Short Paper

**Tunable Bounding Volumes for Elliptic Paraboloids**

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The tightness of the bounding volume is often difficult to adjust to suit different applications. In this paper, we present a technique to derive a tunable bounding volume for elliptic paraboloids, where the tightness can easily be controlled and altered at several levels. Our technique develops such a tunable bounding volume through the optimization process. Bounding volumes thus developed contain the minimal volume at the corresponding level. We implement a geometric application for the elliptic paraboloid, and analyze the tightness of the bounding volumes at different levels. Finally, we demonstrate the feasibility of using our technique in creating new types of geometries as well as several rendered images.

*Keywords:* tunable bounding volumes, geometric applications, elliptic paraboloids, modeling, three-dimensional objects

1. INTRODUCTION

Bounding volumes are often used to improve the computing efficiency in many application domains [1, 2, 4-8]. The basic idea behind the bounding volume is to derive a simple, three-dimensional volume that encloses the original geometry. When enquiring into the original geometry, we explore the bounding volume first, which usually requires much less computationally expensive calculations. In global illumination algorithms, such as Monte Carlo path tracing [3], for example, we can construct a volume which contains a given complex object and permit a simpler ray intersection check than the object. Only if a ray intersects the bounding volume does the object itself need to be checked for intersection. To improve the computing efficiency, a compromise must be reached between the tightness of a bounding volume and the costs of ray-volume intersections. As a result, an object in this kind of applications normally has a single corresponding bounding volume, and this volume need not to be the tightest closure containing the minimal volume of the original geometry.

In this paper, we present the tunable bounding volumes for elliptic paraboloids. To realize the tunable features, an integer is employed to represent the vertex numbers of a bounding volume. Given several integers for different levels, our technique, through the

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optimization process, produces corresponding bounding volumes containing minimal volumes. Finally, we demonstrate the feasibility of our technique in creating new types of geometries.

This paper is organized as follows. Section 2 presents in details the proposed technique. We present a geometric application of our technique in section 3, followed by conclusions and future work in section 4.

2. THE PROPOSED TECHNIQUE

This section presents the approach to determine a tunable bounding volume for an elliptic paraboloid. First, we define the mathematical representation of an elliptic paraboloid. We then describe how to derive desired bounding volumes containing different vertex numbers. We present a tunable parameter to control the vertex numbers. This parameter, in turn, makes it possible to control the tightness of the bounding volume.

2.1 Definition and Assumptions

An elliptic paraboloid is a geometry containing a base and an apex with a finite height. Dissimilar to the paraboloid, the cross sectional view of this geometry is an ellipse with different major and minor radii, depending on its height. This geometry has been used to represent warheads and lampshades in different applications. Without lose of generality, we can define an elliptic paraboloid as a surface in three-dimensional space as shown in Fig. 1 (a). The mathematical representation of an elliptic paraboloid is shown in Eq. (1), where \( X \in [-a, a] \), \( Y \in [-b, b] \), and \( Z \in [0, h] \).

\[
\frac{Z}{h} = \frac{X^2}{a^2} + \frac{Y^2}{b^2}
\] (1)

Fig. 1. (a) Illustration of an elliptic paraboloid; (b) A cross section view with the length of semi-major axis \( a' \) and length of semi-minor axis \( b' \).

In this equation, \( h \) is the height of the elliptic paraboloid, \( a \) is the length of the semi-major axis at \( Z = h \), and \( b \) is the length of the semi-minor axis at \( Z = h \). The elliptic
paraboloid is orthogonal in the XYZ coordinate system, as shown in Fig. 1 (a). When the elliptic paraboloid is not orthogonal, we can always apply a transformation matrix containing translation and rotation vectors to produce an orthogonal one.

