Changing the Diameter in a Diagonal Mesh Network

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Communication speed in a parallel and distributed system is related to the diameter of its underlying graph. The diameter of a graph can be affected by the addition or deletion of edges. In this paper we study how the diameter variability problem arises from change of edges of an \( m \times n \) diagonal mesh, where \( m \) and \( n \) are not both even integers.

We show that the least number of edges whose deletion from an \( m \times n \) diagonal mesh will increase the diameter is no more than 2, and that the least number of edges whose addition to an \( m \times n \) diagonal mesh causes the diameter to decrease is no more than \( \lceil m/2 \rceil n \).

Keywords: diameter, diagonal mesh network, torus, vertex-disjoint path, hypercube

1. INTRODUCTION

Communication efficiency is a critical issue in the design of a parallel and distributed system. When we use a graph to model a parallel and distributed system, where each vertex represents a processor and each edge represents a vertex-to-vertex communication link, the maximum communication delay between any pair of processors in the system can be determined by the diameter of its underlying graph [1]. One way to further reduce the message delay while retaining the existing topology is to add communication links to the system. Besides, it is important to study the degradation of performance with respect to communication delays under maximally faulty edges and/or vertices [3, 4].

A graph \( G = (V, E) \) is defined by a finite vertex set \( V(G) \) and a finite edge set \( E(G) \). An edge is an element in \( V(G) \times V(G) \). The diameter of a graph \( G \), denoted as \( D(G) \), is the maximum distance between any two different vertices, where the distance between \( u \) and \( v \) in a graph \( G \), denoted as \( d_G(u, v) \), is the length of a shortest path from \( u \) to \( v \). The diameter of a graph can be affected by adding or deleting edges [2, 3, 7]. For example, let \( m \)-cycle \( C_m \) be a graph with vertex set \( \{0, 1, 2, ..., m - 1\} \) and edge set \( \{(i, i + 1) | 0 \leq i \leq m - 1\} \), where addition is in integer modulo \( m \). It is known that \( D(C_m) = \lceil m/2 \rceil \). Let \( P_m \) be a

Received February 10, 2012; revised May 3, 2012; accepted July 23, 2012.
Communicated by Hee-Kap Ahn.
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graph with vertex set \{0, 1, 2, ..., m - 1\} and edge set \{(i, i + 1)|0 \leq i \leq m - 2\}, it is known that \(D(P_m) = m - 1\). It is easy to check that deleting any edge renders \(C_m\) to a path of \(m\) vertices and increases the diameter to \(m - 1\). On the other hand, a cycle \(C_m\) can be obtained by adding the edge \((0, m)\) to path \(P_m\). The diameter decreases to \(\lfloor m/2 \rfloor\).

In graph-theoretic terms, the communication issues in a parallel and distributed system as we stated in the first paragraph are closely related to the diameter variability problem arising from change of edges of its underlying graph. Graham and Harary [3] formalized them as follows:

\[D'(G): \text{the least number of edges whose addition to } G \text{ decreases the diameter},\]
\[D^3(G): \text{the maximum number of edges whose deletion from } G \text{ does not change the diameter},\]
\[D'(G): \text{the least number of edges whose deletion from } G \text{ increases the diameter}.\]

For \(n\)-dimensional hypercube \(Q_n\), Graham and Harary [3] proved that \(D'(Q_n) = 2\) by adding two edges in any 2-dimensional subcube with Hamming distance 2 to \(Q_n\). \(D'(Q_n) = n - 1\) by deleting \(n - 1\) of the \(n\) edges incident at any specific vertex, and \(D^3(Q_n) \geq (n - 3)2^{n-1} + 2\). Bouabdallah et al. [2] improved the lower bound on \(D^3(Q_n)\) presented in [3] and furthermore gave an upper bound, \((n-2)2^{n-1} - \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq D^3(n-2)2^{n-1} - \frac{2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1}{2n-1} + 1\).

The \(m \times n\) torus \(T_{m \times n}\) (also known as toroidal mesh) is a popular and well-studied network. Wang et al. [7] showed that \(D'(T_{m \times n}) = 2\) and \(D^3(T_{m \times n}) = 2\) if both \(m\) and \(n\) are odd, 4 if both \(m\) and \(n\) are even, 3 otherwise. In this paper, we study the diameter variability problem in an \(m \times n\) diagonal mesh arising from change of edges. The topology of an \(m \times n\) diagonal mesh, the subject studied in this work, is a graph in which the vertex set is defined by \(V(C_m) \times V(C_n)\) and the edge set is defined by \{\((u_1, u_2), (v_1, v_2)\)\(\in \) \(E(C_m)\) and \((u_2, v_2) \in E(C_n)\)\}. The two networks have some common features such as degree-4, vertex-transitive [8], wirable, and scalable networks [5]. Tang and Padubidri [5, 6] showed that the diagonal mesh has a potentially better performance than the torus. Specifically, the diagonal mesh has a smaller diameter and a larger bisection width.

In this paper, we use \(C_m \otimes C_n\) to present an \(m \times n\) diagonal mesh. We show that \(D'(C_m \otimes C_n) = 1\) (if both \(m\) and \(n\) are odd and \(n = 2m + 1\) or if \(m\) is even and \(n\) is odd and \(n = m/2\)) or 2 (if both \(m\) and \(n\) are odd and \(n \neq 2m + 1\) or if \(m\) is even and \(n\) is odd and \(n \neq m/2\)), and that \(D^3(C_m \otimes C_n)\) is at most \(mn/2\).

In this paper, a concept of the farthest neighbors of a given vertex in \(C_m \otimes C_n\) plays an important role in studying the \(D'(C_m \otimes C_n)\) and \(D^3(C_m \otimes C_n)\). A vertex in a graph \(G\) is said to be a farthest neighbor of a given vertex if the distance between them is \(D(G)\). In order to obtain the farthest neighbors of any vertex in \(C_m \otimes C_n\), we develop a simple approach to compute the diameter of \(C_m \otimes C_n\). The presented approach is different from that in [5]. The approach collects a set of vertices in \(C_m \otimes C_n\) that can be reached from vertex \((0, 0)\) at the \(i\)th step. By the vertex-transitivity of \(C_m \otimes C_n\), the diameter of \(C_m \otimes C_n\) is the smallest \(i\) such that a set of vertices reached from \((0, 0)\) at the \(i\)th step covers the vertex set of \(C_m \otimes C_n\). Before studying \(D'(C_m \otimes C_n)\), we show that each pair of vertices in \(C_m \otimes C_n\) can be connected by two inner vertex-disjoint paths whose lengths are at most \(D(C_m \otimes C_n)\). Finally, in order to obtain \(D'(C_m \otimes C_n)\), we again employ the idea of farthest neighbor to construct a supergraph of \(C_m \otimes C_n\). In the construction we add a set of edges to \(C_m \otimes C_n\) in which each edge connects two farthest vertices.
2. THE DIAMETER OF A DIAGONAL MESH NETWORK

For two cycles $C_m$ and $C_n$, $C_m \otimes C_n$ is connected if and only if $C_m$ or $C_n$ is an odd cycle [8]. Hence, in this paper, for an $m \times n$ diagonal mesh network we assume that at least one of $m$ and $n$ is odd.

