TR-92-002
Corrections to "Scheduling Algorithms for Multiprogramming in a Hard-Real-Time Environment"\textsuperscript{1}

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November 15, 1991

\textsuperscript{1}C. L. Liu and J. Layland, J. ACM, 20(1), 1973
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I. Introduction

In the original paper[1], there are two errors in the proof of theorem 4. The first lies in the following statements copied from the original paper.

Suppose that

\[ C_1 = T_2 \cdot T_1 \cdot \Delta, \quad \Delta > 0 \]

Let

\[ C_1'' = T_2 - T_1 \]
\[ C_2'' = C_2 - 2 \Delta \]
\[ C_3'' = C_3 \]
\[ \vdots \]
\[ C_m'' = C_m \]

Again, \( C_1'', C_2'', C_3'', \ldots, C_m'' \) fully utilize the processor.

We find that equation (1) holds only when \( C_2 > 2 \Delta \). Furthermore, the resulting task set does not always fully utilize the processor. Consider the task set \( T = \{ \tau_1, \tau_2, \tau_3 \} \), where

\[ \tau_1 : T_1 = 3, \quad C_1 = 0.5 \]
\[ \tau_2 : T_2 = 4, \quad C_2 = 1.5 \]
\[ \tau_3 : T_3 = 5, \quad C_3 = 1.5 \]
Obviously, this task set fully utilizes the processor. By changing $C_1$, $C_2$ to $C_1''$, $C_2''$ according to equation (1), we have a new task set $T'' = \{\tau_1'', \tau_2'', \tau_3''\}$

\begin{align*}
\tau_1'': T_1 &= 3, \quad C_1'' = 1 \\
\tau_2'': T_2 &= 4, \quad C_2'' = 0.5 \\
\tau_3'': T_3 &= 5, \quad C_3'' = 1.5
\end{align*}

Note that we can increase $C_3''$ by 0.5 while the resultant task set remains feasible. Thus, task set $T''$ does not fully utilize the processor.

The second error lies in the proof of the utilization bound $U = m (2^{1/m} - 1)$. In the original paper, the authors use partial derivatives to find the minimum. We know that the point at which all partial derivatives equal zero is a local extreme, and not necessarily a global minimum.

Following is my proof of theorem 4.
II. Correction to Original Proof

Theorem 4. For a set of \( m \) tasks with fixed priority order, and the restriction that the ratio between any two request periods is less than 2, the least upper bound to the processor utilization factor is \( U = m \left( 2^{1/m} - 1 \right) \).

Proof. Let \( T \) denote the task set consisting of \( \tau_1, \tau_2, \tau_3, \ldots, \tau_m \), and \( C_1, C_2, C_3, \ldots, C_m \) be the run-times of the task that fully utilize the processor. Assume that \( T_m > T_{m-1} > \ldots > T_2 > T_1 \). Let \( U \) denote the processor utilization factor.

First, we wish to show that when the following conditions are met, the utilization factor is minimum under the assumption that the periods are fixed and the run-times are allowed to change.

\[
C_1 = T_2 - T_1 \\
C_2 = T_3 - T_2 \\
C_3 = T_4 - T_3 \\
\vdots \\
C_m = T_{m+1} - T_m, \text{ where } T_{m+1} = 2T_1
\]

The proof is divided by two stages.

Stage 1. We are going to show that if \( C_j > T_{j+1} - T_j \) for some \( j, 1 \leq j \leq m-1 \), then there exists \( T' \) that fully utilizes the processor with

\[
C_i' \leq T_{i+1} - T_i, \text{ for all } 1 \leq i \leq m-1
\]

and

\[
U' < U,
\]

where \( U' \) is the utilization factor of \( T' \) and \( U \) is the utilization factor of \( T \).

(i) Starting from \( \tau_1 \), find the first task satisfying \( C_i = T_{i+1} - T_i + \Delta \) and \( \Delta > 0 \). Let \( j \)-th task be this task. We define \( C'_i \) for \( 1 \leq i \leq m \) as follows

\[
C_1' = C_1 \\
C_2' = C_2 \\
\vdots \\
C_j' = T_{j+1} - T_j \\
C_{j+1}' = C_{j+1} + \Delta \\
\vdots \\
C_m' = C_m
\]

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Clearly, $C'_1, C'_2, C'_3, \ldots, C'_m$ also fully utilize the processor. Let $U'$ denote the corresponding utilization factor. It follows that

$$U - U' = \Delta/T_j - \Delta/T_{j+1} > 0.$$ 

That is

$$U' < U.$$

(ii) Starting from the $j+1$-th task, continue the search, and do the same transformation as (i) until the $m$-1-th task. Finally, we get a new task set with run-time $C'_i$ of each task $i$ except $\tau'_m$ satisfying

$$C'_i \leq (T_{i+1} - T_i), \text{ for all } 1 \leq i \leq m-1$$

and still fully utilizes the processor. According to the proof of (i), the utilization factor of this new task set is also less than the original task set.

Stage 2. Let $\mathcal{T}'$ denote the resulting task set of stage 1. In this stage, we show that if $C'_j < T_{j+1} - T_j$ for some $j$, $1 \leq j \leq m-1$, then there exists $\mathcal{T}''$ that fully utilizes the processor with

$$C''_i = T_{i+1} - T_i, \text{ for all } 1 \leq i \leq m, T_{m+1} = 2T_1$$

and

$$U'' < U',$$

where $U''$ is the utilization factor of $\mathcal{T}''$ and $U'$ is the utilization factor of $\mathcal{T}'$.