For an arbitrary point \( P(x, y, z) \) on an elliptic paraboloid, we can find an ellipse centered at \( C(0, 0, h') \) as shown in Fig. 1 (b). Clearly, this ellipse is the orthogonal cross section of the elliptic paraboloid passing through \( P(x, y, z) \). In this ellipse, we further define \( a' \) as the length of semi-major axis, \( b' \) as the length of semi-minor axis \((0 < a' \leq a \) and \( 0 < b' \leq b)\), and \( w \) as the distance between \( P(x, y, z) \) and \( C(0, 0, h') \). Finally, let \( \theta \) be the angle between the vector \( CP \) and the unit X-axis vector. We can represent a point \( P(x, y, z) \) on an elliptic paraboloid using the notations given above, as shown in Eqs. (2), (3) and (4).

\[
P(x, y, z) = (w \cos \theta, w \sin \theta, h')
\]

\[
w = \frac{a' b'}{\sqrt{a'^2 \sin^2 \theta + b'^2 \cos^2 \theta}}
\]

\[
a' = \frac{h'}{h - a}, \quad b' = \frac{h'}{h - b}
\]

### 2.2 A Technique for Developing Tunable Bounding Volume

This section illustrates how to derive a tunable bounding volume that can be controlled by an integer to determine its tightness with respect to the elliptic paraboloid. Note that since the cross section of the elliptic paraboloid is an ellipse, we can first consider a bounding “volume” for an ellipse. We utilize a \( 2n \)-sided polygon to enclose the ellipse, where \( n (n \geq 2) \) represents the vertex number. Surely, \( n \) is a tunable value that can control the tightness. Note that we can also impose a \( 2n \)-sided polygon in some specific cross sections along the height of the paraboloid.

Eventually, this will result in a \( 2n \)-sided polygon at multiple layers. When connecting vertices at each layer together, we can construct a polyhedron that is a tight bounding volume as desired. In this paper, we employ three layers (top, middle and bottom), which is a compromise between the vertex numbers and the tightness. Each layer has the same numbers of vertices. We derive several tunable bounding volumes in the following sections.

#### 2.2.1 8-Vertex cuboid (2-layer)

The most common bounding volume for an elliptic paraboloid is the 8-Vertex cuboid \((n = 2 \) at two layers). This cuboid has two same rectangles at the top and bottom layer. Thus, the volume of the 8-Vertex cuboid is \( 4abh \).

#### 2.2.2 8-Vertex frustum (2-layer)

An 8-Vertex frustum is the second bounding volume we derive. Dissimilar to an 8-Vertex cuboid, the top and the bottom layers contain rectangles with different lengths
and widths, as shown in Figs. 2 (a) and (b). The crucial issue is thus to determine four planes tangent to the surface of the elliptic paraboloid so that the constructed 8-Vertex frustum has the minimal volume.

The frustum with the minimal volume can be derived by considering the cross section of the elliptic paraboloid in the X-Z and Y-Z plane. Let \( P_a(x_0, z_0) \) and \( P_b(y_0, z_0) \) be two points tangent to the surface of the elliptic paraboloid in the X-Z and Y-Z plane, respectively. Along each tangent line, accordingly, we can find \( T_a(x_2, z_2) \) at the top layer, \( B_a(x_3, z_3) \) at the bottom layer, and \( C_a(x_1, z_1) \) on the plane \( X = 0 \). Their coordinates can be represented in terms of \( x_0 \) as below:

\[
P_a(x_0, z_0) = (x_0, \frac{h}{a^2} x_0^2), \quad C_a(x_1, z_1) = (0, -\frac{h}{a^2} x_0^2), \quad T_a(x_2, z_2) = \left( \frac{\frac{x_0^2}{2} + \frac{a^2}{4}}, h \right), \quad \text{and} \quad B_a(x_3, z_3) = \left( \frac{x_0}{2}, 0 \right).
\]

Similarly, we can represent the coordinates for \( P_b, C_b, T_b \), and \( B_b \) in terms of \( y_0 \). Immediately, we have \( y_0 = \frac{(b/a)x_0} \) since \( C_a(x_1, z_1) \) and \( C_b(y_1, z_1) \) must have the same Z coordinates. Let \( F_8 \) represent the volume of the 8-Vertex frustum. \( F_8 \) can be determined by deducting the volume of the cap from that of the pyramid. After some calculations, it can be expressed in terms of \( x_0 \) as \( \frac{bh}{a} x_0^2 + abh + \frac{a^2bh}{3x_0} \).
Once the expression of $V_8$ is available, we can determine a bounding volume using the optimization process to minimize $V_8$. We differentiate $V_8$ with respect to $x_0$ so as to determine the extreme. When $x_0 = 3 - 0.25a$, $V_8$ has the minimal volume. Obviously, once $x_0$ is available, all of the vertices in an 8-Vertex frustum with some further calculations.

