Hamiltonicity of the Pyramid Network with or without Fault*

RUEI-YU WU[1] AND DYI-RONG DUH[2]
[1]*Department of Management Information Systems*
Hwa Hsia Institute of Technology
Taipei Hsien, 235 Taiwan
[2]*Department of Computer Science and Information Engineering*
National Chi Nan University
Nantou Hsien, 545 Taiwan

Sarbazi-Azad, Ould-Khaoua, and Mackenzie proved in 2001 that there exists a Hamiltonian cycle in a pyramid network and they also constructed a Hamiltonian path between apex and each of 4 frontiers of a pyramid network. The fault tolerance is a crucial matter for parallel computing, especially in a large network. This work improves Sarbazi-Azad *et al.*'s result and considers other relative problems in pyramid networks such as the fault tolerant Hamiltonian problem and the Hamiltonian-connected problem. The problem of finding Hamiltonian cycles in a pyramid network with one faulty node (link) is investigated. Additionally, the Hamiltonian-connectedness of a pyramid network can be shown by constructing a Hamiltonian path between any two distinct nodes in it.

Keywords: pyramid networks, Hamiltonian cycle, Hamiltonian-connectedness, fault tolerance, interconnection networks

1. INTRODUCTION

For the great improvement in the technology of designing and manufacturing of semiconductor, the architecture of parallel computing machines can be implemented, and lots of different architectures of parallel computing machines have been proposed in literature [1-3]. The main problem of designing a parallel computing machine lies in how to connect the processors in it such that the machine has the most efficiency, and this is the problem of interconnection networks. This kind of problem is often viewed and analyzed in the way of solving a graph theory problem [6, 12, 15]. For this reason, a network is conveniently represented by a graph whose vertices represent the nodes (*i.e.*, processors) of the network and whose edges represent the communication links of the network. This investigation uses the terms "network" and "graph"; "node" and "vertex", and "link" and "edge", interchangeably.

Let $G = (V, E)$ be a connected graph, where V and E represent the vertex set and edge set of G, respectively. The *degree* of a vertex in G is the number of edges incident with it. If all vertices have the same degree d, then G is *regular* or d-*regular*. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path between u and v. The *diameter* of G is the maximal distance between any two vertices. The *vertex* (*edge*) *connectivity* of G is the minimal number of vertices (edges) in G whose removal can cause G to be disconnected or trivial [20].

Received May 14, 2007; revised September 19, 2007 & March 18, 2008; accepted April 17, 2008.
Communicated by Tsan-sheng Hsu.
* This work was also partially supported by the National Science Council of Taiwan, R.O.C. under contract No. NSC 91-2213-E-260-003-.

The pyramid network (pyramid for short) is one of the important network topologies as it has been used as both hardware architecture and software structure for parallel and network computing [16, 18], image processing [17], and machine vision [22]. In parallel and network computing, a lot of parallel algorithms are efficiently implemented on pyramid networks. Furthermore, some parallel algorithms are implemented in supercomputers like Cray T3D and T3E and each processor acts as a node in a pyramid network. Some topological properties such as diameter, vertex connectivity, and fault diameter of a pyramid network have been investigated in [6]. Additionally, a pyramid network is Hamiltonian [19] and pancyclic [21]. Their results show that the pyramid network has very good fault-tolerant properties.

The cycle (ring) and path topologies are used frequently due to their good properties such as simplicity, expansibility and easiness of implementation. The Hamiltonian property is one of major requirements for designing the topology of a network. The *Hamiltonicity* of a network indicates its capability of embedding cycles (paths) [4, 9]. A cycle that contains every node of a network is called a *Hamiltonian cycle*. A network is *Hamiltonian* if it has a Hamiltonian cycle. An n-node network is Hamiltonian if the network contains a cycle of length n. A network is *Hamiltonian-connected* if the network contains a Hamiltonian path between every two distinct nodes [5, 8]. A Hamiltonian-connected network can embed a longest path between every two distinct nodes.

Linear arrays and rings are suitable for developing simple algorithms with low communication costs. Many efficient algorithms that were designed on linear arrays and rings for solving a variety of algebraic problems and graph problems can be found in [9, 16]. They can also be used as a control/data flow structure for distributed computation in arbitrary networks. There is an application of longest paths to a practical problem that was encountered in the on-line optimization of a complex Flexible Manufacturing System [11]. These motivate the embedding of paths and cycles in networks. In fact, an n-node network containing a Hamiltonian cycle represents that it can embed a ring of length n with dilation 1 and congestion 1 [16]. This work concentrates on the problem of finding a Hamiltonian cycle in a pyramid and shows that a pyramid with a faulty node or a faulty link is also Hamiltonian. This work also demonstrates that a pyramid is Hamiltonian-connected by constructing a Hamiltonian path between any two distinct nodes in it.

The rest of this work is organized as follows. Section 2 formally defines pyramid networks in terms of graphs. Section 3 constructs a Hamiltonian cycle in a fault free pyramid network and builds a Hamiltonian cycle in a pyramid network with a faulty node or a faulty link. Moreover, Hamiltonian-connectedness of a pyramid network is also presented. Finally, conclusions are made in section 4.