Given a cycle $C_m$, let $N_m(i)$ denote the set of the vertices in $C_m$ that can be reached from vertex 0 at the $i$th step. Then

$$N_m(i) = \begin{cases} \{1, 3, \ldots, i, m-i, m-i+2, \ldots, m-1\} & \text{for } i \text{ odd,} \\ \{0, 2, 4, \ldots, i, m-i, m-i+2, \ldots, m-2\} & \text{for } i \text{ even.} \end{cases}$$

For example, given $C_4$, we have $N_4(2) = N_4(6) = \{0, 2, 4, 6, 8\}$, $N_4(5) = N_4(7) = N_4(9) = \{1, 3, 5, 7, 9\}$. Given $C_5$, we have $N_5(4) = \{0, 2, 4, 7, 9\}$, $N_5(5) = \{1, 3, 5, 6, 8, 10\}$ and $N_5(10) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. With careful observation, we have the following remarks:

(R1) $i = \lfloor m/2 \rfloor$ is the smallest integer satisfying $\bigcup_{i=0}^{m} N_m(j) = \{0, 1, 2, \ldots, m-1\}$, and in particular, $N_m(\lfloor m/2 \rfloor - 1) \cup N_m(\lfloor m/2 \rfloor) = \{0, 1, 2, \ldots, m-1\}$.

(R2) $\lfloor m/2 \rfloor \not\in N_m(i)$ for all $i \leq \lfloor m/2 \rfloor - 1$.

(R3) $N_m(i) \subseteq N_m(i + 2)$ for $1 \leq i \leq m - 3$, and in particular, $N_m(i) = N_m(i + 2)$ for $m$ even and $m/2 \leq i \leq m - 3$.

Since $C_m \otimes C_n$ is vertex-transitive, the diameter of $C_m \otimes C_n$ is the longest distance among the shortest paths from $(0, 0)$ to all of the other vertices. We use $N(i)$ to denote the set of vertices in $C_m \otimes C_n$ that can be reached from $(0, 0)$ at the $i$th step. It follows that $N(i) = N_m(i) \times N_n(i)$. The diameter of $C_m \otimes C_n$ is the smallest integer $i$ satisfying $\bigcup_{i=0}^{N(i)} = V(C_m) \times V(C_n)$. By the remarks, it is sufficient to show that the diameter of $C_m \otimes C_n$ is the smallest integer $i$ satisfying $\bigcup_{i=0}^{N(i)} = V(C_m) \times V(C_n)$. And it can be easily verified that $D(C_m \otimes C_n) \geq \max\{\lfloor m/2 \rfloor, \lfloor n/2 \rfloor\}$.

Lemma 1 Assume that both $m$ and $n$ are odd. Then,

$$D(C_m \otimes C_n) = \begin{cases} m & \text{for } m = n, \\ m-1 & \text{for } m + 2 \leq n \leq 2m + 1, \\ \lfloor n/2 \rfloor & \text{for } 2m + 3 \geq n. \end{cases}$$

Proof: We distinguish the following cases:

Case 1: $m = n$. Since $N_m(m-1) = V(C_n)$ and $N_n(i) = N_n(i)$, $N(m - 1) = V(C_n) \times V(C_n)$ and $D(C_m \otimes C_n) \leq m - 1$. To show $D(C_m \otimes C_n) \geq m - 1$, it should be noted that $N_m(m-2) = V(C_m) - \{0\}$ and $N_n(m-3) = V(C_n) - \{1, m - 1\}$. Then it follows that $N(m - 2) = N_m(m-2) \times N_n(m-2) = \{0, i\}, (i, 0) \mid 0 \leq i \leq m-1 \}$ and $N(m - 3) = \{V(C_m) \times V(C_n) - \{(1, i), (m-1, i), (i, m-1) \mid 0 \leq i \leq m-1 \}$ Hence $N(m - 2) \cup N(m - 3) = \{0, 1, (m-1, 0), (1, 0) \mid \text{as a consequence, } D(C_m \otimes C_n) \geq m - 1.}$
Case 2: $m + 2 \leq n \leq 2m + 1$. Since $m \geq \lfloor n/2 \rfloor$, $N_a(m - 1) \cup N_a(m) = V(C_n)$. Furthermore, since $N_a(m) = V(C_n)$, and $D(C_m \otimes C_n) \leq m$. To show $D(C_m \otimes C_n) \geq m$, by noting that $N_a(m - 1) = \{0, 2, 4, ..., m - 1, n - m + 1, n - m + 3, ..., n - 2\}$, $N_a(m - 2) = \{1, 3, 5, ..., m - 2, n - m + 2, n - m + 4, ..., n - 1\}$, and $N_a(m - 1) = V(C_n)$, we have $N(m - 1) = N_a(m - 1) \times N_a(m - 1) = V(C_n) \times \{0, 2, 4, ..., m - 1, n - m + 1, n - m + 3, ..., n - 2\}$ and $N(m - 2) = N_a(m - 2) \times N_a(m - 2) = \{1, 2, 3, ..., n - 1\} \times \{1, 3, 5, ..., n - 2, n - m + 2, n - m + 4, ..., n - 1\}$. Hence $N(m - 2) \cup N(m - 1) = V(C_n) \times V(C_n) = \{(0, i) \mid i \in N_a(m) - N(m - 1)\}$. Since $m + 2 \leq n$, $N_a(m) - N_a(m - 1) \neq \emptyset$ and furthermore $N_a(m) - N_a(m - 1) = \{1, 3, ..., m, n - m, n - m + 2, ..., n - 1\}$ and $N(m - 2) \cup N(m - 1) = \{1, 3, ..., m, n - m, n - m + 2, ..., n - 1\} - \{0, 2, 4, ..., n - (m - 1), n - (m - 1) + 2, ..., n - 2\} = \{1, 3, ..., n - m - 1, m + 1, m + 3, ..., n - 1\}$. Consequently, $D(C_m \otimes C_n) \geq m$.

Case 3: $2m + 3 \leq n$, i.e., $\lfloor n/2 \rfloor \geq m + 1$. Note that $N_a(n/2) = V(C_n)$. Hence $D(C_m \otimes C_n) \geq \lfloor n/2 \rfloor$. To show the converse, since $N_a(n/2) - 1 \cup N_a(n/2) = V(C_n)$, $\bigcup_{i=1}^{\lfloor n/2 \rfloor} (N_a(i) \times N_a(i)) = V(C_m) \times V(C_n)$. Thus $D(C_m \otimes C_n) = \lfloor n/2 \rfloor$. The lemma follows.