### 2.2.3 12-Vertex frustum (3-layer)

A 12-Vertex frustum is the third bounding volume we derive, as shown in Figs. 2 (c) and (d). It has three layers, each of which contains four vertices, forming a rectangle with different lengths and widths.

Dissimilar to the 8-Vertex frustum, the length and width of the top and the middle rectangles are $2a$ and $2b$, respectively. Again, we need to determine four tangent planes around the surface of the elliptic paraboloid so that it has the minimal volume.

Similar to the case we derive for the 8-Vertex frustum, we assume a line is tangent to paraboloid at $P_a$ and this tangent line intersects the X-Y plane at $B_a(x_1, z_1)$, and the Y-Z plane at $C_a(x_3, z_3)$, respectively. Furthermore, $M_a(x_2, z_2)$ is the intersection point of the above tangent line and the plane parallel to the Y-Z plane and passing through $T_a(x_4, z_4)$. Again, the coordinates for $P_a, B_a, M_a, C_a,$ and $T_a$ can be represented in terms of $x_0$, where

$$P_a(x_0, z_0) = (x_0, \frac{h}{a^2} x_0^2), \quad B_a(x_1, z_1) = (\frac{x_0}{2}, 0), \quad M_a(x_2, z_2) = (a, \frac{2h}{a^2} x_0(a - x_0) + \frac{h}{a^2} x_0^2),$$

$$C_a(x_3, z_3) = (0, -\frac{h}{a^2} x_0^2), \quad T_a(x_4, z_4) = (a, h).$$

Similarly, in the Y-Z plane, the coordinates for $P_b, B_b, M_b, C_b,$ and $T_b$ can be represented in terms of $y_0$. Once again, we obtain the following relation: $y_0 = (b/a)x_0$.

Now, let $V_{12}$ represent the volume of the 12-Vertex frustum. This volume can be calculated by adding the volume of the cuboid on the top part and the volume of the 8-Vertex frustum on the middle part. As a result, $V_{12}$ can be expressed in terms of $x_0$ as

$$V_{12} = 4abh - 16hh x_0 + \frac{4bh}{a} x_0^3 - \frac{bh}{3a} x_0^4.$$  

We differentiate $V_{12}$ with respect to $x_0$ so as to determine the extreme. As a result, when $x_0 = -2\sqrt{2}a\cos(17\pi/12)$, $V_{12}$ has the minimal volume.

### 2.2.4 16-Vertex frustum (2-layer) and 24-Vertex frustum (3-layer)

For the 16-Vertex frustum and the 24-Vertex frustum, they are an extension of the 8-Vertex frustum and 12-Vertex frustum separately, where the top and the bottom layers are octagons, instead of rectangles, with different lengths and widths. Note that apart from the vertex of the paraboloid, there are 8 (16) tangent points in the perimeters of the 16-Vertex frustum (the 24-Vertex frustum). Clearly, a 16-Vertex frustum (24-Vertex frustum) is tighter than a 8-Vertex frustum (12-Vertex frustum).

Table 1 shows the features of the four minimal bounding volumes for an elliptic paraboloid. The second column lists the X coordinates of the crucial tangent point $P_a(x_0, z_0)$ in various types of frustum (see Fig. 2). As expected, both 8-Vertex and 16-Vertex frustums have similar X coordinates at $P_a(x_0 = 3 - 0.25a)$, while 12-Vertex and 24-Vertex
Table 1. Features of four minimal bounding polyhedra for the elliptic paraboloid.