2. BACKGROUND AND NOTATIONS

This section investigates the structure of a pyramid network more deeply, and explains some of its topological properties. A pyramid, which is constructed similar to a rooted tree, has a hierarchical structure. Only one node, the *apex*, is at layer 0. At layer 1, there are four nodes connected as a 2-dimensional mesh and all the four nodes are the *children* of the root (the apex), and each node has four children at layer 2. At layer 2, there are sixteen nodes connected as a 2-dimensional mesh and each node also has four children at layer 3, and so on. All nodes at layer k, $1 \le k < n$, are connected as a 2-dimensional mesh and each node has four child nodes at layer $k + 1$.

Let $V(G)$ denote the node set of network G. A $i \times j$ mesh, denoted by $M[i, j]$, is a set of nodes $V(M[i, j]) = \{(x, y) \mid 1 \le x \le i, 1 \le y \le j\}$ and any two nodes are joined by a link if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. A pyramid network is one of hierarchical structures based on meshes. An n-layer pyramid network, denoted by PM[n], is a set of nodes $V(\text{PM}[n]) = \{(k; x, y) \mid 0 \le k \le n, 1 \le x, y \le 2^k\}$. Node $(k; x, y) \in V(\text{PM}[n])$ is a node in the mesh of layer k with the coordinate (x, y). All nodes at layer k form a mesh $M[2^k, 2^k]$ and node $(k; x, y)$ is also connected to $(k + 1; 2x - 1, 2y - 1)$, $(k + 1; 2x - 1, 2y)$, $(k + 1; 2x, 2y - 1)$, and $(k + 1; 2x, 2y)$. Node $(k; x, y)$ is the *parent* of $(k + 1; 2x - 1, 2y - 1)$, $(k + 1; 2x - 1, 2y)$, $(k + 1; 2x, 2y - 1)$, and $(k + 1; 2x, 2y)$. Conversely, $(k + 1; 2x - 1, 2y - 1)$, $(k + 1; 2x - 1, 2y)$, $(k + 1; 2x, 2y - 1)$, and $(k + 1; 2x, 2y)$ are the four children of $(k; x, y)$. Two nodes are *adjacent* if they are endnodes of a link. In a PM[n], the maximum degree is 9 since a node at layer k, $1 \le k < n$, is connected to its parent at layer $k - 1$, the four children at layer $k + 1$, and the four adjacent nodes in the mesh $M[2^k, 2^k]$ the node resides. Clearly, the degree of the apex is 4. Excluding a trivial node, the minimum degree of a PM[n] is 3, which appears at the four *frontiers* $(n; 1, 1)$, $(n; 1, 2^n)$, $(n; 2^n, 1)$, and $(n; 2^n, 2^n)$. Fig. 1 shows the structure of a PM[2]. Note that the degrees of nodes $(2; x, y)$ in a PM[2] are ranging from 3 to 5, where $1 \le x, y \le 4$.

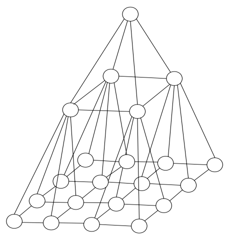


Fig. 1. The structure of a PM[2].

For convenience, some notations should be defined as follows in advance.

Definition 1 The links of a PM[n] are classified into two types: If the endnodes of a link belong to different layers, the link is a *layer-link*, otherwise it is a *mesh-link*.

Let (u, v) denote a link of two endnodes u and v in a PM[n]. Then, ((1; 1, 1), (2; 1, 1)) is a layer-link and ((2; 1, 1), (2; 2, 1)) is a mesh-link.

Definition 2 A PM[n] can be partitioned into the apex and four $(n - 1)$-layer subpyramid PM[$(n - 1)_{ij}$]s by removing the layer-links jointed to the apex and some mesh-links not in PM[$(n - 1)_{ij}$]s, and each node of a PM[$(n - 1)_{ij}$] can be addressed by $(l; x, y)_{ij}$, where $1 \le i, j \le 2$, $1 \le l < n$, and $1 \le x, y \le 2^l$.

For example, a PM[2] can be partitioned into the apex (0; 1, 1) and four subpyramids PM[1_{11}], PM[1_{12}], PM[1_{21}], and PM[1_{22}].

Definition 3 Three kinds of Hamiltonian paths can be constructed in a PM[1] as follows.

(1) M-path: (1; 1, 2) → (1; 1, 1) → (0; 1, 1) → (1; 2, 1) → (1; 2, 2);
(2) N-path: (1; 1, 2) → (1; 1, 1) → (0; 1, 1) → (1; 2, 2) → (1; 2, 1);
(3) r-path: (0; 1, 1) → (1; 2, 1) → (1; 1, 1) → (1; 1, 2) → (1; 2, 2).

By the definition, as shown in Fig. 2, M-path and N-path contain two layer-links but an r-path contains only one layer-link. Thick lines in Fig. 2 represent the links of paths. Obviously, a Hamiltonian cycle can be constructed in a PM[1] by joining two endnodes of M-path or r-path.

Definition 4 The *rotation* operation, denoted by $\Re_t(n, q)$, rotates a path (cycle) q clockwise in a PM[n], where t is the degree of rotation and $t \in \{90, 180, 270\}$.

