the fuzziness of $R(X)$ and the fuzziness of $R(Y)$ are the same, which is shown in the following property:

Property 3.1 The linear fuzziness $(g_X^R)_l$ and the quadratic fuzziness $(g_X^R)_q$ of a rough set are unique. That is, if $R(X) = R(Y)$, then the fuzziness of $R(X)$ and the fuzziness of $R(Y)$ are the same.

Proof: If $R(X) = R(Y)$, then

$$\mu_{G_X^R}(u) = \frac{\sum_{v \in B_{R(X)}^u} v}{|B_{R(X)}^u|} = \frac{\sum_{v \in B_{R(Y)}^u} v}{|B_{R(X)}^u|} = \mu_{G_Y^R}(u).$$

That is, if $R(X) = R(Y)$, then $G_X^R = G_Y^R$. Hence, the fuzziness of $R(X)$ and the fuzziness of $R(Y)$ are the same. ❑

Property 3.2 For any X in an approximation space (U, R), the following holds:

$$B_{R(X)}^u = \begin{cases} \{\frac{j}{|[u]_R|} : 1 \le j \le |[u]_R| - 1\} & u \in \overline{X} - \underline{X} \\ \{1\} & u \in \underline{X} \\ \{0\} & u \in U - \overline{X}. \end{cases}$$

Proof: If $u \in \underline{X}$, then $[u]_R \subseteq \underline{X}$. Hence $[u]_R \subseteq Y$ for $Y \in R(X)$. This yields $\frac{|[u]_R \cap Y|}{|[u]_R|} = \frac{|[u]_R|}{|[u]_R|} = 1$ for $Y \in R(X)$.

If $u \in U - \overline{X}$, then $[u]_R \cap Y = \phi$ for $Y \in R(X)$. Therefore, $\frac{|[u]_R \cap Y|}{|[u]_R|} = 0$ for $Y \in R(X)$.

If $u \in \overline{X} - \underline{X}$, then $[u]_R \cap Y \subset [u]_R$ for $Y \in R(X)$. Hence,

$$\left\{\frac{|[u]_R \cap Y|}{|[u]_R|} : Y \in R(X)\right\} = \left\{\frac{j}{|[u]_R|} : 1 \le j \le |[u]_R| - 1\right\}.$$ ❑

From property 3.2, we have the following:

Property 3.3 For any X in an approximation space (U, R),

$$|B_{R(X)}^u| = \begin{cases} |[u]_R| - 1 & \text{if } u \in \overline{X} - \underline{X} \\ 1 & \text{otherwise.} \end{cases}$$

From the definition of G_X^R and properties above, we have the following:

Property 3.4 For any X in an approximation space (U, R), the following holds:

$$\mu_{G_X^R}(u) = \begin{cases} 0.5 & u \in \overline{X} - \underline{X} \\ 1 & u \in \underline{X} \\ 0 & u \in U - \overline{X}. \end{cases}$$

Proof: If $u \in \overline{X} - \underline{X}$, then

$$\frac{\sum_{j=1}^{|[u]_R|-1} j}{|[u]_R|(|[u]_R|-1)} = 0.5. \qquad \square$$

Property 3.4 shows that the membership function of G_X^R coincides with the rough membership function proposed by Pawlak in [15].

Property 3.5 For any approximation space (U, R), we have

1. $G_U^R = U$,
2. $G_\phi^R = \phi$.

Proof: It follows immediately from property 3.4, because $\underline{U} = U$ and $U - \overline{\phi} = U$. $\square$

Property 3.6 For $X, Y \subseteq U$, if $X \subseteq Y$, then $G_X^R \subseteq G_Y^R$.

Proof: If $u \in \underline{X}$, then $[u]_R \subseteq \underline{X}$. Hence, $[u]_R \subseteq \underline{Y}$. Therefore, $G_Y^R(u) = 1 \geq G_X^R(u)$ for $u \in \underline{X}$.