<table>
<thead>
<tr>
<th>Frustum Type</th>
<th>The X Coordinate of ( P_{a}(x_0, z_0) )</th>
<th>Frustum Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-Vertex</td>
<td>( x_0 = 3^{0.25} a )</td>
<td>( bh \frac{z}{a} + abh + \frac{a^3 bh}{3x^3} )</td>
</tr>
<tr>
<td>12-Vertex</td>
<td>( x_0 = -2\sqrt{2} a \cos(17\pi/12) )</td>
<td>( 4abh - \frac{16bh}{3} x_0 + \frac{4bh}{a} \frac{1}{3} )</td>
</tr>
<tr>
<td>16-Vertex</td>
<td>( x_0 = 3^{0.25} a )</td>
<td>( (2\sqrt{2} - 2)(\frac{bh}{a} \frac{1}{3} + abh + \frac{a^3 bh}{3x^3}) )</td>
</tr>
<tr>
<td>24-Vertex</td>
<td>( x_0 = -2\sqrt{2} a \cos(17\pi/12) )</td>
<td>( (2\sqrt{2} - 2)(4abh - \frac{16bh}{3} x_0 + \frac{4bh}{a} \frac{1}{3} )</td>
</tr>
</tbody>
</table>

Table 2. The comparison of volume ratios for different bounding volumes.

<table>
<thead>
<tr>
<th>Frustum Type</th>
<th>Bounding Volume</th>
<th>Volume Ratio (while ( n \to \infty ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-Layer Cuboid</td>
<td>( 2^n ab \tan(\frac{\pi}{2^n}) )</td>
<td>2</td>
</tr>
<tr>
<td>2-Layer Frustum</td>
<td>( (2^n \tan(\frac{\pi}{2^n})) \frac{2\sqrt{3} + 3}{12} )</td>
<td>1.07735</td>
</tr>
<tr>
<td>3-Layer Frustum</td>
<td>( (2^n \tan(\frac{\pi}{2^n}))(1 + \frac{8\sqrt{2}}{3} \cos \theta + 8\cos^2 \theta - \frac{16\cos^3 \theta}{3} )abh, \theta = 17\pi/12 )</td>
<td>1.07180</td>
</tr>
</tbody>
</table>

frustums have corresponding X coordinates at \( P_a(x_0 = -2\sqrt{2} a \cos(17\pi/12)) \). Similarly, the volume of the 16-Vertex frustum is \( 2\sqrt{2} - 2 \) times that of the 8-Vertex frustum, and a similar relationship can be observed between the 12-Vertex and 24-Vertex frustum. Clearly, a 24-Vertex frustum is tighter than a 12-Vertex one.

We define the volume ratio as the tunable bounding volume over the volume of the elliptic paraboloid when \( n \) approaches infinity. Clearly, the smaller the volume ratio is, the tighter the bounding volume is. Table 2 shows a comparison of volume ratios for different tunable bounding volumes. In the 2-Layer cuboid or frustum, vertex numbers are 8, 16, and 24. However, in a 3-Layer frustum, vertex numbers are 12, 24, and 36. A comparison shows that the 3-Layer frustum has the minimal volume ratio, while it also has the largest numbers of vertices. The volume ratio also indicates that when using the 3-Layer frustum, the minimum volume difference is 7.1 %. Obviously, a 3-Layer frustum is tighter than a 2-Layer frustum.

2.3 The Algorithm

The pseudo code of our technique is listed below. For simplicity, we present this code separately for cases of 2-Layer and 3-Layer. The basic idea behind these procedures is to determine the coordinates of the vertices. This can be done by determining the coordinates of the first vertex and then applying rotation transformation with respect to an angle parameterized by the tunable variable \( n \) to find the coordinates of the other vertices.
**Procedure1:** Compute the 2-Layer bounding frustum of an elliptic paraboloid.

**Input:** An ellipse, height \( h \), length of semi-major axis \( a \) and length of semi-minor axis \( b \). Tunable vertex numbers \( 2^n \) for the bounding volume.

**Output:** Vertex coordinates of the minimal bounding volume.

**Procedure2:** Compute the 3-Layer bounding frustum of an elliptic paraboloid.

**Input:** An ellipse, height \( h \), length of semi-major axis \( a \) and length of semi-minor axis \( b \). Tunable vertex numbers \( 2^n \) for the bounding volume.