Let $v$ be any vertex in $C_m \otimes C_n$. We use $F(v)$ to denote the set of farthest neighbors of $v$ in $C_m \otimes C_n$. Let $X = \{0, 1, 2, ..., m - 1\}$ and $Y = \{j + 1, j + 2, ..., j + (n - m - 1), j + (m + 1), j + (m + 3), j + (m + 5), ..., j + (n - 1)\}$. Using the proof of Lemma 1, we have the following corollary.

Corollary 1 Assume that both $m$ and $n$ are odd. Let $v = (i, j)$ be a vertex in $C_m \otimes C_n$. Then

$$F(v) = \begin{cases} 
(i, j), (i, j - 1), (i + 1, j), (i - 1, j) & \text{if } m = n, \\
(i, j) \in Y & \text{if } m + 2 \leq n \leq 2m - 1, \\
(i, j, x, j + m), (x, j + m + 1) \in X \times Y & \text{if } 2m + 1 = n, \\
(i, j + \lfloor n/2 \rfloor, x, j + \lfloor n/2 \rfloor) \in X & \text{if } 2m + 3 \leq n,
\end{cases}$$

where both addition and subtraction are performed with modulus $m$ in the first coordinate and with modulus $n$ in the second coordinate.

Proof: By the vertex-transitivity of $C_m \otimes C_n$, the cases $m = n$ and $m + 2 \leq n \leq 2m - 1$ can be directly derived from the proof of Lemma 1.

For $n = 2m + 1$, the farthest neighbors of a given vertex $j$ in $C_n$ are $j + m$ and $j + m + 1$. Note that $N_a(m) = N_a(m + 1) = V(C_n)$. Hence, by the Case 2 of Lemma 1 the farthest neighbors of $(i, j)$ are $(i, j), (i, j + m)$, and $(x, j + m + 1)$, where $x \in X$ and $y \in Y$.

For $n \geq 2m + 3$, it should be noted that $N_a(n/2) - 1 = V(C_n)$, $N_a(n/2 - 1) = V(C_n)$, and $N_a(n/2) = V(C_n)$. Since the diameter of $C_m \otimes C_n$ is $\lfloor n/2 \rfloor$, the farthest neighbors of $(i, j)$ in $C_m \otimes C_n$ are $(x, j + \lfloor n/2 \rfloor) \in X \times Y$, where $x \in X$.

The corollary follows.

For example, given $C_5 \otimes C_8$, $F(1, 3) = \{(1, 2), (1, 4)\}$. Let $v = (2, 7)$ be a vertex in $C_7 \otimes C_{19}$, $F(v) = \{(2, 0), (2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (2, 12), (2, 14)\}$. Given $C_5 \otimes C_{13}$, $F(1, 8) = \{(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2)\}$.
Lemma 2 Assume that \( m \) is even and \( n \) is odd. Then, \( D(C_m \otimes C_n) = \begin{cases} n & \text{for } m/2 \leq n, \\ m/2 & \text{for } m/2 > n. \end{cases} \)

**Proof:** Suppose that \( m/2 \leq n \). Then, \( N_m(n-1) = \{0, 2, 4, \ldots, m-2\}, N_m(n) = \{1, 3, 5, \ldots, m-1\}, N_n(n-2) = V(C_n) - \{0\}, \) and \( N_n(n-1) = N_n(n) = V(C_n) \). We have

\[
\bigcup_{i=1}^n (N_m(i) \times N_n(i)) = (N_m(n-1) \times N_n(n-1)) \bigcup (N_m(n) \times N_n(n)) = V(C_m) \times V(C_n)
\]

and thus \( D(C_m \otimes C_n) \leq n \). On the other hand,

\[
\bigcup_{i=1}^n (N_m(i) \times N_n(i)) = N(n-2) \bigcup N(n-1) = V(C_m \otimes C_n) - N_m(n-1) \times \{0\}
\]

and thus \( D(C_m \otimes C_n) > n - 1 \). Hence, \( D(C_m \otimes C_n) = n \).

Suppose that \( m/2 > n \). Then we have \( N_m(m/2 - 1) = N_m(m/2) = V(C_m) \) and \( D(C_m \otimes C_n) \geq m/2 \). On the other hand, we have \( (N_m(m/2 - 1) \times N_n(m/2 - 1)) \bigcup (N_m(m/2) \times N_n(m/2)) = V(C_m) \times V(C_n) \). Hence, \( D(C_m \otimes C_n) = m/2 \). The lemma follows.

**Corollary 2** Assume \( m \) is even and \( n \) is odd. Let \( v = (i, j) \) be a vertex in \( C_m \otimes C_n \). Then,

\[
F(v) = \begin{cases} \{(x, 0) | 1 \leq x \leq m - 1 \text{ and } x \text{ is odd} \} & \text{for } m/2 \leq n - 1, \\ \{(x, 0) | 1 \leq x \leq m - 1 \text{ and } x \text{ is odd} \} \bigcup \{(n, y) | 1 \leq y \leq n - 1\} & \text{for } m/2 = n, \\ \{(m/2, y) | 0 \leq y \leq n - 1\} & \text{for } m/2 > n. \end{cases}
\]

3. TWO INNER VERTEX-DISJOINT PATHS IN \( C_m \otimes C_n \)

Before establishing the result we require some notations. Let \( v_0 \) and \( v_1 \) be two distinct vertices in a graph \( G \). A walk from \( v_0 \) to \( v_k \) is a sequence of vertices \( v_0, v_1, v_2, \ldots, v_k \) such that \((v_i, v_{i+1})\) is an edge of \( G \) for \( 0 \leq i \leq k - 1 \). We use \( \langle \) and \( \rangle \) to delimit a walk. The length of a walk is the number of edges in it. A path is a walk with no repeated vertices. Let \( W = \langle w_0, w_1, \ldots, w_k \rangle \) and \( Z = \langle z_0, z_1, \ldots, z_k \rangle \) be walks in a graph \( G \). When \( w_i = z_i \), we use \( WZ \) to denote the walk from \( w_0 \) to \( z_k \) by concatenating \( W \) with \( Z \). Let \( W = \langle w_0, w_1, \ldots, w_k \rangle \) and \( Q = \langle q_0, q_1, \ldots, q_l \rangle \) be walks of length \( k \) in \( C_m \) and \( C_n \), respectively. We define \( W \odot Q = \langle (w_0, q_0), (w_1, q_1), \ldots, (w_k, q_l) \rangle \). By the definition of \( C_m \odot C_n, W \odot Q \) is a walk of length \( k \) from \((w_0, q_0)\) to \((w_k, q_l)\) in \( C_m \odot C_n \). For \( r \geq 1 \), a walk passing \( W \) back and forth \( r \) times is denoted as \( W' \). For example, \( \langle 0, 1 \rangle^3 = \langle 0, 1, 0, 1, 0, 1 \rangle \) and \( \langle 0, 1, 0 \rangle^2 = \langle 0, 1, 0, 1, 0, 1 \rangle \).

