A Near-Quadratic Algorithm for the Alpha-Connected Two-Center Problem*

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Given a set S of n points in the plane and a constant α, the alpha-connected two-center problem is to find two congruent closed disks of the smallest radius covering S, such that the distance of the two centers is at most $2(1 - \alpha)r$. We present an $O(n^2 \log^2 n)$ expected-time algorithm for this problem, improving substantially the previous $O(n^5)$-time solution. The algorithm translates the alpha-connected two-center problem into a distance problem between two circular hulls.

Keywords: computational geometry, algorithm, p-center problem, alpha-connected two-center problem, circular hull

1. INTRODUCTION

Given n demand points in the plane, the *Euclidean p-center problem* is to find p supply points (anywhere in the plane) so as to minimize the maximum distance from a demand point to its respective nearest supply point. This problem has many important applications such as transportation and locating stations and facilities [5, 6, 17]. When parameter p is a part of the input, the problem has been proved to be *NP-complete* [21]; so researchers are interested in finding algorithms which solve the p-center problem in polynomial time, for any fixed p. A recent best result in this direction, presented in [16], is an $O(n^{O(\sqrt{p})})$-time algorithm for solving the p-center problem. The *one-center problem* is an extreme case of particular interest, which can be solved in $O(n)$ time [20]. The next easiest case, the *two-center problem*, can be viewed less abstractly as one of covering a set of points in the plane by two congruent closed disks, in such a way as to minimize the radius of the covering disks. Although the two-center problem is a special case

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of the p-center problem, researchers took a long time to find an efficient algorithm [1, 3, 5, 6, 9, 10, 14, 17, 18, 23]. In a major breakthrough, Sharir [23] gave an $O(n \log^9 n)$-time algorithm. And shortly after the announcement of Sharir's result, Eppstein [10] improved it with a randomized algorithm running in $O(n \log^2 n)$ expected time.

In [15], we study the *alpha-connected two-center decision problem*: Given a set S of n points in the plane, a radius r and a constant α, determine whether S can be covered by two congruent closed disks of radius r, such that the distance of the two centers is at most $2(1 - \alpha)r$. An $O(n^2 \log^2 n)$-time algorithm for solving this problem is also presented.

This paper considers the *alpha-connected two-center problem* (abbreviated to an *α-C2C problem*) which is a variant of the two-center problem and is the optimization version (find two disks of the smallest radius) of the alpha-connected two-center decision problem. One motivation of the alpha-connected two-center problem is the establishment of medical centers. Consider the problem of placing two emergency medical units in an area of n given sites, such that the worst-case response distance to any site does not exceed r. This problem is just the ordinary planar two-center problem in which the two centers are wholly independent. However, in practice, one unit may request blood or technical support from the other one for emergency surgical operations. Accordingly, limiting the distance of the two units may be advantageous. Another application of the alpha-connected two-center problem is designing the backbone of a hierarchical radio network such that the maximum transmission power used by the backbone is minimized and the service stations should be connected for exchanging control and data packets [13].

The *connected two centers problem* presented by Huang [13] is a special case of the alpha-connected two-center problem. In their case, the two centers are separated just by distance r (i.e., $\alpha = 1/2$). Huang had proposed an $O(n^5)$-time algorithm to solve the problem. The main contribution of our work is a randomized algorithm which solves the alpha-connected two-center problem in $O(n^2 \log^2 n)$ time, improving substantially the previous result.

2. PRELIMINARIES AND PROBLEMS

We begin with notations, most of which were noted in the earlier paper [15]. Given two points p and q in the plane, we denote the Euclidean distance between the two points by $d(p, q)$. Given a point p in the plane and a radius r, we denote the circle (disk) with center p and radius r by $C(p, r)$ ($D(p, r)$). A pair of circles $C(p, r)$ and $C(q, r)$ that intersect each other, $d(p, q) \leq 2r$, is called a *connected two-circle* and is represented by $C2C(p, q, r)$. Given a constant α, a pair of circles $C(p, r)$ and $C(q, r)$ is called *alpha-connected* and denoted as $\alpha\text{-}C2C(p, q, r)$ if the distance between the two centers, $d(p, q)$, is at most $2(1 - \alpha)r$.