**Output:** Vertex coordinates of the minimal bounding volume.

**Procedure1 2_Layer_Frustrum_Coordinates**

\[(a, b, n, h)\]

\( n \) is an integer and \( n \geq 2 \)

\( x_i = 3 \cdot a \)

\( y_i = 3 \cdot b \)

For \( i = 1 \) to \( 2^n \)

\[ B_x(i) = \frac{x_i}{2} \cdot \cos\left(\frac{\pi}{2^n} \cdot i\right) \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ B_y(i) = \frac{y_i}{2} \cdot \sin\left(\frac{\pi}{2^n} \cdot i\right) \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ B_z(i) = 0 \]

\[ T_x(i) = \frac{a^2 + a^2 \cdot \cos\left(\frac{\pi}{2^n} \cdot i\right)}{2y_i} \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ T_y(i) = \frac{a^2 + b^2 \cdot \sin\left(\frac{\pi}{2^n} \cdot i\right)}{2y_i} \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ T_z(i) = h \]

**Procedure2 3_Layer_Frustrum_Coordinates**

\[(a, b, n, h)\]

\( n \) is an integer and \( n \geq 2 \)

\( x_0 = -2 \sqrt{3} \cdot a \cdot \cos(17\pi/12) \)

\( y_0 = -2 \sqrt{3} \cdot b \cdot \cos(17\pi/12) \)

For \( i = 1 \) to \( 2^n \)

\[ B_x(i) = \frac{x_i}{2} \cdot \cos\left(\frac{\pi}{2^n} \cdot i\right) \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ B_y(i) = \frac{y_i}{2} \cdot \sin\left(\frac{\pi}{2^n} \cdot i\right) \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ B_z(i) = 0 \]

\[ M_x(i) = a \cdot \cos\left(\frac{\pi}{2^n} \cdot i\right) \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ M_y(i) = b \cdot \sin\left(\frac{\pi}{2^n} \cdot i\right) \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ M_z(i) = \frac{2h_i}{a} \cdot h - \frac{h_i}{a} \cdot x_i \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ T_x(i) = a \cdot \cos\left(\frac{\pi}{2^n} \cdot i\right) \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ T_y(i) = b \cdot \sin\left(\frac{\pi}{2^n} \cdot i\right) \cdot \left\lfloor \cos\left(\frac{\pi}{2^n}\right) \right\rfloor \]

\[ T_z(i) = h \]

### 3. AN APPLICATION TO GEOMETRIC MODELING

This section presents a geometric application using the technique we propose. Given an elliptic paraboloid, one may intend to develop a "terrain" on the surface of the basis geometry, constructing new types of geometries. We impose a constraint where the summits of the terrain can not exceed the boundaries of the bounding volumes for an elliptic paraboloid. In other words, the maximal height of the terrain is the distance from a point on the elliptic paraboloid, along its normal direction, to the boundaries of the bounding volume. One approach to constructing such geometries is first to select a random point, say \( A \), on the surface of the elliptic paraboloid. This is shown in Fig. 3. We then determine a normalized normal vector, say \( N \), corresponding to the point \( A \), and the maximal distance, \( DIS_A \), which is the distance between \( A \) and the intersection point \( C \) at the boundaries of the bounding volume. Note we denote the maximal distance with a suffix
because this distance varies greatly with respect to the location of the point $A$ as well as the bounding volumes we employ. Finally, we select a second random point, say $B$, along the normal vector $N$ between $A$ and $C$. Mathematically, we can represent point $B$ using the expression $B = A + \xi(D\text{IS}_A)N$, where $\xi \in [0, 1)$ represents a random number. Once $B$ is available, we can construct a terrain, where the summit is at $B$ and the mountain ridges are created by connecting $B$ with four neighboring points of $A$. Clearly, $B$ is constrained by the maximal distance, $D\text{IS}_A$, and the latter, in return, is constrained by the bounding volume’s boundaries. Thus, new types of geometries can be generated when we employ different bounding volumes.