(i) Starting from $\tau'_1$, find the first task satisfying $C'_j = T_{j+1} - T_j - \Delta$ and $\Delta > 0$. Let $j$-th task be this task. We define $C''_i$ for $1 \leq i \leq m$ as follows

$$C''_1 = C'_1$$

$$C''_2 = C'_2$$

$$\cdots$$

$$C''_j = T_{j+1} - T_j$$

$$\cdots$$

$$C''_m = C'_m - 2\Delta.$$

For this set of tasks to be feasible and fully utilize the processor, the following two conditions must be met.

1. $C'_m \geq 2\Delta$.

2. Part of the execution time of $\tau'_m$ must be within $[T_j, T_m]$ with elapsed time $\geq \Delta$.

Since

$$C''_i \leq (T_{i+1} - T_i), \text{ for } 1 \leq i \leq m-1, i \neq j$$

and

$$C''_j = T_{j+1} - T_j - \Delta, \Delta > 0,$$ 

it follows that

$$C'_1 + C'_2 + C'_3 + \ldots + C'_{m-1} \leq T_m - T_1 - \Delta < 2T_1 - T_1 - \Delta = T_1 - \Delta.$$
This means that the first jobs of \( \tau_1', \tau_2', \tau_3', \ldots, \tau_{m-1}' \) completed before time \( T_1 - \Delta \). Thus, we can assure that the run-time of \( \tau_m' \) in the time zones \([0,T_1]\) and \([T_j,T_{j+1}]\) is greater than or equal to \( \Delta \), respectively (See remark 1). This guarantees the above two conditions to be met while the resulting task set is still feasible and also fully utilizes the processor (See remark 2).

Let \( U'' \) denote the corresponding utilization factor. Then we have

\[
U' - U'' = -\Delta/T_j + 2\Delta/T_m > 0,
\]

and equivalently

\[
U'' < U'.
\]

(ii) Starting from the \( j+1 \)-th task, continue the search, and do the same transformation as (i) until the \( m-1 \)-th task. Finally, we get a new task set with run-time \( C_i^0 \) of each task \( i \) satisfying

\[
C_i^0 = (T_{i+1} - T_i), \quad \text{for} \quad 1 \leq i \leq m-1
\]

and

\[
C_m^0 = 2T_1 - T_m.
\]

According to (i), this new task set also fully utilizes the processor and its utilization factor is also less than the original task set.

Thus, for a task set that fully utilizes the processor, we have proved that if the periods are fixed and the run-times are allowed to change, the utilization factor is minimized when the following conditions are met

\[
C_1^0 = T_2 - T_1
\]

\[
C_2^0 = T_3 - T_2
\]

\[
C_3^0 = T_4 - T_3
\]

\[
C_m^0 = T_{m+1} - T_m, \quad \text{where} \quad T_{m+1} = 2T_1
\]

and the utilization factor is

\[
U^0 = \sum_{i=1}^{m} \frac{C_i}{T_i} = \sum_{i=1}^{m} \frac{T_{i+1} - T_i}{T_i} - m
\]

(2)

Now, we want to find the minimum value of \( U^0 \). Since the arithmetic mean is greater than or equal to the geometric mean of several positive real numbers, it follows that

\[
U^0 = \sum_{i=1}^{m} \frac{C_i}{T_i} = \sum_{i=1}^{m} \frac{T_{i+1} - T_i}{T_i} - m \geq m \sqrt[m]{\prod_{i=1}^{m} \frac{T_{i+1}}{T_i}} - m.
\]

Consequently,

\[
U^0 \geq m\sqrt[2]{2} - m
\]

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and the equality holds when

\[ \frac{T_{i+1}}{T_i} = \sqrt{2}, \text{ for } i = 1 \text{ to } m. \]

Thus, the proof is completed.

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Remarks

1. In order to guarantee this condition, stage 1 must precede stage 2. That is, stage 2 may proceed only after stage 1 is completed. For example, consider the following intermediate task set obtained from stage 1:

\[ T_1 = 3.5, \quad C_1' = 0.5 = T_2 - T_1 \]
\[ T_2 = 4, \quad C_2' = 0.5 = T_3 - T_2 - 0.5 = T_3 - T_2 - \Delta_2 \]
\[ T_3 = 5, \quad C_3' = 0.5 = T_4 - T_3 - 0.5 = T_4 - T_3 - \Delta_3 \]
\[ T_4 = 6, \quad C_4' = 3. \]

\[ \tau_4' \]
\[ \tau_3' \]
\[ \tau_2' \]
\[ \tau_1' \]

\[ C_1' \]
\[ C_2' \]
\[ C_3' \]
\[ C_4' \]

\[ \tau_4' \] runs in time zones 1, 2, and 3. Time zone 1 is greater than \( \Delta_2 + \Delta_3 \). Thus, there is sufficient time in time zones 1, 2, and 3 for \( C_2' \) and \( C_3' \) to increase. This assures that the task set produced by the manipulations of (i) still fully utilize the processor and is feasible.
2. If the operation of stage 2 starts before stage 1 is completed, the manipulations of (i) do not guarantee to produce a task set that fully utilize the processor and is feasible. Consider the following task set

\[ T_1 = 4, \quad C_1' = 0.5 = T_2 - T_1 - 0.5 = T_2 - T_1 - \Delta_1 \]
\[ T_2 = 5, \quad C_2' = 1.5 = T_3 - T_2 + 0.5 = T_3 - T_2 + \Delta_2 \]
\[ T_3 = 6, \quad C_3' = 0.5 = T_4 - T_3 - 0.5 = T_4 - T_3 - \Delta_3 \]
\[ T_4 = 7, \quad C_4' = 2. \]

According to (i), \( C_3' \) must increase by 0.5 and \( C_4' \) decrease by 1. Thus, time zone [4.5,5] is idle and the resulting task set does not fully utilize the processor.
Reference