We now formulate the problem as follows.

Problem: ALPHA-CONNECTED TWO-CENTER PROBLEM. Given a set S of n points in the plane and a constant α, the *alpha-connected two-center problem* is to find two congruent closed *alpha-connected* disks of the smallest radius covering S.

Next we introduce the *constrained alpha-connected two-center problem* from which our algorithm solves the alpha-connected two-center problem.

Definition 1 Given a set S of n points in the plane, a *linear partition* (S_1, S_2) is a pair of subsets, S_1 and S_2, of S, such that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S$ and S_1 and S_2 are separated by a straight line.

For an n-point set, there are $O(n^2)$ possible linear partitions.

Problem: CONSTRAINED ALPHA-CONNECTED TWO-CENTER PROBLEM. Given a linear partition (S_1, S_2) of a point set S and a constant α, the *constrained alpha-connected two-center problem* is to find an α-connected two-circle α-$C2C(p, q, r)$ of the smallest radius, such that S_1 and S_2 can be covered by $C(p, r)$ and $C(q, r)$, respectively.

Theorem 1 Given a set of points S in the plane and a constant α, there exists a linear partition (S_1, S_2) such that the optimal solution to the *constrained alpha-connected two-center problem* is the same as that of the *alpha-connected two-center problem*.

Proof: Similar to that of Theorem 1 in [15]. ❑

An immediate consequence of this theorem is that the alpha-connected two-center problem can be solved by solving all the $O(n^2)$ corresponding constrained alpha-connected two-center problem (each with a different linear partition).

3. SOLVING THE ALPHA-CONNECTED TWO-CENTER PROBLEM

This section presents a near quadratic algorithm for the alpha-connected two-center problem. The key idea behind the algorithm is to transform the constrained alpha-connected two-center problem into a distance problem between two circular hulls of variable size. Before proceeding with our algorithm, some definitions are needed.

Definition 2 Let S be a set of points in the plane. The *circular hull* of S with radius r, denoted as $CIRH(S, r)$, is the intersection of all the radius-r disks centered at the points of S, i.e., $CIRH(S, r) = \cap_{p \in S} D(p, r)$.

Intuitively, a circular hull is like a convex hull except that its boundary consists of arcs of circles. As an example, Fig. 1 shows the circular hull $CIRH(S, r)$ of the 4-point set $S = \{a, b, c, d\}$. Circular hull has been well-defined and extensively used in solving numerous geometric problems [2, 7, 8, 10-12, 15]. We find it convenient to redefine it as above.

In [15], we have shown the close relationships between the circular hull and the farthest point Voronoi diagram of a point set: For a point set $S = \{s_1, s_2, \ldots, s_n\}$ in the plane, the farthest point Voronoi diagram of S is well-defined and fixed. Within each Voronoi region $V(s_i)$ of $FPVD(S)$, $s_i \in S$, $\partial CIRH(S, r)$ is a *minor arc* of a radius-r circle centered at s_i. As an example, Fig. 2 shows the circular hull and the farthest point Voronoi diagram

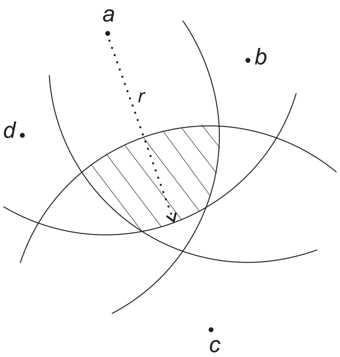

Fig. 1. The circular hull of the point set $S = \{a, b, c, d\}$.