By the vertex-transitivity of \( C_m \odot C_n \), it suffices to construct two inner vertex-disjoint paths between \((0, 0)\) and \((i, j)\) such that their lengths are no more than \( D(C_m \odot C_n) \) for any vertex \((i, j)\). In the following proof, detailed straightforward constructions are presented. In the constructions, each path from \((0, 0)\) to \((i, j)\) is determined by two equal-length walks, where one walk is from \( 0 \) to \( i \) in \( C_m \) and the other is from \( 0 \) to \( j \) in \( C_n \). The walk from \( 0 \) to \( i \) in \( C_m \) may be \( \langle 0, 1, \ldots, i \rangle \) or \( \langle 0, m-1, m-2, \ldots, i \rangle \), and similar for the walk from \( 0 \) to \( j \) in \( C_n \), depending on the parity of \( i \) and \( j \).
**Theorem 1** Let both m and n be odd. Then, each pair of vertices in $C_m \boxtimes C_n$ can be connected by two vertex-disjoint paths whose lengths are at most $D(C_m \boxtimes C_n)$ for $n \neq 2m + 1$.

**Proof:** In this proof, two inner vertex-disjoint paths are presented. There are three cases to consider, namely, $m = n$, $m + 2 \leq n \leq 2m - 1$, and $n \geq 2m + 3$.

**Case 1:** $m = n$. We further distinguish the following cases.

1.1: i and j have the same parity.
   (1) If $i = 0$ and $j > 0$, there are two inner vertex-disjoint paths given by $(0, 1, 0)^{j/2} \oplus (0, 1, 2, ..., j)$ and $(m - 1, m - 2, m - 1)^{(j/2 - 1)}/(m - 1, 0) \oplus (0, 1, 2, ..., j)$. For example, this construction is depicted in Fig. 1.

![Fig. 1. Paths $(0, 0), (1, 1), (0, 2), (1, 3), (0, 4), (1, 5), (0, 6)$ and $(0, 0), (8, 1), (7, 2), (8, 3), (7, 4), (8, 5), (0, 6)$ are defined by $(0, 1, 0, 1, 0, 1) \oplus (0, 1, 2, 4, 5, 6)$ and $(0, 8, 7, 8, 7, 8, 0) \oplus (0, 1, 2, 3, 4, 5, 6)$, respectively.](image1)

(2) If $i > 0$ and $j = 0$, two inner vertex-disjoint paths can be defined as $(0, 1, 2, ..., i) \oplus (0, n - 1, n - 2, ..., n - 1)^{(j/2 - 1)/(n - 1, 0)}$. $j$ are two inner vertex-disjoint paths.

(3) If $i = j$, $(0, 1, 2, ..., i) \oplus (0, m - 1, m - 2, ..., i) \oplus (0, n - 1, n - 2, ..., j)$ are two inner vertex-disjoint paths.

(4) If $i < j$, we define $(0, 1, 2, ..., i) \oplus (j, j + 1, j)^{(j/2)} \oplus (0, 1, 2, ..., j)$ and $(0, m - 1, 0)^{(j/2)} \oplus (0, 1, 2, ..., j)$. For example, an illustration of this construction is depicted in Fig. 2.

(5) If $i > j$, $(0, 1, 2, ..., i) \oplus (j, j + 1, j)^{(j/2)} \oplus (0, 1, 2, ..., i) \oplus (0, n - 1, 0)^{(j/2)} \oplus (0, 1, 2, ..., j)$ are two inner vertex-disjoint paths.

1.2: i and j have different parity.
   (1) If $i = 0$ and $j > 0$, two inner vertex-disjoint paths are defined as $(0, 1, 0)^{(m-j/2)} \oplus (0, n - 1, n - 2, ..., j)$ and $(0, m - 1, m - 2, m - 1)^{(m-j/2 - 1)/(m - 1, 0)} \oplus (0, n - 1, n - 2, ..., j)$. An example of this construction is given in Fig. 3.
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Fig. 2. Paths \((0, 0), (1, 1), (2, 2), (3, 3), (2, 4), (3, 5), (2, 6), (3, 7), (2, 8)\) and \((0, 0), (8, 1), (0, 2), (8, 3), (0, 4), (8, 5), (0, 6), (1, 7), (2, 8)\) are given by \((0, 1, 2, 3, 2, 3, 2, 3, 2)\) \(\otimes\) \((0, 1, 2, 4, 5, 6, 7, 8)\) and \((0, 8, 0, 8, 0, 8, 0, 1, 2)\) \(\otimes\) \((0, 1, 2, 3, 4, 5, 6, 7, 8)\), respectively.

Fig. 3. Paths \((0, 0), (1, 8), (0, 7), (1, 6), (0, 5), (1, 4), (0, 3)\) and \((0, 0), (8, 8), (7, 7), (8, 6), (7, 5), (8, 4), (0, 5)\) are defined by \((0, 1, 0, 1, 0, 1, 0)\) \(\otimes\) \((0, 8, 7, 6, 5, 4, 3)\) and \((0, 8, 7, 8, 7, 8, 0)\) \(\otimes\) \((0, 8, 7, 6, 5, 4, 3)\), respectively.

(2) If \(i > 0\) and \(j = 0\), two paths are given by \((0, m-1, m-2, \ldots, i)\) \(\otimes\) \((0, 1, 0)\) and \((0, m-1, m-2, \ldots, i)\) \(\otimes\) \((0, n-1, n-2, \ldots, i-1)\).

(3) If \(i = n-j\), we define \((0, 1, 2, \ldots, i)\) \(\otimes\) \((0, n-1, n-2, \ldots, j)\) and \((0, m-1, m-2, \ldots, i)\) \(\otimes\) \((0, 1, 2, \ldots, j)\). For example, an illustration of this construction is given in Fig. 4.

(4) If \(i > n-j\), the two paths are \((0, 1, 2, \ldots, i)\) \(\otimes\) \((0, 1, 0)\) and \((0, 1, 2, \ldots, i)\) \(\otimes\) \((0, n-1, n-2, \ldots, j)\) \(\otimes\) \((0, n-1, n-2, \ldots, j)\).