It is not difficult to find the maximal distance, $D\text{IS}_A$, with respect to point $A$, since we can always cast a ray with the origin at $A$ and the normal vector $N$ to a bounding volume’s boundaries to determine the intersection point $C$. Surely, the maximal distance must have the property such that $D\text{IS}_A = |AC|$. The problem remaining is how to generate a number of stratified points on an elliptic paraboloid. We tackle this by using a stratified sampling technique, together with the properties of probability distribution functions.

We describe below our approach to sampling an elliptic paraboloid. We intend to generate uniformly a number of random points on the surface of an elliptic paraboloid. The probability density function (p.d.f.) is thus the inverse of the surface area of the geometry. Note that it is more complex to calculate the surface area of an elliptic paraboloid than that of a circular paraboloid. To simplify the computing we therefore first assume that an elliptic paraboloid is degenerated to a paraboloid. Nevertheless, we will describe later how to extend our approach to sampling an elliptic paraboloid. We represent the p.d.f as $p(x') = 1/S$, where $S$ is the surface area of a paraboloid, and $x'$ is a random point on the paraboloid. Let $\xi_1, \xi_2 \in [0, 1)$ be two random numbers. We can calculate $S$ by revolving a parabolic with respect to the Z-axis.

$$S = 2\pi \int_0^w x\sqrt{1 + 4a^2x^2} \, dx = \frac{2\pi}{8a^2} \left( \frac{2}{3} w^{1.5} \right)$$

$$= \frac{\pi}{6a^2} \left( (1 + 4a^2w^2)^{1.5} - 1 \right) = \frac{\pi}{6a^2} \left( (1 + 4ah)^{1.5} - 1 \right)$$

where $a = h/w^2$, $w$ represents the radius of the circle at the plane $Z = h$, and $h$ indicates the height of a paraboloid (see Fig. 1). Let $d'$ be the radius of the circle at the plane $Z = h'$, where $0 < h' < h$ is an arbitrary height of the paraboloid. Let $S'$ be the surface area of the
paraboloid with arbitrary height $h'$, then

$$S' = \frac{\pi}{6a^2} ((1 + 4ah^{1.5}) - 1).$$

Clearly, $S'$ is a function of $h'$ since $a$, the radius of the circle at the plane $Z = h$, is a given constant. In addition, we can verify from the expression that when $h' = h$, $S'$ represents the entire surface area of a paraboloid. Now, select a random number $\xi_1$ and let $S'/S = \xi_1$. Then $h'$ can be calculated by the inverse function $S'^{-1}(\xi_1)$ in terms of $\xi_1$, $a$ and $h$, where

$$h' = \frac{(((\xi_1)((1 + 4ah^{1.5}) + 1)^{\frac{3}{2}}) - 1)}{4a},$$

$$d' = \frac{\sqrt{h'}}{a} = \frac{\sqrt{(((\xi_1)((1 + 4ah^{1.5}) + 1)^{\frac{3}{2}}) - 1)}}{2a}.$$ 

Once $d'$ is available, the circle at the plane $Z = h'$ is also determined. Under this situation, we can easily generate a number of random points from the circumference by selecting another random number $\xi_2$. Assume $\xi_2$ is the selected random number and let $\xi_2 = \theta 2\pi$. Therefore, a sample point $x'$ on the surface of a paraboloid can be generated using two random numbers $\xi_1$ and $\xi_2$, and its coordinates can expressed in the local Cartesian coordinates as $x' = (d'\cos(2\pi\xi_2), d'\sin(2\pi\xi_2), ad')$.

For stratification, we can first subdivide the interval between $[0, 1)$ into $M$ intervals and select $M$ random number for $\xi_1$ from each interval. We can apply a similar procedure to the second random number $\xi_2$, but select $N$ random numbers from each interval. Thus a total of $MN$ pairs of random numbers $(\xi_1, \xi_2)$ can be selected, and this, in return, leads to generating $MN$ random samples on the surface of a paraboloid.