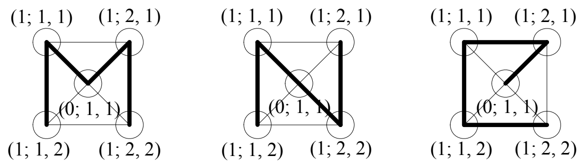


Fig. 2. Hamiltonian paths in a PM[1].

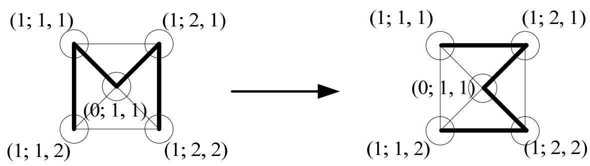


Fig. 3. To rotate the M-path in a PM[1] with degree 90°.

Notably, operations $\Re_{90}(n, q)$, $\Re_{180}(n, q)$ and $\Re_{270}(n, q)$ are used to image that nodes $(k; 2^k - y + 1, x)$, $(k; 2^k - x + 1, 2^k - y + 1)$ and $(k; y, 2^k - x + 1)$, respectively, are node $(k; x, y)$ in a PM[n]. Significantly, after executing a rotation operation, a mesh-link (layer-link) is still a mesh-link (layer-link). For example, Fig. 3 (a) is the original M-path in a PM[1]. After executing the operation $\Re_{90}(1, M\text{-path})$, the new M-path in a PM[1] is shown as Fig. 3 (b). In other words, the coordinates of the original M-path are (1; 1, 2) → (1; 1, 1) → (0; 1, 1) → (1; 2, 1) → (1; 2, 2). After executing $\Re_{90}(1, M\text{-path})$, the coordinates of the new M-path are (1; 1, 1) → (1; 2, 1) → (0; 1, 1) → (1; 2, 2) → (1; 1, 2).

M-path, N-path, and r-path in a PM[1] can be rotated 90°, 180°, and 270° and the resulting paths are still Hamiltonian paths because a PM[1] is a 4-ary rooted tree and all nodes at layer 1 are connected as a 2-dimentional mesh.

3. HAMILTONICITY

Sarbazi-Azad *et al*. proved that there exists a Hamiltonian cycle in a pyramid network and they also constructed a Hamiltonian path between the apex and each of four frontiers [19]. Farahabady *et al.* attempt to construct a Hamiltonian path between any two distinct nodes in a pyramid network [7]. Unfortunately, there are many errors in the proofs of Lemmas 2 and 3 in [7] such as the induction bases of these two lemmas are missed, Lemma 2 never considers the case that two terminal nodes are in two non-adjacent subpyramids, respectively, and Lemma 3 does not show that a Hamiltonian path between any two nodes in a pyramid can contain the desired link ((0; 1, 1), (1; 2, 1)). This work improves their results to consider the fault tolerant Hamiltonian problem and the Hamiltonian-connected problem in a pyramid network. Sarbazi-Azad *et al.* graphically built Hamiltonian paths in a pyramid network case by case. Since a pyramid network is a recursive structure and the expansion of the pyramid network is quite regular, this work constructs a Hamiltonian cycle in a PM[n] mathematically. Additionally, a Hamiltonian cycle is constructed in a PM[n] with one faulty node or one faulty link. Thus, a PM[n] is Hamiltonian even with one node or one link fault. Finally, Hamiltonian-connectedness of a PM[n] is shown by constructing a Hamiltonian cycle between any two distinct nodes.

3.1 Hamiltonian Cycle

Let HP(k; u, v) (respectively, HP(k_{ij}; u, v)), $1 \leq k < n$, denote a Hamiltonian path between two distinct nodes u and v in a PM[k] (respectively, PM[k_{ij}]). Also let HC(k) (respectively, HC(k_{ij})), $1 \leq k < n$, denote a Hamiltonian cycle in a PM[k] (respectively, PM[k_{ij}]). Because the pyramid is a recursive structure, the following theorem can be proved by induction on layer n.

Theorem 1 A PM[n] is Hamiltonian.

Proof: This theorem is proved by induction on layer n. If $n = 1$, either an M-path or an r-path can be constructed in a PM[1] according to Definition 3. Due to the starting node and ending node of an M-path or an r-path are adjacent, a link is added to connect them to form a Hamiltonian cycle. Clearly, there is one mesh-link unused for constructing the Hamiltonian cycle in the PM[1]. By the rotation operation, an unused link can be rotated to link ((1; 1, 1), (1; 2, 1)).

Suppose it is true for $n = 1$ to k. Then, the PM[n] exists an HC(n) which contains an M-path if the four PM[$(n-1)_{ij}$], $1 \leq i, j \leq 2$, are regarded as four supernodes. By induction hypothesis, there exists an HC(k_{ij}) which contains an M-path in each PM[k_{ij}], where $1 \leq i, j \leq 2$. As mentioned above, the unused mesh-link is rotated to ($(1; 1, 1)_{ij}$, $(1; 2, 1)_{ij}$) in each PM[k_{ij}].