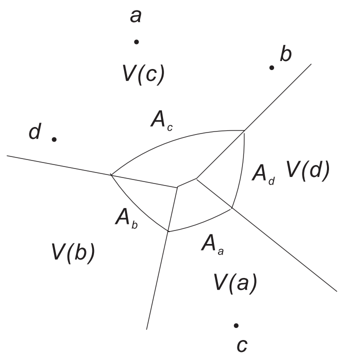

Fig. 2. The circular hull and the farthest-point Voronoi diagram of the point set $S = \{a, b, c, d\}$.

of the point set $S = \{a, b, c, d\}$, where arcs A_a, A_b, A_c and A_d are within Voronoi regions $V(a)$, $V(b)$, $V(c)$ and $V(d)$, and both are associated with the *defining points* (farthest neighbors in S) a, b, c and d, respectively. We have shown that $\partial CIRH(S, r)$ is a circular list of arcs of radius-r circles centered at the associated farthest neighbors in S (Lemma 4 in [15]), thus the $CIRH(S, r)$ can be described with a circular list of the defining points and the radius r.

Here we would like to point out that if we gradually increase the radius r, a *critical radius* (the radius that $CIRH(S, r)$ undergoes a combinatorial change) will occur while the circular hull intersects with a vertex of the farthest point Voronoi diagram. Thus, we can find all the critical radii of the circular hull by tracing along the edges of the farthest point Voronoi diagram. Since the Voronoi diagram of an n-point set has at most $3n - 6$ edges [22], the number of critical radii is $O(n)$. Then, sorting the values of the critical radii will lead to a *critical radius sequence*.

Definition 3 The *distance* between two circular hulls, $CIRH(S_1, r)$ and $CIRH(S_2, r)$, is defined as $\min\{d(p, q) \mid p \in CIRH(S_1, r), q \in CIRH(S_2, r)\}$; p and q are called the *nearest points of the circular hulls*. If $CIRH(S_1, r)$ and $CIRH(S_2, r)$ intersect each other, then $p = q$ may be any point in the intersection of the two circular hulls.

Notably, it is obvious that p and q must lie on the boundaries of the circular hulls.

Theorem 2 If r^* is the minimum radius such that the distance between $CIRH(S_1, r^*)$ and $CIRH(S_2, r^*)$ is at most $2(1 - \alpha)r^*$, then there exists an α-connected two-circle α-$C2C(p, q, r^*)$ that is a solution to the constrained α-connected two-center problem with linear partition (S_1, S_2).

Proof: Similar to that of Theorem 2 in [15]. ❑

Theorem 2 involves the idea of solving the constrained α-connected two-center problem by solving the distance problem between two circular hulls. However, since the shapes of the two circular hulls may vary with the value of r, how to find the minimum radius r^* of the optimal solution from the two non-fixed-shape circular hulls becomes the key problem.

Given a linear partition (S_1, S_2) of a point set S and the radius r, the two circular hulls $CIRH(S_1, r)$ and $CIRH(S_2, r)$ are well defined. Since circular hulls obey *convexity* and *unimodality* like convex polygons, we can use Chin's algorithm [4] to compute the nearest points of $CIRH(S_1, r)$ and $CIRH(S_2, r)$.

We now describe the algorithm for solving the constrained alpha-connected two-center problem. Let S be a set of points in the plane. Given a linear partition (S_1, S_2) of S and a constant α, the *constrained alpha-connected two-center algorithm* consists of the following steps.