The above approach can be extended when we consider an elliptic paraboloid. The surface area of an elliptic paraboloid, $S$, can be expressed as:

$$S = \int \int_D h \left( \frac{(2x/a)^2}{a} + \frac{(2y/b)^2}{b} + 1 \right) dS.$$

In this expression, $D$ represents the corresponding elliptic area of an elliptic paraboloid projected on the X-Y plane, and $a$, $b$ represent the length of the semi-major and semi-minor axis, respectively. Note that in a circular paraboloid, $a$, $b$ have the same value since they represent a single radius. Unfortunately, $a$ and $b$ are different in an elliptic paraboloid, and as a result, $S$ can not be computed analytically. Instead, it must be computed numerically since there is no explicit anti-derivative form for the integrand. Despite the numerical computing, a number of random points can be generated using a similar approach. Thus, given a random number $\xi_1$, we must first determine $d'$ (the radius of the circle at the plane $Z = h'$) such that $S'/S = \xi_1$, where $S'$ represents the surface area integrated under $h'$. Once $d'$ is available, the elliptic circumference is determined. Once again, we can derive a random point on this ellipse by selecting another random number $\xi_2$ and
generating a random sample on the elliptic circumference. This process also requires numerical computing. Finally, it is trivial to extend the above approach for stratified sampling. As a result, we can generate a number of stratified random points on an elliptic paraboloid, though numerical computing seems to be necessary.

Fig. 4 shows the results of developing a terrain on an elliptic paraboloid given tunable bounding volumes as different constraints. We also demonstrate the boundaries of the tunable bounding volumes. When no constraints are applied, a new geometry can be generated as shown in Fig. 4 (a). Here, the basis geometry is shaded in green, and the terrain is in red. We generate 1,600 stratified random points on an elliptic paraboloid using numerical computing. In contrast, when we impose constraints, the summits cannot exceed the boundaries of the bounding volume, and images (b)-(f) demonstrate different terrains with such constraints. As expected, the height of the terrains in these geometries is smaller when tighter bounding volumes are employed. Image (f) has the smallest height in the terrain where we employ the tightest bounding volume, restricting the range that can be selected for the summit.

![Fig. 4. An elliptic paraboloid with different terrains is constructed using tunable bounding volumes.](image1)

(a) No constraint. (b) Cuboid constraint. (c) 8-vertex frustum constraint.

(d) 12-vertex frustum constraint. (e) 16-vertex frustum constraint. (f) 24-vertex frustum constraint.

Fig. 4. An elliptic paraboloid with different terrains is constructed using tunable bounding volumes.

![Fig. 5. New geometries are created using a stratified sampling technique on a paraboloid.](image2)

(a) 1,600 uniform sample points. (b) Monte Carlo direct lighting. (c) Monte Carlo path tracing.

Fig. 5. New geometries are created using a stratified sampling technique on a paraboloid.
Fig. 5 shows the results of developing new types of geometries directly on the bounding volume of a circular paraboloid. Fig. 5 (a) displays a total number of 1,600 samples points using a uniform sampling approach. Fig. 5 (b) displays a rendered image using Monte Carlo direct lighting algorithm [7]. This rendering algorithm generates images with soft shadows on the floor. Finally, Fig. 5 (c) demonstrates images rendered with Monte Carlo path tracing algorithm [3]. We utilize an 8-Vertex, 2-Layer frustum as a bounding volume. Since the central geometry has a property of specular reflection, inter-reflection effects are perceived at different mountain ridges: the mountain ridges become red when they are near the red wall. Also, the ceiling demonstrates a color bleeding effect where the red and green colors are mixed around the rectangular light source.

4. CONCLUSIONS AND FUTURE WORK

We have presented a technique to derive a tunable bounding volume for an elliptic paraboloid. This technique employs an optimization process, developing a bounding volume with the minimal volume at different levels. Through an integer, a user can control and alter the tightness of a bounding volume. We further analyze the tightness of the bounding volumes at different levels so that a user can select bounding volumes with distinct tightness to suit the application. We demonstrated the feasibility of our technique in geometric applications, where new types of geometries are created. Combined with two rendering algorithms, we presented new geometries with visually plausible appearances.

Future work can be conducted by applying our technique to other applications. Another possible direction is to approximate the solid angle when computing the radiance for global illumination calculation.

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