To establish an HC($k + 1$) in a PM[$k + 1$], one suitable layer-link is removed from each PM[k_{ij}] to form a Hamiltonian path. The removed layer-links are ($(0; 1, 1)_{11}$, $(1; 2, 1)_{11}$), ($(1; 1, 1)_{21}$, $(0; 1, 1)_{21}$), ($(0; 1, 1)_{22}$, $(1; 1, 1)_{22}$), and ($(1; 2, 1)_{12}$, $(0; 1, 1)_{12}$). Moreover, the constructed Hamiltonian paths are HP(k_{11}; $(0; 1, 1)_{11}$, $(1; 2, 1)_{11}$), HP(k_{21}; $(1; 1, 1)_{21}$, $(0; 1, 1)_{21}$), HP(k_{22}; $(0; 1, 1)_{22}$, $(1; 1, 1)_{22}$), and HP(k_{12}; $(1; 2, 1)_{12}$, $(0; 1, 1)_{12}$), respectively. Note that two layer-links are connected to $(0; 1, 1)_{ij}$ and the removed layer-link in each PM[k_{ij}] is one of them. One endnode of the removed layer-link in the PM[k_{i1}] (respectively, PM[k_{1j}]) is adjacent to one endnode of the removed layer-link in the PM[k_{i2}] (respectively, PM[k_{2j}]). Finally, Hamiltonian cycle HC($k + 1$) is constructed as (0; 1, 1) → HP(k_{11}; (0; 1, 1)$_{11}$, (1; 2, 1)$_{11}$) → HP(k_{21}; (1; 1, 1)$_{21}$, (0; 1, 1)$_{21}$) → HP(k_{22}; (0; 1, 1)$_{22}$, (1; 1, 1)$_{22}$) → HP(k_{12}; (1; 2, 1)$_{12}$, (0; 1, 1)$_{12}$) → (0; 1, 1). Clearly, an *M*-path and 4 *r*-paths are included in the constructed Hamiltonian cycle. ❑

Taking a PM[2] as an example, the resulting Hamiltonian cycle HC(2), which is established according to the proof of Theorem 1, is illustrated in Fig. 4.

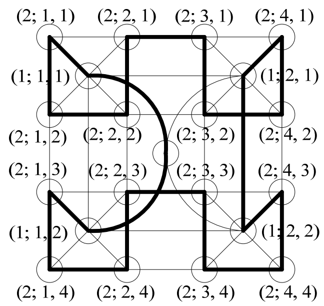


Fig. 4. An HC(2) in a PM[2].

Since node and/or link faults may happen in a practical network and it is practically important to consider faulty networks. The node (link) connectivity of a PM[n] is 3 [6]. Since node connectivity and link connectivity of cycles is 2, under the worst case, only one node (link) can be failure or no Hamiltonian cycle can be constructed in a faulty pyramid network. The existence of a Hamiltonian cycle in such a faulty network is proved constructively. Recalling the recursive property of a pyramid network, the following theorem can be proved by induction again.

Theorem 2 A PM[n] with one faulty link is Hamiltonian.

Proof: This theorem is proved by induction on layer n. If n = 1, there are unused layer-links and mesh-links in a PM[1]. Operation $\Re_t$(1, HC(1)) can be applied to ensure that a Hamiltonian cycle does not contain the faulty link.

Suppose it is true for n = 1 to k and assume that $\Re_t$(n, HC(n)) can lead to a Hamiltonian cycle without a faulty link in a faulty PM[n]. Now, it is tenable to prove for n = k + 1. Since $\Re_t$(k + 1, HC(k + 1)) would involve in the computations of $\Re_t$(k_{ij}, HC(k_{ij})), $\Re_t$(k + 1, HC(k + 1)) vanishes the faulty link in HC(k + 1) when the fault is one of layer-links or mesh-links by which the PM[k_{ij}] are connected. There are four layer-links used to connect (0; 1, 1) to four PM[k_{ij}]s to form the PM[k + 1]. Two of these layer-links are not in HC(k + 1). Fortunately, only one of mesh-links at layer 1 and two of eight mesh-links at layer 2 are used in HC(k + 1). Thus, $\Re_t$(k + 1, HC(k + 1)) can be applied to guarantee that the resulting Hamiltonian cycle contains no faulty link. ❑

For example, suppose that link ((1; 2, 1), (1; 2, 2)) is failure. After applying $\Re_{90}$(2, HC(2)) on the HC(2) as shown in Fig. 4, the new HC(2) avoiding the faulty link is illustrated in Fig. 5 and the faulty link is marked by the symbol "✕".

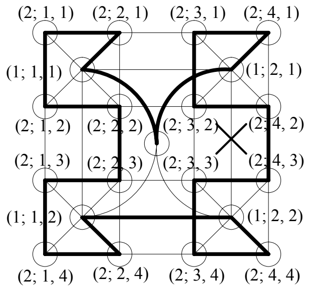
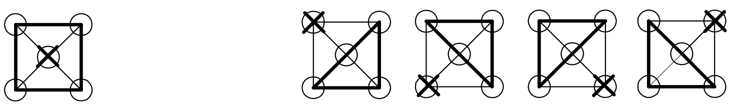


Fig. 5. A Hamiltonian cycle in a PM[2] with a faulty link.

Theorem 3 A PM[n] with one faulty node is Hamiltonian.

Proof: Three cases are needed to be considered when a node in a PM[n] is faulty. First, the faulty node is the apex. Second, the faulty node is (n; x, y), where $1 \leq x, y \leq 2^n$. Third, the faulty node is one of the others. Let f denote the faulty node and HC′(n) denote a Hamiltonian cycle in the PM[n] in which one node is faulty. This theorem is proved by induction on layer n.