(5) If \(i < n-j\), \((0, 1, 2, \ldots, i)\) \(\otimes\) \((0, n-1, n-2, \ldots, j)\) and \((0, m-1, 0)\) are two inner vertex-disjoint paths.
Fig. 4. Paths \(\langle 0, 0 \rangle, \langle 1, 8 \rangle, \langle 2, 7 \rangle, \langle 3, 6 \rangle\) and \(\langle 0, 0 \rangle, \langle 8, 1 \rangle, \langle 7, 2 \rangle, \langle 6, 3 \rangle, \langle 5, 4 \rangle, \langle 4, 5 \rangle, \langle 3, 6 \rangle\) are defined by \((0, 1, 2, 3) \otimes (0, 8, 7, 6)\) and \((0, 8, 7, 6, 5, 4, 3) \otimes (0, 1, 2, 3, 4, 5, 6)\), respectively.

Case 2: \(m + 2 \leq n \leq 2m - 1\). We consider the following cases:

2.1: \(i\) and \(j\) have the same parity. We further consider the following cases:

2.1.1: \(i = 0\) or \(j = 0\).
(1) If \(i = 0\) and \(0 < j < m\), two inner vertex-disjoint paths are similar to 1.1(1).
(2) If \(i = 0\) and \(j > m\), \(m \geq n - j\). Two inner vertex-disjoint paths are given by \((0, 1, 2, ..., m - 1, 0) \otimes (0, n - 1, n - 2, ..., j) \langle j, j - 1, j \rangle^{(m - j)/2}\) and \((0, m - 1, m - 2, ..., 1, 0) \otimes (0, n - 1, n - 2, ..., 1) \langle j, j - 1, j \rangle^{(m - j)/2}\).
(3) If \(i > 0\) and \(j = 0\), the construction is similar to 1.1(2).

2.1.2: \(0 < i \leq j \leq m - 2\). The constructions are similar to 1.1(3) and (4).

2.1.3: \(i > j > 0\). Two inner vertex-disjoint paths are similar to 1.1(5).

2.1.4: \(0 < i \leq m - 1 \leq j\). Two cases \(m - i = n - j\) and \(m - i \neq n - j\) to consider.
(1) If \(m - i = n - j\) with \(i = 1, j = n - m + 1 \leq m\). Two paths are defined by \((0, m - 1, m - 2, ..., 1) \otimes (0, n - 1, n - 2, ..., j)\) and \((0, 1)^{(n - j)/2} \otimes (0, 1, 2, ..., j)\).
(2) If \(m - i = n - j\) with \(i > 1\), we define \((0, m - 1, m - 2, ..., i) \otimes (0, n - 1, n - 2, ..., j)\) and \((0, 1, 0, m - 1, m - 2, ..., i) \otimes (0, n - 1, n - 2, ..., j, j - 1, j)\).
(3) If \(m - i \neq n - j\), for \(m - i < (n - j)\), we have \((0, m - 1, m - 2, ..., i) \langle i, i - 1, j \rangle^{(n - j)/2} \otimes (0, n - 1, n - 2, ..., j)\) and \((0, 1, 0, m - 1, m - 2, ..., i) \langle i, j - 1, j \rangle^{(n - j)/2}\).

2.2: \(i\) and \(j\) have different parity. We further consider the following cases:

2.2.1: \(i = 0\) or \(j = 0\).
(1) If \(i = 0\) and \(0 < j \leq m\), two inner vertex-disjoint paths are defined by \((0, 1, 2, ..., m, 1, 0) \otimes (0, 1, 2, ..., j) \langle j, j + 1, j \rangle^{(m - j)/2}\) and \((0, m - 1, m - 2, ..., 1, 0) \otimes (0, 1, 2, ..., j) \langle j, j + 1, j \rangle^{(m - j)/2}\).
(2) If \(i > 0\) and \(j > m\), two inner vertex-disjoint paths are similar to 1.2(1).
(3) If \(i > 0\) and \(j = 0\), the construction is similar to 1.2(2).

2.2.2: \(i > j > 0\).
(1) If \( m - i \geq j \), we have \( m - i \leq m - 2 \). Two paths are defined by \( (0, m - 1, m - 2, \ldots, i) \ominus \langle 0, 1, 2, \ldots, j \rangle \) and \( (0, m - 1, m - 2, \ldots, i, i - 1, i) \ominus \langle 0, n - 1, 0 \rangle \). If \( m - i < j \), we define \( (0, m - 1, m - 2, \ldots, j) \) and \( (0, 1, 0, 1, 2, \ldots, j) \).

2.2.2: \( 0 < i < j < m \). We distinguish two cases \( m - i = j \) and \( m - i \neq j \).

(1) If \( m - i = j \) with \( i = 1, n - j = n - m + 1 \leq m \). Two inner vertex-disjoint paths are defined by \( (0, m - 1, m - 2, \ldots, 1) \ominus \langle 0, 1, 2, \ldots, j \rangle \) and \( (0, 1, 0, m - 1, m - 2, \ldots, j) \).

(2) If \( m - i = j \) with \( i > 1 \), two paths are defined as \( (0, m - 1, m - 2, \ldots, i) \ominus \langle 0, 1, 2, \ldots, j \rangle \) and \( (0, 1, 0, m - 1, m - 2, \ldots, j) \).

(3) If \( m - i > j \), we define \( (0, m - 1, m - 2, \ldots, i) \ominus \langle 0, 1, 2, \ldots, j \rangle \) and \( (0, 0, m - 1, m - 2, \ldots, i) \).

(4) If \( m - i < j \), two paths are given by \( (0, m - 1, m - 2, \ldots, j) \ominus \langle 0, 1, 2, \ldots, j \rangle \) and \( (0, 1, 0, m - 1, m - 2, \ldots, j) \).

2.2.3: \( 0 < i \leq m \). We consider \( i = n - j \) and \( i \neq n - j \).

(1) If \( i = n - j \) with \( j = m \), two paths are defined by \( (0, 1, 2, \ldots, 1) \ominus \langle 0, n - 1, 0 \rangle \) and \( (0, 0, m - 1, m - 2, \ldots, j) \).

(2) If \( i = n - j \) with \( j > m \), two paths are given by \( (0, 1, 2, \ldots, i) \ominus \langle 0, n - 1, 0 \rangle \) and \( (0, m - 1, 0, 1, 2, \ldots, 1) \).

(3) If \( i > n - j \), the construction is similar to 1.2(4).

(4) If \( i < n - j \), the construction is similar to 1.2(5).

Case 3: \( n \geq 2m + 3 \). We consider the following cases:

3.1: \( i \) and \( j \) have the same parity. We further consider the following cases:

3.1.1: \( i = 0 \) or \( j = 0 \). It is similar to 2.1.1.

3.1.2: \( 0 < i \leq \lfloor n/2 \rfloor \).

