Optimal Time-Convex Hull under the L_p Metrics*

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Abstract. We consider the problem of computing the time-convex hull of a point set under the general L_p metric in the presence of a straight-line highway in the plane. The traveling speed along the highway is assumed to be faster than that off the highway, and the shortest time-path between a distant pair may involve traveling along the highway. The time-convex hull TCH(P) of a point set P is the smallest set containing both P and *all* shortest time-paths between any two points in TCH(P). In this paper we give an algorithm that computes the time-convex hull under the L_p metric in optimal $\mathcal{O}(n \log n)$ time for a given set of n points and a real number p with $1 \leq p \leq \infty$.

1 Introduction

Path planning, in particular, shortest time-path planning, in complex transportation networks has become an important yet challenging issue in recent years. With the usage of heterogeneous moving speeds provided by different means of transportation, the *time-distance* between two points, i.e., the amount of time it takes to go from one point to the other, is often more important than their straight-line distance. With the reinterpretation of distances by the time-based concept, fundamental geometric problems such as convex hull, Voronoi diagrams, facility location, etc. have been reconsidered recently in depth and with insights [1, 4, 6].

From the theoretical point of view, straight-line highways which provide faster moving speed and which we can enter and exit at any point is one of the simplest transportation models to explore. The speed at which one can move along the highway is assumed to be $v > 1$, while the speed off the highway is 1. Generalization of convex hulls in the presence of highways was introduced by Hurtado et al. [8], who suggested that the notion of convexity be defined by the inclusion of shortest time paths, instead of straight-line segments, i.e., a set S is said to be **convex** if it contains the shortest time-path between any two points of S. Using

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this new definition, the time-convex hull TCH(P) for a set P is the closure of P with respect to the inclusion of shortest time-paths.

In following work, Palop [10] studied the structure of TCH(P) in the presence of a highway and showed that it is composed of convex clusters possibly together with segments of the highway connecting all the clusters. A particularly interesting fact implied by the hull-structure is that, the shortest time-path between each pair of inter-cluster points must contain a piece of traversal along the highway, while similar assertions do not hold for intra-cluster pairs of points: A distant pair of points (p, q) whose shortest time-path contains a segment of the highway could still belong to the same cluster, for there may exist other points from the same cluster whose shortest time-path to either p or q does not use the highway at all. This suggests that, the structure of TCH(P) in some sense indicates the degree of convenience provided by the underlying transportation network. We are content with clusters of higher densities, i.e., any cluster with a large ratio between the number of points of P it contains and the area of that cluster. For sparse clusters, we may want to break them and benefit distant pairs they contain by enhancing the transportation infrastructure.

The approach suggested by Palop [10] for the presence of a highway involves enumeration of shortest time-paths between all pairs of points and hence requires $\Theta(n^2)$ time, where n is the number of points. This problem was later studied by Yu and Lee [11], who proposed an approach based on incremental point insertions in a highway-parallel monotonic order. However, the proposed algorithm does not return the correct hull in all circumstances as particular cases were overlooked. The first sub-quadratic algorithm was given by Aloupis et al. [3], who proposed an $\mathcal{O}(n \log^2 n)$ algorithm for the L_2 metric and an $\mathcal{O}(n \log n)$ algorithm for the L_1 metric, following the incremental approach suggested by [11] with careful case analysis. To the best of our knowledge, no previous results regarding metrics other than L_1 and L_2 were presented.

Our Focus and Contribution. In this paper we address the problem of computing the time-convex hull of a point set in the presence of a straight-line highway under the L_p metric for a given real number p with $1 \leq p \leq \infty$. First, we adopt the concept of *wavefront propagation*, a notion commonly used for path planning [2, 6], and derive basic properties required for depicting the hull structure under the general L_p metric. When the shortest path between two points is not uniquely defined, e.g., in L_1 and L_∞ metrics, we propose a re-evaluation on the existing definition of convexity. Previous works concerning convex hulls under metrics other than L_2, e.g., Ottmann et al. [9] and Aloupis et al. [3], assume a particular path to be taken when multiple choices are available. However, this assumption allows the boundary of a convex set to contain reflex angles, which in some sense deviates from the intuition of a set being convex.

In this work we adopt the definition that requires a convex set to include *every shortest path* between any two points it contains. Although this definition fundamentally simplifies the shapes of convex sets for L_1 and L_∞ metrics, we show that the nature of the problem is not altered when time-based concepts are considered. In particular, the problem of deciding whether any pair of the given points belong to the same cluster under the L_p metric requires $\Omega(n \log n)$ time under the algebraic computation model [5], for all $1 \leq p \leq \infty$.

Second, we provide an optimal $\mathcal{O}(n \log n)$ algorithm for computing the time-convex hull for a given set of points. The known algorithm due to Aloupis et al. [3] stems from a scenario in the cluster-merging step where we have to check for the existence of intersections between a line segment and a set of convex curves composed of parabolae and line segments, which leads to their $\mathcal{O}\left(n \log^2 n\right)$ algorithm. In our paper, we tackle this situation by making an observation on the duality of cluster-merging conditions and reduce the problem to the geometric query of deciding if any of the given points lies above a line segment of an arbitrary slope. This approach greatly simplifies the algorithm structure and can be easily generalized to other L_p-metrics for $1 \leq p \leq \infty$. For this particular geometric problem, we use a data structure due to Guibas et al [7] to answer this query in logarithmic time. All together this yields our $\mathcal{O}(n \log n)$ algorithm. We remark that, although our adopted definition of convexity simplifies the shape of convex sets under the L_1 and the L_∞ metrics, the algorithm we propose does not take advantage of this specific property and also works for the original notion for which only a particular path is to be included.