When $n = 1$, f is either the apex or (1; x, y), where $1 \leq x, y \leq 2$. If it is the apex, then an HC′(1) can be constructed along mesh-link as shown in Fig. 6 (a). Otherwise, f = (1; x, y), an HC′(1) is built by using two mesh-links and two layer-links as shown in Fig. 6 (b). Thus, there exists a Hamiltonian cycle in a PM[1] with one faulty node. The faulty node f in Fig. 6 is marked by symbol "✕" again.



(a) An HC′(1) without layer-links. (b) HC′(1)s with layer-links.

Fig. 6. HC′(1)s in a PM[n] with a faulty node.

For n = 1 to k, suppose that a PM[n] with a faulty node contains an HC′(n). That PM[k + 1] with a faulty node contains an HC′(k) is proved as follows.

Case 1: (f = (0; 1, 1)): By Theorem 1, an HC(k + 1) is constructed as (0; 1, 1) → HP(k_{11}; $(0; 1, 1)_{11}$, $(1; 2, 1)_{11}$) → HP(k_{21}; $(1; 1, 1)_{21}$, $(0; 1, 1)_{21}$) → HP(k_{22}; $(0; 1, 1)_{22}$, $(1; 1, 1)_{22}$) → HP(k_{12}; $(1; 2, 1)_{12}$, $(0; 1, 1)_{12}$) → (0; 1, 1). Let every PM[k_{ij}] be regarded as a super-node, the PM[k + 1] can be regarded as a PM[1] and can construct an HC′(1) in it as described above. Since the source of HP(k_{11}; $(0; 1, 1)_{11}$, $(1; 2, 1)_{11}$) and the destination of HP(k_{12}; $(1; 2, 1)_{12}$, $(0; 1, 1)_{12}$) are adjacent, the node (0; 1, 1) is removed from HC(k+1), and the source node of HP(k_{11}; $(0; 1, 1)_{11}$, $(1; 2, 1)_{11}$) and the destination node of HP(k_{12}; $(1; 2, 1)_{12}$, $(0; 1, 1)_{12}$) is connected to form HC′(k + 1) as HP(k_{11}; $(0; 1, 1)_{11}$, $(1; 2, 1)_{11}$) → HP(k_{21}; $(1; 1, 1)_{21}$, $(0; 1, 1)_{21}$) → HP(k_{22}; $(0; 1, 1)_{22}$, $(1; 1, 1)_{22}$) → HP(k_{12}; $(1; 2, 1)_{12}$, $(0; 1, 1)_{12}$) → $(0; 1, 1)_{11}$.

Case 2: ($f = (k + 1; x, y)$, where $1 \leq x, y \leq 2^n$): By induction hypothesis and Fig. 6, there exists an HC(k_{ij}) which contains two layer-links with the endnode $(0; 1, 1)_{ij}$ in each PM[k_{ij}], where $1 \leq i, j \leq 2$. Then, by the proof of Theorem 1 an HC′($k + 1$) can be built in the PM[$k + 1$].

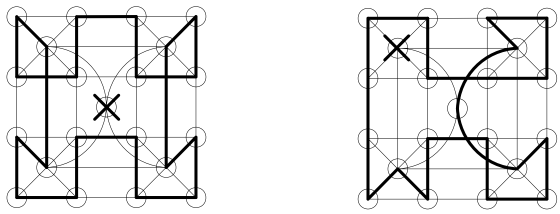
Without loss of generality, suppose that the faulty node is in the PM[k_{11}]. By induction hypothesis, there exists an HC′(k_{11}). The HC′(k_{11}) contains a layer-link ($(0; 1, 1)_{11}$, y_{11}) and y_{11} is either $(1; 2, 1)_{11}$ or $(1; 2, 2)_{11}$. After removing the layer-link ($(0; 1, 1)_{11}$, y_{11}), there exists an HP′(k_{11}; y_{11}, $(0; 1, 1)_{11}$). The HC′($k + 1$) can then be built as follows:

If $y_{11} = (1; 2, 1)_{11}$, $(0; 1, 1) \to$ HP′(k_{11}; $(0; 1, 1)_{11}$, y_{11}) $\to$ HP(k_{21}; $(1; 1, 1)_{21}$, $(0; 1, 1)_{21}$) $\to$ HP(k_{22}; $(0; 1, 1)_{22}$, $(1; 1, 1)_{22}$) $\to$ HP(k_{12}; $(1; 2, 1)_{12}$, $(0; 1, 1)_{12}$)$\to$$(0; 1, 1)$.

If $y_{11} = (1; 2, 2)_{11}$, $(0; 1, 1) \to$ HP′(k_{11}; $(0; 1, 1)_{11}$, y_{11}) $\to$ HP(k_{21}; $(1; 1, 2)_{21}$, $(0; 1, 1)_{21}$) $\to$ HP(k_{22}; $(0; 1, 1)_{22}$, $(1; 1, 1)_{22}$) $\to$ HP(k_{12}; $(1; 2, 1)_{12}$, $(0; 1, 1)_{12}$) $\to$ $(0; 1, 1)$.