(1) If \( j = \lfloor n/2 \rfloor - 1 \) or \( j = \lfloor n/2 \rfloor \), we have \( i \leq m - 1 \leq \lfloor n/2 \rfloor - 2 \). Two inner vertex-disjoint paths are \( (0, 1, 2, \ldots, i) \ominus \langle 0, 1, 2, \ldots, j \rangle \) and \( (0, m - 1, 0, 1, 2, \ldots, i) \ominus \langle 0, m - 1, 0 \rangle \).

(2) If \( j \leq \lfloor n/2 \rfloor - 2 \), the construction is similar to 2.1.2.

3.1.3: \( i > j \).

3.1.4: \( 0 < i \leq m - 1 \) and \( j > \lfloor n/2 \rfloor \). It is similar to 2.1.4. Note that there is no need to divide the case \( m - i = n - j \) into \( i = 1 \) and \( i > 1 \).

3.2: \( i \) and \( j \) have different parity. We further consider the following cases:

3.2.1: \( i = 0 \) or \( j = 0 \). It is similar to 2.2.1.

3.2.2: \( i > j \). It is similar to 2.2.2.

3.2.3: \( i < j \leq \lfloor n/2 \rfloor \). It is similar to 2.2.3.

3.2.4: \( i \leq m - 1 \) and \( j > \lfloor n/2 \rfloor \). It is similar to 2.2.4.

It can be easily checked that the length of each path defined in each case is no more than the diameter of \( C_m \otimes C_n \). That completes the proof.

The next corollary is the directed result of Theorem 1.

Corollary 3 Let \( m \) and \( n \) be odd. Then \( D^+(C_m \otimes C_n) \geq 2 \) for \( n \neq 2m + 1 \).
Theorem 2  Let $m$ be even and $n$ be odd. Then, for $n \neq m/2$, each pair of vertices in $C_m \otimes C_n$ can be connected by two vertex-disjoint paths whose lengths are at most $D(C_m \otimes C_n)$.

Proof: Similar to the proof in Theorem 1, we shall find two vertex-disjoint paths from $(0, 0)$ to $(i, j)$ for any vertex $(i, j)$. We distinguish two cases, namely, $m/2 > n$ and $m/2 < n$.

Case 1: $m/2 > n$. We further distinguish the following subcases.

1.1: $i = j$. In this case, since $j < n$ and $n < m/2$, we have $j \leq m/2 - 2$. We can define two inner vertex-disjoint paths as follows: $(0, 1, 2, ..., i) \otimes (0, 1, 2, ..., j)$ and $(0, 1, 2, ..., i, i + 1, i) \otimes (0, n - 1, 0, 1, 2, ..., j)$.

1.2: $i$ and $j$ have the same parity. We further distinguish the following subcases.

1.2.1: $i = n - j$. In this case, $i \geq m/2$ is excluded from consideration. For $i < m/2$, we define $(0, 1, 2, ..., i) \otimes (0, n - 1, n - 2, ..., j)$ and $(0, 1, 2, ..., i, i + 1, i) \otimes (0, 1, 0, n - 1, n - 2, ..., j)$. However, a special case is $n = m/2 + 1$ with $i = n = 0$. Two inner vertex-disjoint paths are defined as $(0, 1, 2, ..., n) \otimes (0, 1, 2, ..., 0)$ and $(0, 1, 2, ..., n) \otimes (0, n - 1, n - 2, ..., 0)$.

1.3: $i$ and $j$ have different parity. We further distinguish the following three cases.

1.3.1: $i < n - j$. The two paths are defined by $(0, 1, 2, ..., i) \otimes (0, n - 1, n - 2, ..., j)$ and $(0, m - 1, 0, 1, 2, ..., i) \otimes (0, n - 1, n - 2, ..., j)$.

1.3.3: $i > n - j$. We further distinguish the following subcases.

2.1: $i = j$. The two paths are defined by $(0, 1, 2, ..., i) \otimes (0, 1, 2, ..., j)$ and $(0, m - 1, 0, 1, 2, ..., i) \otimes (0, 1, 2, ..., j)$.

2.2: $i \neq j$ and $i, j$ having the same parity. We further distinguish the following subcases.

2.3: $i$ and $j$ have different parity. We further distinguish the following subcases.
2.3.1: \( i = n - j \).
1. If \( i = m/2 \), two inner vertex-disjoint paths are defined by \( (0, 1, 2, \ldots, i) \otimes (0, n-1, n-2, \ldots, j) \) and \( (0, m-1, m-2, \ldots, i) \otimes (0, n-1, n-2, \ldots, j) \).
2. If \( i < m/2 \), the construction is similar to 1.3.1.
3. If \( i > m/2 \), we define \( (0, 1, 2, \ldots, i) \otimes (0, n-1, n-2, \ldots, j) \) and \( (0, m-1, m-2, \ldots, i)(i + 1, \ldots, j) \otimes (0, n-1, n-2, \ldots, j) \).

2.3.2: \( i < n - j \). In this case, the construction is similar to 1.3.2.

2.3.3: \( i > n - j \). In this case, we consider the following subcases.
1. If \( n - j < i < m/2 \), the construction is similar to 1.3.3(1).
2. If \( n - j < m/2 < i \), we first consider \((m - i) \leq (n - j)\). In this case, \( n - j \leq n - 2 \). We define \( (0, 1, 0)^{(m-i)-(n-j)/2+1}(0, m-1, m-2, \ldots, i) \otimes (0, n-1, n-2, \ldots, j) \) and \( (0, m-1, m-2, \ldots, i)(i+1, \ldots, j) \otimes (0, n-1, n-2, \ldots, j) \). Second, for \( (m - i) > (n - j) \), it is similar to 1.3.3(2).
3. If \( m/2 < n - j < i \), we have \( (0, m-1, m-2, \ldots, i)(i+1, i-1, \ldots, j) \otimes (0, n-1, n-2, \ldots, j) \) and \( (0, 1, 0)^{(m-i)-(n-j)/2}(0, m-1, m-2, \ldots, i) \otimes (0, n-1, n-2, \ldots, j) \).

It can be easily verified that the length of each path defined in each case is no more than the diameter of \( C_n \otimes C_n \). That completes the proof.

4. THE CALCULATION OF \( D'(C_n \otimes C_n) \)

Let \( P = (x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k) \) be a path in \( C_n \otimes C_n \). For the sake of simplicity, we say that \( P \) takes \( k \) steps passing the vertex sequence \( x_0, x_1, \ldots, x_k \) in \( C_n \) or the vertex sequence \( y_0, y_1, \ldots, y_k \) in \( C_n \).