We first construct the farthest point Voronoi diagrams of S_1 and S_2 and the critical radius sequences, R_1 and R_2, of these two diagrams. This step takes $O(n \log n)$ time. Next we sort R_1 and R_2 into one critical radius sequence R by *merge sort* (linear time) and apply *binary search* on R to find an interval, $[r_{low}, r_{high}]$, containing the minimum radius r^* of the optimal solution, where r_{low} and r_{high} are two consecutive elements in R. The interval containing r^* implies that we can not find an α-connected two-circle covering S_1 and S_2 and with a radius $r < r_{low}$; and for radius $r > r_{high}$, we can always find an α-connected two-circle that covers S_1 and S_2. Each search takes $O(n)$ time for constructing the two circular hulls [2, 7, 8, 10-12] and $O(\log n)$ time for finding their nearest points [4]. This step can thus be executed in totally $O(n \log n)$ time. Notably, r_{high} may not be the minimum radius of the optimal solution to the constrained alpha-connected two-center problem, the algorithm thus has to find the minimum radius r^* in the range (r_{low}, r_{high}). Fortunately, *parametric search* [19] is easily applicable here to find r^* in $O(\log^2 n)$ time, and this completes our algorithm.

Theorem 3 The constrained alpha-connected two-center algorithm takes $O(n \log n)$ time.

Based on the above algorithm, we can solve the alpha-connected two-center problem by a randomized approach like the one in the appendix of [3]: Set $r = \infty$. For a fixed point, says s_i, in S, we generate all $O(n)$ linear partitions by a rotational line sweep around s_i. Let U_{ij} be the set of points in S that is below the line passing through s_i and s_j (including s_i and s_j); let V_{ij} be the set of points in S strictly above this line. Then we check all the linear partitions (U_{ij}, V_{ij}), for $j = 1, \ldots, n$, in the order generated by the rotational line sweep, to test whether both $CIRH(U_{ij}, r)$ and $CIRH(V_{ij}, r)$ have nonempty interiors and the distance between $CIRH(U_{ij}, r)$ and $CIRH(V_{ij}, r)$ is at most $2(1 - \alpha)r$. If so, apply the constrained alpha-connected two-center algorithm to evaluate the smallest radius r_{ij} for this constrained alpha-connected two-center problem with linear partition (U_{ij}, V_{ij}). Let $r_i = \min_{1 \le j \le n} r_{ij}$. If $r_i < r$, reset $r = r_i$ and do the next repetition. The next i should be chosen in random order.

It is clear that this algorithm computes the optimal solution r^* of the alpha-connected two-center problem. For a linear partition (U, V) of S, constructing the two circular hulls and checking whether both have nonempty interiors takes $O(n)$ time [2, 7, 8, 10-12]; computing the distance of two circular hulls and testing whether it is at most $2(1 - \alpha)r$ takes $O(\log n)$ time [4]. As a result, for a fixed point s_i in S and the $O(n)$ linear partitions corresponding to the line passing through s_i, the test would take $O(n^2)$ time. Fortunately, we can use the offline data structure of Hershberger and Suri to maintain the circular hulls $CIRH(U_{ij}, r)$ and $CIRH(V_{ij}, r)$ as in [3]. Thus, this part can be improved to $O(n \log n)$ time, and the overall running time in the testing part of this algorithm is $O(n^2 \log n)$. On the other hand, each r_i can be evaluated in $O(n^2 \log n)$ time by n applications of Theorem 3. Let t be the number of indices i for which r_i is evaluated by this procedure. The expected value of t is bounded by the nth harmonic number $H_n = O(\log n)$ [3]. It follows that the algorithm runs in $O(n^2 \log^2 n)$ expected time.

Theorem 4 The alpha-connected two-center problem, for a set of n points in the plane, can be solved in $O(n^2 \log^2 n)$ expected time.

4. CONCLUSIONS

In this paper, we study the alpha-connected two-center problem which is a variant of the planar two-center problem and is the optimization version of the alpha-connected two-center decision problem. In addition, we have presented an algorithm for solving this problem in $O(n^2 \log^2 n)$ expected time, improving substantially the previous best result.

A remaining open problem is whether a near-linear algorithm is possible. In addition, solving this problem in higher dimensions may have more practical value and hence be a good direction.