Case 3: ($f = (l; x, y)$, where $1 \leq x, y \leq 2^l$ and $1 \leq l \leq k$): The faulty node f must belong to one PM[$(k + 1 - l)_{ij}$] in a PM[$k + 2 - l$]. Without loss of generality, assume that f is in the PM[$(k + 1 - l)_{11}$]. By Case 1, construct an HC′($k + 1 - l$) in the PM[$(k + 1 - l)_{11}$]. Moreover, by Theorem 1, an HC($(k + 1 - l)_{ij}$) is established in each of the other three PM[$(k + 1 - l)_{ij}$]s. Remove four links ($(l + 1; 2x - 1, 2y)$, $(l + 1; 2x, 2y)$), ($(l + 1; 2x + 1, 2y)$, $(l; x + 1, y)$), ($(l + 1; 2x + 1, 2y + 1)$, $(l; x + 1, y + 1)$), and ($(l + 1; 2x - 1, 2y + 1)$, $(l + 1; 2x, 2y + 1)$) from the constructed HC′($k + 1 - l$) and HC($k + 1 - l$)s. Then five links, ($(l + 1; 2x - 1, 2y)$, $(l + 1; 2x - 1, 2y + 1)$), ($(l + 1; 2x, 2y)$, $(l + 1; 2x + 1 , 2y)$), ($(l; x + 1, y)$, $(l - 1; \lceil x/2 \rceil, \lceil y/2 \rceil)$), ($(l - 1; \lceil x/2 \rceil, \lceil y/2 \rceil)$, $(l; x + 1, y + 1)$), and ($(l + 1; 2x + 1, 2y + 1)$, $(l + 1; 2x, 2y + 1)$), are added to form an HC′($k + 2 - l$) in a PM[$k + 2 - l$] in which the faulty node $f = (l; x, y)$ and $1 \leq l \leq k$. Now, an HC′($k + 1$) can be easily constructed in the PM[$k + 1$] with $f = (l; x, y)$ by Case 2 if the PM[$k + 2 - l$]s are regarded as supernodes. ❑

For example, by Theorem 3, the constructed HC′(2) in a PM[2] with the faulty node $(0; 1, 1)$ is shown in Fig. 7 (a). Fig. 7 (b) shows the constructed HC′(2) in a PM[2] with $f = (1; 1, 1)$.



(a) (b)

Fig. 7. An HC′(2) in a PM[2] with one faulty node; the faulty node f is (a) the apex (b) not the apex.

3.2 Hamiltonian-connectedness

A *Hamiltonian-connected* network is a network which contains at least one Hamiltonian path between every two distinct nodes. In the following, the Hamiltonian-connectedness property of a pyramid network is revealed. In a PM[k + 1], 1 ≤ k < n, two PM[k_{ij}]s are *adjacent* if their apexes are adjacent, otherwise the two PM[k_{ij}]s are *nonadjacent*.

Theorem 4 A PM[n] is Hamiltonian-connected, where $n \geq 1$.

Proof: To show a PM[n] is Hamiltonian-connected, a path, which traverses every node exactly once between any two distinct nodes u and v in the PM[n], is devised. Four cases are needed to be considered in this proof. First, both u and v are in the same subpyramid. Second, u or v is the apex. Third, u and v are in two adjacent subpyramids. Last, u and v are in two nonadjacent subpyramids. This theorem is proved by induction on layer n.

If n = 1, only three cases are needed to be considered as follows. If u or v is the apex in a PM[1], then r-path is applied to construct a Hamiltonian path. Otherwise, because nodes u and v are in adjacent/nonadjacent subpyramids, M-path or N-path is adopted to achieve a Hamiltonian path. Additionally, these constructed paths all include at least one layer-link (x, y) where x is (0; 1, 1) and y is either (1; 2, 1) or (1; 2, 2).

For n = 1 to k, assume that a PM[n] is Hamiltonian-connected and each Hamiltonian path in it includes a layer-link (x, y) where x is (0; 1, 1) and y is either (1; 2, 1) or (1; 2, 2).

Let x_{ij} and y_{ij} represent the nodes x and y, respectively, in the PM[k_{ij}], $1 \leq i, j \leq 2$. Also let $a \xrightarrow{P} b$ represent the path P from node a to node b. The Hamiltonian-connectedness of the PM[k + 1] is shown as follows.

Case 1: (u and v are in the same subpyramid): By induction assumption, there exists a Hamiltonian path between any two nodes in a PM[k_{ij}]. Without loss of generality, suppose u and v are in the PM[k_{11}]. An HP(k_{11}; u, v) is constructed and link (x_{11}, y_{11}) is removed from it such that the path is divided into two subpaths P' and P''. Two endnodes of P' (P'') are u (v) and x_{11} (y_{11}). In fact, the length of P' or P'' may be 0. Because, in each of the other three PM[k_{ij}]s, there exists a Hamiltonian path between two arbitrary nodes u_{ij} and v_{ij}, the HP(k + 1; u, v) can be formed as follows:

If y_{11} = (1; 2, 1)$_{11}$, $u \xrightarrow{P'} x_{11} \to$ (0; 1, 1) $\to$ HP(k_{12}; u_{12}, v_{12}) $\to$ HP(k_{22}; u_{22}, v_{22}) $\to$ HP(k_{21}; u_{21}, v_{21}) $\to y_{11} \xrightarrow{P''} v$.