**Theorem 3** Let both \( m \) and \( n \) be odd. Then, \( D'(C_n \otimes C_n) = \begin{cases} 1 & \text{if } n = 2m + 1, \\ 2 & \text{otherwise.} \end{cases} \)

**Proof:** We first consider the case \( n = 2m + 1 \). By noting that \( D(C_n \otimes C_n) = m \), each path from \( (0, 0) \) to \( (m - 1, m - 1) \) with length no more than \( m \) must take \( m - 1 \) steps passing the unique vertex sequence \( 0, 1, \ldots, m - 1 \) in \( C_n \). Otherwise, it will take at least \( m + 1 \) steps. Suppose that we delete the edge \( ((0, 0), (1, 1)) \) from \( C_m \otimes C_m \). Then any path from \( (0, 0) \) to \( (m - 1, m - 1) \) takes at least \( m + 1 \) steps to pass the vertex sequence starting with \( 0 \) and ending at \( m - 1 \) in \( C_m \). It follows that the distance is at least \( m + 1 \). Thus \( D'(C_m \otimes C_m) = 1 \) for \( n = 2m + 1 \).

For \( n \neq 2m + 1 \), based on Theorem 1, it is sufficient to prove that the distance of a specific path will be greater than \( D(C_m \otimes C_m) \) when we delete two specific edges from \( C_m \otimes C_m \). We distinguish the following cases:

**Case 1:** \( m = n \). It is shown in Corollary 1 that the distance between \( (0, 0) \) and \( (0, 1) \) is exactly equal to the diameter of \( C_n \otimes C_n \), i.e., \( m - 1 \). It follows that each shortest path from \( (0, 0) \) to \( (0, 1) \) takes \( m - 1 \) steps to pass the vertex sequence from vertex 0 to vertex 1 in \( C_n \). Therefore, all of the shortest paths from \( (0, 0) \) to \( (0, 1) \) contain exactly one of the edges \( ((0, 0), (1, m - 1)) \) and \( ((0, 0), (m - 1, m - 1)) \). As a consequence, deleting \( ((0, 0), (1, m - 1)) \) and \( ((0, 0), (m - 1, m - 1)) \) from \( C_n \otimes C_n \) causes the distance between \( (0, 0) \) and \( (0, 1) \) to be greater than \( D(C_m \otimes C_m) \), and thus \( D'(C_m \otimes C_m) = 2 \).
Case 2: \( m + 2 \leq n \leq 2m - 1 \). Note that any shortest path from \((0, 0)\) to \((m - 1, 0)\) in \(C_m \otimes C_n\) passing the vertex sequence \(0, 1, \ldots, 0, n - 1, \ldots, 0\) in \(C_n\) takes even steps. It follows that the shortest path passes the vertex sequence \(0, 1, 2, \ldots, m - 1\) in \(C_m\). Let \(G\) denote the graph obtained from \(C_m \otimes C_n\) by removing \((0, 0), (1, n - 1)\) and \((0, 0), (1, 1)\). Then, any shortest path from \((0, 0)\) to \((m - 1, 0)\) in \(G\) passing the vertex sequence \(0, 1, 2, \ldots, m - 1\) in \(C_m\) takes odd steps. It results in that any shortest path passing vertex sequences in \(C_m\) starting with vertex 0 and ending up with vertex 0 takes odd steps. Thus the distance from \((0, 0)\) to \((m - 1, 0)\) is at least \(m + 1\) and \(D'(C_m \otimes C_n) = 2\).

Case 3: \( n \geq 2m + 3 \). By the Corollary 1 the farthest neighbors of \((0, 0)\) are \((i, \lfloor n/2 \rfloor)\) for \(0 \leq i \leq m - 1\), it follows that any shortest path from \((0, 0)\) to \((i, \lfloor n/2 \rfloor)\) follows that the shortest path passes the vertex sequence \(0, 1, \ldots, 1\) in \(C_n\). Similar to Case 1, the diameter of \(C_m \otimes C_n\) will increase when we delete the two edges \((0, 0), (1, 1)\) and \((0, 0), (m - 1, 1)\) from \(C_m \otimes C_n\). Thus, \(D'(C_m \otimes C_n) = 2\).

Hence the theorem follows.

**Theorem 4** Let \(m\) be even and \(n\) be odd. Then \(D'(C_m \otimes C_n) = \begin{cases} 1 & \text{if } n = m/2, \\ 2 & \text{otherwise.} \end{cases}\)

**Proof:** We distinguish the following cases:

Case 1: \( n = m/2 \). In this case, \(D(C_m \otimes C_n) = n\). Any shortest path from \((0, 0)\) to \((n - 1, n - 1)\) is determined by the vertex sequence \(0, 1, 2, \ldots, n - 1\) in \(C_n\). When the edge \((0, 0), (1, 1)\) is deleted from \(C_m \otimes C_n\), each path from \((0, 0)\) to \((n - 1, n - 1)\) must pass the vertex sequence \(m - 1, m - 2, \ldots, n - 1\) in \(C_m\). And the length will be at least \(n + 1\). Thus \(D'(C_m \otimes C_n) = 1\).

Case 2: \( n < m/2 \). We consider any path from \((0, 0)\) to \((m/2 - 1, 1)\). The shortest path from 0 to \(m/2 - 1\) in \(C_m\) is determined by the unique vertex sequence \(0, 1, 2, \ldots, m/2 - 1\). Thus, when we remove the edges \((0, 0), (1, 1)\) and \((0, 0), (1, n - 1)\) from \(C_m \otimes C_n\), any path from \((0, 0)\) to \((m/2 - 1, 1)\) must pass the vertex sequence \(m - 1, m - 2, \ldots, m/2 - 1\) in \(C_m\). And the length will be at least \(m/2 + 1\). Thus \(D'(C_m \otimes C_n) = 2\).

Case 3: \( n > m/2 \). We consider any path from \((0, 0)\) to \((0, 1)\). Since the length of any walk from 0 to 0 in \(C_m\) is even, any walk from 0 to 1 in \(C_n\) with even length is determined by the vertex sequence \(0, n - 1, \ldots, 1\). When we delete the edges \((0, 0), (1, n - 1)\) and \((0, 0), (m - 1, n - 1)\) from \(C_m \otimes C_n\), each path from \((0, 0)\) to \((0, 1)\) must pass the vertex sequence \(0, 1, \ldots, 1\) in \(C_n\). And the length will be at least \(n + 1\). Thus \(D'(C_m \otimes C_n) = 2\).

5. AN UPPER BOUND FOR \(D'(C_m \otimes C_n)\)

In section, we use Corollary 1 and 2 to compute \(D'(C_m \otimes C_n)\).

**Theorem 5** Let both \(m\) and \(n\) be odd. Then,

\[
D'(C_m \otimes C_n) \leq \begin{cases} \left\lceil \frac{m/2}{\lceil n/2 \rceil} \right\rceil & \text{for } m \leq n \leq 2m - 1, \\ 2 \left\lceil \frac{m/2}{\lceil n/2 \rceil} \right\rceil & \text{for } 2m + 1 \leq n. \end{cases}
\]
**Proof:** In order to decrease the diameter of \( C_m \otimes C_n \), we construct a supergraph \( H \) of \( C_m \otimes C_n \) by adding a set of edges in which each edge connects one vertex and its farthest neighbor. We consider two cases as follows.