If $u \in \overline{X} - \underline{X}$, then either $[u]_R \subseteq \overline{Y} - \underline{Y}$ or $[u]_R \subseteq \underline{Y}$. Hence, $u \in \overline{Y}$. By property 3.4, we have

$$\mu_{G_Y^R}(u) = \begin{cases} 0.5 & \text{if } u \in \overline{Y} - \underline{Y} \\ 1 & \text{if } u \in \underline{Y} \end{cases}$$

and

$$\mu_{G_X^R}(u) = 0.5 \quad \text{if } u \in \overline{X} - \underline{X}.$$

Therefore, $\mu_{G_Y^R}(u) \geq \mu_{G_X^R}(u)$ for $u \in \overline{X} - \underline{X}$.
If $x \in U - \overline{X}$, then $\mu_{G_Y^R}(u) \geq \mu_{G_X^R}(u) = 0$. $\square$

Property 3.7 For $X, Y \subseteq U$, the following holds:

1. $G_X^R \cup G_Y^R \subseteq G_{X\cup Y}^R$;
2. $G_X^R \cup G_Y^R = G_{X\cup Y}^R$ if $R(X) = R(Y)$ and either $X \subseteq Y$ or $Y \subseteq X$.

Proof: Since $X \subseteq X \cup Y$, by property 3.6, we have

$$G_X^R \subseteq G_{X\cup Y}^R. \tag{2}$$

Similarly, since $Y \subseteq X \cup Y$, we have

$$G_Y^R \subseteq G_{X\cup Y}^R. \tag{3}$$

Combining (2) and (3), we have

$$G_X^R \cup G_Y^R \subseteq G_{X\cup Y}^R.$$

If $R(X) = R(Y)$ and either $X \subseteq Y$ or $Y \subseteq X$, then either $R(X \cup Y) = R(Y)$ or $R(X \cup Y) = R(X)$. That is, $R(X \cup Y) = R(X) = R(Y)$. Hence,

$$G_X^R \cup G_Y^R = G_{X\cup Y}^R.$$ ❑

Property 3.8 For X, $Y \subseteq U$, the following holds:

1. $G_X^R \cap G_Y^R \supseteq G_{X\cap Y}^R$;
2. $G_X^R \cap G_Y^R = G_{X\cap Y}^R$ if $R(X) = R(Y)$ and either $X \subseteq Y$ or $Y \subseteq Y$.

Proof: Since $X \cap Y \subseteq X$, by property 3.6, we have

$$G_{X\cap Y}^R \subseteq G_X^R. \tag{4}$$

Similarly, since $Y \subseteq X \cup Y$, we have

$$G_{X\cap Y}^R \subseteq G_Y^R. \tag{5}$$

Combining (4) and (5), we have

$$G_{X\cap Y}^R \subseteq G_X^R \cap G_Y^R.$$

If $R(X) = R(Y)$ and either $X \subseteq Y$ or $Y \subseteq X$, then either $R(X \cap Y) = R(X)$ or $R(X \cap Y) = R(Y)$. That is $R(X \cap Y) = R(X) = R(Y)$. Hence,

$$G_X^R \cap G_Y^R = G_{X\cap Y}^R. \qquad \square$$

Property 3.9 For any approximation space (*U*, *R*),

$$\mu_{G_{X'}^R}(u) = \begin{cases} \mu_{G_X^R}(u), & \text{if } u \in \overline{X'} - \underline{X'} \\ 1 - \mu_{G_X^R}(u), & \text{otherwise} \end{cases},$$

where *X'* is the complement of *X*.

Proof: Since

$$\overline{X'} - \underline{X'} = (\underline{X})' - (\overline{X})' = U - \underline{X} - (U - \overline{X}) = \overline{X} - \underline{X},$$

by property 3.4, we have

$$\mu_{G_{X'}^R}(u) = \mu_{G_X^R}(u)$$

for $u \in \overline{X'} - \underline{X'}$.

Since $\underline{X'} = U - \overline{X}$, we have $\mu_{G_X^R}(u) = 0$ for $u \in \underline{X'}$. Hence,

$$\mu_{G_{X'}^R}(u) = 1 - \mu_{G_X^R}(u)$$

for $x \in \underline{X'}$.