If y_{11} = (1; 2, 2)$_{11}$, $u \xrightarrow{P'} x_{11} \to$ (0; 1, 1) $\to$ HP(k_{21}; u_{21}, v_{21}) $\to$ HP(k_{22}; u_{22}, v_{22}) $\to$ HP(k_{12}; u_{12}, v_{12}) $\to y_{11} \xrightarrow{P''} v$.

Case 2: (u or v is the apex): Without loss of generality, assume that u is in the PM[k_{11}] and v is the apex. An HP(k_{11}; u, y_{11}) is first constructed. If $u = x_{11}$, apply the scheme of r-path in the PM[k_{11}]. Otherwise, apply either the M-path or N-path scheme in the PM[k_{11}]. Because, in each of the other three PM[k_{ij}]s, there exists a Hamiltonian path between two arbitrary nodes u_{ij} and v_{ij}, the HP(k + 1; u, v) is built as follows:

If v_{11} = (1; 2, 1)$_{11}$, HP(k_{11}; u, v_{11}) $\to$ HP(k_{21}; u_{21}, v_{21}) $\to$ HP(k_{22}; u_{22}, v_{22}) $\to$ HP(k_{12}; u_{12}, v_{12}) $\to$ (0; 1, 1).

If $v_{11} = (1; 2, 2)_{11}$, HP(k_{11}; u, v_{11}) → HP(k_{12}; u_{12}, v_{12}) → HP(k_{22}; u_{22}, v_{22}) → HP(k_{21}; u_{21}, v_{21}) → (0; 1, 1).

Case 3: (u and v are in two adjacent subpyramids): Without loss of generality, assume that u is in the PM[k_{11}] and v is in the PM[k_{21}]. An HP(k_{11}; u, v_{11}) first constructed. If $u = x_{11}$, r-path is adopted. Otherwise either M-path or N-path is adopted in the PM[k_{11}]. In the PM[k_{21}], an HP(k_{21}; u_{21}, v) is constructed similarly. Because, in each of the other two PM[k_{ij}]s, there exists a Hamiltonian path between two arbitrary nodes u_{ij} and v_{ij}, the HP($k + 1$; u, v) is built as follows:

HP(k_{11}; u, v_{11}) → HP(k_{12}; u_{12}, v_{12}) → (0; 1, 1) → HP(k_{22}; u_{22}, v_{22}) → HP(k_{21}; u_{21}, v).

Case 4: (u and v are in two nonadjacent subpyramids): Without loss of generality, assume that u is in the PM[k_{11}] and v is in the PM[k_{22}]. An HP(k_{11}; u, v_{11}) is first constructed. If $u = x_{11}$, then r-path is adopted. Otherwise, N-path is adopted in the PM[k_{11}]. Similarly, an HP(k_{22}; u_{11}, v) can be constructed in the PM[k_{22}]. Then, the HP($k + 1$; u, v) is built as follows:

HP(k_{11}; u, v_{11}) → HP(k_{12}; u_{12}, v_{12}) → (0; 1, 1) → HP(k_{21}; u_{21}, v) → HP(k_{22}; u_{22}, v).

Consequently, an HP($k + 1$; u, v) can be built in a PM[n] between two arbitrary nodes u and v, and the HP($k + 1$; u, v) contains the *layer-link* (x, y). Therefore, a PM[n] is Hamiltonian-connected for $n \geq 1$. ❑

4. CONCLUSION

This work constructs a Hamiltonian cycle in a pyramid network. Since the defection of nodes or links might occur in a practical network, it is practically important to consider faulty networks. This work also shows that a pyramid network with one faulty node (link) is Hamiltonian. The result is optimal because node connectivity and link connectivity of a pyramid are both 3, and at most one node or one link can be faulty in the worst case. In a pyramid network, a Hamiltonian path between any two distinct nodes can be constructed. In other words, a pyramid network is Hamiltonian-connected. Conditional fault-tolerant properties of a network have been defined and investigated in literature [10, 13, 14]. Under the assumption that each node has at least two fault-free links, how many link faults in a PM[n] can tolerate and still remain a Hamiltonian cycle (path)? Since the node degree of a PM[n] ranges from 3 to 9, there are nodes whose degrees are greater than 3. Future work will explore the conditional fault-tolerant Hamiltonicity of a PM[n].

ACKNOWLEDGMENTS

The anonymous referees are greatly appreciated for their constructive comments, which strongly enhanced the structure and readability of the paper.