**Case 1:** \( m \leq n \leq 2m - 1 \). In this case, we add the set of edges

\[
E_1 = \{(x, y), (x, y + 1)\} \mid 0 \leq x \leq m - 2, 0 \leq y \leq n - 1 \text{ and } x, y \text{ even}\]

to \( C_m \otimes C_n \). Then, \( |E_1| = \lceil m/2 \rceil \lceil n/2 \rceil \). Let \( v = (i, j) \) be a vertex in \( C_m \otimes C_n \). Then, it suffices to show that \( d_H(v, v') \leq D(C_m \otimes C_n) - 1 \), where \( v' \) is a farthest neighbor of \( v \).

When \( m = n \), as stated in Corollary 1, \( v' \) is one of \((i, j + 1), (i, j - 1), (i - 1, j), (i + 1, j)\). Hence, we can show that \( d_H(v, v') \leq 3 \) when at least one of \( i \) and \( j \) is odd.

When \( m + 2 \leq n \leq 2m - 1 \), as stated in Corollary 1, \( F(v) \) is the set \{\((i, y) | y = j + 1, j + 3, \ldots, j + (n - m - 1), j + (m + 1), j + (m + 3), j + (m + 5), \ldots, j + (n - 1)\}\). Similar to the above case, we can show that \( d_H(v, v') \leq 3 \).

**Case 2:** \( n \geq 2m + 1 \). Let

\[
E_2 = \{(x, y), (x + \lfloor m/2 \rfloor, y + \lfloor n/2 \rfloor), (x, y + 1), (x + \lfloor m/2 \rfloor, y + \lceil n/2 \rceil)\} \mid 0 \leq x < m - 1, 0 \leq y \leq \lfloor n/2 \rfloor \text{ and } x, y \text{ even}\}

Then, \( |E_2| = \lceil m/2 \rceil \lceil n/2 \rceil \). In this case, a supergraph \( H_1 \) of \( C_m \otimes C_n \) is constructed by adding \( E_1 \cup E_2 \). Similar to Case 1, we can show that the diameter of \( H_1 \) is less than \( D(C_m \otimes C_n) \) and we omit the proof here.

**Theorem 6** Let \( m \) be even and \( n \) be odd. Then

\[
D(C_m \otimes C_n) \leq \begin{cases} \lfloor m/2 \rfloor \lceil n/2 \rceil & \text{for } m/2 < n - 1, \\ \lfloor m/2 \rfloor n & \text{for } m/2 \geq n - 1. \end{cases}
\]

**Proof:** We distinguish the following cases.

**Case 1:** \( m/2 < n - 1 \). As stated in Corollary 2, this case is similar to the case \( m + 2 \leq n < 2m + 1 \) in the proof of Theorem 5, hence, we add the set of edges

\[
E_1 = \{(x, y), (x, y + 1)\} \mid 0 \leq x \leq m - 2, 0 \leq y \leq n - 1 \text{ and } x, y \text{ even}\]

to \( C_m \otimes C_n \). Then, \( |E_1| = \lfloor m/2 \rfloor \lceil n/2 \rceil \). It is available to decrease the diameter of \( C_m \otimes C_n \).

**Case 2:** \( m/2 = n - 1 \). In this case, the diameter of \( C_m \otimes C_n \) is \( n \). It can be checked that the diameter is not available to decrease when the set of edges \( E_1 \) is added to \( C_m \otimes C_n \). The reason is that the distance between \((x, y)\) and \((x + m/2, y)\) is still \( n \) for some even \( y \) with \( 0 \leq y \leq n - 1 \) and \( 0 \leq x \leq m - 1 \). Hence, we add another edges to \( C_m \otimes C_n \) to decrease the diameter of \( C_m \otimes C_n \). Let
\[ E_2 = E_1 \cup \{(x, y), (x + m/2, y + 2)\} \mid 0 \leq x \leq m - 2, 0 \leq y \leq n - 2 \text{ and } x, y \text{ even}. \]

Then, \[ |E_2| = (m/2 \lfloor n/2 \rfloor + (m/2 \lfloor n/2 \rfloor) = (m/2)n \text{ and the distance between } (i, j) \text{ and } (i + m/2, j) \text{ is less than or equal to 5.} \]

**Case 3:** \(m/2 = n\). By following the Corollary 2, the distance between \((n, y_1)\) and \((n, y_2)\) is less than \(n\) after adding the edges \((n, y), (n, y + 1)\) to \(C_m \odot C_n\) for \(0 \leq y \leq n - 1\) with \(y\) even. Hence, adding the edges in \(E_2\) is available to decrease the diameter of \(C_m \odot C_n\).

**Case 4:** \(m/2 \geq n\). Using the similar arguments as described in **Case 2**, adding the edges in \(E_2\) is available to decrease the diameter of \(C_m \odot C_n\).

This theorem holds.

### 6. CONCLUSION

In this paper, we have studied the diameter variability problem arising from the change of edges for the graph \(C_m \odot C_n\). We show that there are two vertex-disjoint paths connecting two different vertices whose length are at most \(D(C_m \odot C_n)\). We provide a simple approach to compute the diameter of \(C_m \odot C_n\). A consequence of our approach is that we can easily find the farthest neighbors for any vertex in \(C_m \odot C_n\). A concept of the farthest neighbors is an important role in studying the \(D'(C_m \odot C_n)\) and \(D(C_m \odot C_n)\). In addition, we construct a supergraph of \(C_m \odot C_n\) by adding edges such that the diameter is decreased.

We conjecture that the values for \(D(C_m \odot C_n)\) in Theorems 5 and 6, respectively, are the exact values of \(D'(C_m \odot C_n)\).

In the future, we are interested in considering the extent of the change of diameter of graphs, i.e., given an arbitrary positive integer \(k\), we define

- \(D^k(G)\): the least number of edges whose addition to \(G\) decreases the diameter by at least \(k\).
- \(D^k(G)\): the least number of edges whose deletion from \(G\) increases the diameter by at least \(k\).

It is easy to check that \(D^1(C_m) = D^2(C_m) = D^{(m-1-[m/2])}(C_m) = 1\) and \(D(P_m) = D^2(P_m) = \ldots = D^{(m-1-[m/2])}(P_m) = 1\). For any graph \(G\), \(D^i(G) \geq \kappa'(G)\) for all positive integers \(i\), where \(\kappa'(G)\) is the edge connectivity of \(G\). Note that \(0 \leq D^i(G) \leq D^j(G)\) and \(0 \leq D^i(G) \leq D^j(G)\) if \(i \leq j\).

### REFERENCES

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