Since $U - \overline{X'} = \underline{X}$, we have $\mu_{G_X^R}(u) = 1$ for $u \in U - \overline{X'}$. Hence,

$$\mu_{G_{X'}^R}(u) = 1 - \mu_{G_X^R}(u)$$

for $u \in U - \overline{X'}$. $\square$

The linear index of fuzziness of G_X^R and the quadratic index of fuzziness of G_X^R can be computed according to the following property:

Property 3.10 For any *X* in an approximation space (*U*, *R*), we have

$$(g_X^R)_l = \frac{\left|\overline{X} - \underline{X}\right|}{|U|}$$

and

$$(g_X^R)_q = \sqrt{\frac{\left|\overline{X} - \underline{X}\right|}{|U|}}.$$

That is,

$$(g_X^R)_l = \sqrt{(g_X^R)_q}.$$

Proof: Following definition 2.5 and property 3.4, we have

$$(g_X^R)_l = \frac{2}{|U|}\left|\overline{X} - \underline{X}\right| * 0.5 = \frac{\left|\overline{X} - \underline{X}\right|}{|U|}$$

and

$$(g_X^R)_q = \frac{2}{\sqrt{|U|}}\sqrt{\left|\overline{X} - \underline{X}\right| * \left(\frac{1}{2}\right)^2} = \sqrt{\frac{\left|\overline{X} - \underline{X}\right|}{|U|}}.$$ ❑

Property 3.10 shows that the fuzziness of $R(X)$ is proportional to the size of the boundary of $R(X)$, $|\overline{X} - \underline{X}|$. Moreover, the quadratic fuzziness of a rough set is the square of its linear fuzziness.

Example 3.3: Let (U, R) be an approximation space, where $U = \{1, 2, 3, 4, 5, 6\}$ and $R = \{\{1, 2\}, \{3, 4, 5, 6\}\}$. Let us consider a subset $X = \{1, 2, 3\}$ of U. Then the rough set of X is $R(X) = (\underline{X}, \overline{X})$, where $\underline{X} = \{1, 2\}$ and $\overline{X} = \{1, 2, 3, 4, 5, 6\}$. The boundary of $R(X)$ is $\{3, 4, 5, 6\}$. Therefore, the linear fuzziness in $R(X)$ is

$$\frac{\left|\overline{X} - \underline{X}\right|}{|U|} = \frac{4}{6} = 0.666667,$$

and the quadratic fuzziness in $R(X)$ is

$$\sqrt{\frac{\left|\overline{X} - \underline{X}\right|}{|U|}} = \sqrt{\frac{4}{6}} = 0.816497.$$

From property 3.11, we have the following:

Property 3.11 For any X in an approximation space (U, R), we have

$$(g_{X'}^R)_l = (g_X^R)_l$$

and

$$(g_{X'}^R)_q = (g_X^R)_q,$$

where X' is complement of X.

Property 3.11 says that the linear fuzziness and the quadratic fuzziness in the rough set of X are the same as those of X's complement. This can be demonstrated by the following example.

Example 3.4: Let (U, R) be an approximation space, where U = {1, 2, 3, 4, 5, 6} and R = {{1, 2}, {3, 4, 5, 6}}. Let us consider a subset X = {1, 2, 3} of U. Then the rough set of X is $R(X) = (\underline{X}, \overline{X})$, where $\underline{X}$ = {1, 2} and $\overline{X}$ = {1, 2, 3, 4, 5, 6}. The boundary of $R(X)$ is {3, 4, 5, 6}. The complement of X is

X' = {4, 5, 6}.

Since $\underline{X'} = \phi$ and $\overline{X'}$ = {3, 4, 5, 6}, the boundary of $R(X')$ is

{3, 4, 5, 6}.

Therefore, the linear fuzziness in $R(X')$ is

$$\frac{\left|\overline{X'} - \underline{X'}\right|}{|U|} = \frac{4}{6} = 0.666667,$$

and the quadratic fuzziness in $R(X')$ is

$$\sqrt{\frac{\left|\overline{X'} - \underline{X'}\right|}{|U|}} = \sqrt{\frac{4}{6}} = 0.816497.$$

4. CONCLUSIONS

Rough set theory has been considered as a useful means of modelling vagueness and has been successfully applied in many fields. Every rough set is associated with some amount of fuzziness. In this paper, we have proposed a new measure of fuzziness of rough sets. Our measure prevents the problems found in the measure proposed by Chakrabarty *et al*. For a rough set, starting with different members in that rough set, we have the same fuzziness with our measure. In contrast to Chakrabarty *et al*.'s measure which depends on the chosen member of the rough set, the fuzziness of the rough set obtained with our measure depends on the boundary of the rough set. Moreover, the fuzziness of a rough set can computed based on its boundary.

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