REFERENCES

1. P. Agarwal and M. Sharir, "Planar geometric location problems," *Algorithmica*, Vol. 11, 1994, pp. 185-195.
2. M. de Berg, P. Bose, D. Bremner, S. Ramaswami, and G. Wilfong, "Computing constrained minimum-width annuli of point sets," *Computer-Aided Design*, Vol. 30, 1998, pp. 267-275.
3. T. M. Chan, "More planar two-center algorithms," *Computational Geometry: Theory and Applications*, Vol. 13, 1999, pp. 189-198.
4. F. Chin and C. A. Wang, "Optimal algorithms for intersection and minimum distance problems between planar polygons," *IEEE Transactions on Computers*, Vol. C-32, 1983, pp. 1203-1207.
5. Z. Drezner, *Facility Location: A Survey of Applications and Methods*, Springer Verlag, New York, 1995.
6. Z. Drezner, "The planer two-center and two-median problems," *Transportation Science*, Vol. 18, 1984, pp. 351-361.
7. H. Edelsbrunner, D. G. Kirkpatrick, and R. Seidel, "On the shape of a set of points in the plane," *IEEE Transactions on Information Theory*, Vol. IT-29, 1983, pp. 551-559.
8. H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, *EATCS Monographs on Theoretical Computer Science*, V.10, Springer Verlag, New York, 1987.
9. D. Eppstein, "Dynamic three-dimensional linear programming," *ORSA Journal on Computing*, Vol. 4, 1992, pp. 360-368.

10. D. Eppstein, "Faster construction of planar two-centers," in *Proceedings of the 8th ACM-SIAM Symposium on Discrete Algorithms*, 1997, pp. 131-138.
11. J. Hershberger, "A faster algorithm for the two-center decision problem," *Information Processing Letters*, Vol. 47, 1993, pp. 23-29.
12. J. Hershberger and S. Suri, "Finding tailored partitions," *Journal of Algorithms*, Vol. 12, 1991, pp. 431-463.
13. C. H. Huang, "Some problems on radius-weighted model of packet radio network," Ph. D. Dissertation, Department of Computer Science, Tsing Hua University, Taiwan, 1992.
14. N. F. Huang, C. H. Huang, and Y. L. Wang, "A sweepline algorithm to solve the two-center problem," *Information Processing Letters*, Vol. 49, 1994, pp. 171-177.
15. P. H. Huang, Y. T. Tsai, and C. Y. Tang, "A fast algorithm for the alpha-connected two-center decision problem," *Information Processing Letters*, Vol. 85, 2003, pp. 205-210.
16. R. Z. Hwang, R. C. T. Lee, and R. C. Chang, "The slab dividing approach to solve the euclidean P-center problem," *Algorithmica*, Vol. 9, 1993, pp. 1-22.
17. J. Jaromczyk and M. Kowaluk, "An efficient algorithm for the Euclidean two-center problem," in *Proceedings of the 10th ACM Symposium on Computational Geometry*, 1994, pp. 303-311.
18. M. Katz and M. Sharir, "An expander-based approach to geometric optimization," in *Proceedings of the 9th ACM Symposium on Computational Geometry*, 1993, pp. 198-207.
19. N. Meggido, "Applying parallel computation algorithms in the design of serial algorithms," *Journal of the ACM*, Vol. 30, 1983, pp. 852-865.
20. N. Megiddo, "Linear-time algorithms for linear programming in R^3 and related problems," *SIAM Journal of Computing*, Vol. 12, 1983, pp. 759-776.
21. N. Megiddo and K. Supowit, "On the complexity of some common geometric location problems," *SIAM Journal of Computing*, Vol. 13, 1984, pp. 1182-1196.
22. F. P. Preparata and M. I. Shamos, *Computational Geometry: An Introduction*, Springer Verlag, New York, 1985.
23. M. Sharir, "A near-linear algorithm for the planar 2-center problem," in *Proceedings of the 12th ACM Symposium on Computational Geometry*, 1996, pp. 106-112.

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