REFERENCES

1. Y. A. Ashir and I. A. Stewart, "Fault-tolerant embedding of hamiltonian circuits in *k*-ary *n*-cube," *SIAM Journal on Discrete Mathematics*, Vol. 15, 2002, pp. 317-328.
2. L. N. Bhuyan and D. P. Agrawal, "Generalized hypercube and hyperbus structures for a computer network," *IEEE Transactions on Computers*, Vol. C-33, 1984, pp. 323-333.
3. J. Bruck, R. Cypher, and C. T. Ho, "Fault-tolerant meshes and hypercubes with minimal numbers of spares," *IEEE Transactions on Computers*, Vol. C-42, 1993, pp. 1089-1104.
4. M. Y. Chan and S. J. Lee, "On the existence of hamiltonian circuits in faulty hypercubes," *SIAM Journal on Discrete Mathematics*, Vol. 4, 1991, pp. 511-527.
5. J. M. Chang, J. S. Yang, Y. L. Wang, and Y. W. Cheng, "Panconnectivity, fault-tolerant hamiltonicity and hamiltonian-connectivity in alternating group graphs," *Networks*, Vol. 44, 2004, pp. 302-310.
6. F. Cao, D. Z. Du, D. F. Hsu, and S. H. Teng, "Fault tolerance properties of pyramid networks," *IEEE Transactions on Computers*, Vol. 48, 1999, pp. 88-93.
7. M. H. Farahabady and H. Sarbazi-Azad, "On the hamiltonian connectedness of pyramids," in *Proceedings of the IASTED International Conference on Networks and Communication Systems*, 2005, pp. 398-402.
8. J. S. Fu, "Hamiltonian-connectedness of the wk-recursive network," in *Proceedings of International Symposium on Parallel Architectures*, *Algorithms and Networks*, 2004, pp. 569-574.
9. J. S. Fu, "Longest fault-free paths in hypercubes with vertex faults," *Information Sciences*, Vol. 176, 2006, pp. 759-771.
10. J. S. Fu, "Conditional fault-tolerant hamiltonicity of twisted cubes," in *Proceedings of the 7th International Conference on Parallel and Distributed Computing*, *Applications and Technologies*, 2006, pp. 5-10.
11. S. Y. Hsieh, G. H. Chen, and C. W. Ho, "Longest fault-free paths in star graphs with edge faults," *IEEE Transactions on Computers*, Vol. 50, 2001, pp. 960-971.
12. D. F. Hsu, "On container width and length in graphs, groups, and networks," *IEICE Transactions on Fundamentals of Electronics*, Vol. E77A, 1994, pp. 668-680.
13. H. S. Hung, J. S. Fu, and G. H. Chen, "Pancycle problem of crossed cube with conditional fault," in *Proceedings of the International Computer Symposium*, 2006, pp. 153-158.
14. H. S. Hung, J. S. Fu, and G. H. Chen, "Conditional fault-tolerant cycle-embedding of crossed cube," in *Proceedings of the 7th International Conference on Parallel and Distributed Computing*, *Applications and Technologies*, 2006, pp. 90-95.
15. C. N. Lai, G. H. Chen, and D. R. Duh, "Constructing one-to-many disjoint paths in folded hypercubes," *IEEE Transactions on Computers*, Vol. 51, 2002, pp. 33-45.
16. F. T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays*, *Trees*, *Hypercubes*, Morgan Kaufmann, CA, 1992.
17. S. A. Levialdi, *Pyramid Project Using Integrated Technology*, *Integrated Technology for Parallel Image Processing*, Academic Press, New York, 1985.
18. M. J. Quinn, *Designing Efficient Algorithms for Parallel Computers*, McGraw-Hill, New York, 1987.

19. H. Sarbazi-Azad, H. M. Ould-Khaoua, and L. M. Mackenzie, “Algorithmic construction of hamiltonians in pyramids,” *Information Processing Letters*, Vol. 80, 2001, pp. 75-79.
20. D. B. West, *Introduction to Graph Theory*, 2nd eds., Prentice-Hall, NJ, 2001.
21. R. Y. Wu and D. R. Duh, “Pancyclicity of pyramid networks,” in *Proceedings of International Conference on Parallel and Distributed Processing Techniques and Applications*, 2004, pp. 1523-1526.
22. S. G. Ziavras and M. A. Siddiqui, “Pyramid mapping onto hypercubes for computer vision: Connection machine comparative study,” *Concurrency: Practice and Experience*, Vol. 5, 1993, pp. 471-489.

Ruei-Yu Wu (吳瑞瑜) received the B.S. degree in Information Management from the Fu Jen Catholic University, Taipei, Taiwan, in 1999 and the M.S. degree in Computer Science and Information Engineering from the National Chi Nan University, Nantou, Taiwan, in 2001. She is currently a Ph.D. candidate in the Department of Computer Science and Information Engineering at the National Taiwan University and she is also a lecturer in the Department of Management Information Systems Hwa Hsia Institute of Technology. Her currently research interests focus on graph theoretic interconnection networks and algorithms.

Dyi-Rong Duh (杜迪榕) received the B.S. degree in Electronics Engineering from the National Taiwan Institute of Technology, Taipei, Taiwan, in 1983, the M.S. degree in Computer Engineering from the Tamkang University, Taipei, Taiwan, in 1988, and the Ph.D. degree in Computer Science and Information Engineering from the National Taiwan University, Taipei, Taiwan, in 1994. He worked for the Sysgration Ltd., Taiwan as the R&D manager of the computer peripherals division from June 1988 to July 1989. He was a teaching assistant from August 1989 to July 1990 in the Department of Computer Science and Information Engineering at the National Taiwan University. From August 1990 to July 1994, he was a lecturer in the Department of Electronics Engineering at the Hwa Hsia Institute of Technology, Taipei Hsien, Taiwan. He is currently a professor in the Department of Computer Science and Information Engineering at the National Chi Nan University, Nantou Hsien, Taiwan. His research interests include graph-theoretic interconnection networks, graph theory, embedded systems, wireless sensor networks, and computer architectures.