2 Preliminaries

In this section, we give precise definitions of the notions as well as sketches of previously known properties that are essential to present our work. We begin with the general L_p distance metric and basic time-based concepts.

Definition 1 (Distance in the L_p-metrics). *For any real number $p \geq 1$ and any two points $q_i, q_j \in \mathbb{R}^n$ with coordinates $(i_1, i_2, \ldots, i_n)$ and $(j_1, j_2, \ldots, j_n)$, the distance between q_i and q_j under the L_p-metric is defined to be $d_p(q_i, q_j) = \left(\sum_{k=1}^{n} |i_k - j_k|^p\right)^{\frac{1}{p}}$.*

Note that when p tends to infinity, $d_p(q_i, q_j)$ converges to $\max_{1 \leq k \leq n} |i_k - j_k|$. This gives the definition of the distance function in the L_∞-metric, which is $d_\infty(q_i, q_j) = \max_{1 \leq k \leq n} |i_k - j_k|$. For the rest of this paper, we use the subscript p to indicate the specific L_p-metric, and the subscript p will be omitted when there is no ambiguity.

A *transportation highway* $\mathcal{H}$ in $\mathbb{R}^n$ is a hyperplane in which the moving speed in $\mathcal{H}$ is $v_\mathcal{H}$, where $1 < v_\mathcal{H} \leq \infty$, while the moving speed off $\mathcal{H}$ is assumed to be *unit*. Given the moving speed in the space, we can define the *time-distance* between any two points in $\mathbb{R}^n$.

Definition 2. *For any $q_i, q_j \in \mathbb{R}^n$, a continuous curve $\mathcal{C}$ connecting q_i and q_j is said to be a* shortest time-path *if the traveling time required along $\mathcal{C}$ is minimum among all possible curves connecting q_i and q_j. The traveling time required along $\mathcal{C}$ is referred to as the time-distance between q_i and q_j, denoted $\hat{d}(q_i, q_j)$.*

For any two points q_i and q_j, let $\mathrm{STP}(q_i, q_j)$ denote the set of shortest time-paths between q_i and q_j. For any $\mathcal{C} \in \mathrm{STP}(q_i, q_j)$, we say that $\mathcal{C}$ *enters the highway* $\mathcal{H}$ if $\mathcal{C} \cap \mathcal{H} \neq \emptyset$. The *walking-region* of a point $q \in \mathbb{R}^n$, denoted $\mathrm{WR}(q)$, is defined to be the set of points whose set of shortest time-paths to q contains a time-path that does not enter the highway $\mathcal{H}$. For any $\mathcal{C} \in \mathrm{STP}(q_i, q_j)$, we say that $\mathcal{C}$ *uses the highway* $\mathcal{H}$ if $\mathcal{C} \cap \mathcal{H}$ contains a piece with non-zero length, i.e., at some point $\mathcal{C}$ enters the highway $\mathcal{H}$ and walks along it.

Convexity and Time-Convex Hulls. In classical definitions, a set of points is said to be *convex* if it contains every line segment joining each pair of points in the set, and the convex hull of a set of points $Q \subseteq \mathbb{R}^n$ is the minimal convex set containing Q. When time-distance is considered, the concept of convexity as well as convex hulls with respect to time-paths is defined analogously. A set of points is said to be *convex with respect to time*, or, *time-convex*, if it contains every shortest time-path joining each pair of points in the set.

Definition 3 (Time-convex hull). *The time-convex hull, of a set of points* $Q \subseteq \mathbb{R}^n$, *denoted* $\mathrm{TCH}(Q)$, *is the minimal time-convex set containing* Q.

Although the aforementioned concepts are defined in $\mathbb{R}^n$ space, in this paper we work in $\mathbb{R}^2$ plane with an axis-parallel highway placed on the x-axis as higher dimensional space does not give further insights: When considering the shortest time-paths between two points in higher dimensional space, it suffices to consider the specific plane that is orthogonal to $\mathcal{H}$ and that contains the two points.

Time-Convex Hull under the L_1 and the L_2 Metrics. The structure of time-convex hulls under the L_1 and the L_2 metrics has been studied in a series of work [3, 10, 11]. Below we review important properties. See also Fig. 1 for an illustration.

Proposition 1 ([10, 11]). *For the L_2-metric and any point $q = (x_q, y_q)$ with $y_q \geq 0$, we have the following properties.*

1. *If a shortest time-path starting from q uses the highway $\mathcal{H}$, then it must enter the highway with an incidence angle $\alpha = \arcsin 1/v_{\mathcal{H}}$ toward the direction of the destination.*
2. *The walking region of a point $q \in \mathbb{R}^2$ is characterized by the following two parabolae: (a)* right discriminating parabola, *which is the curve satisfying*

$$\begin{cases} x \geq x_q + y_q \tan\alpha, \quad \text{and} \\ \sqrt{(x - x_q)^2 + (y - y_q)^2} = y_q \sec\alpha + y \sec\alpha + \frac{1}{v_{\mathcal{H}}}((x - y\tan\alpha) - (x_q + y_q \tan\alpha)). \end{cases}$$

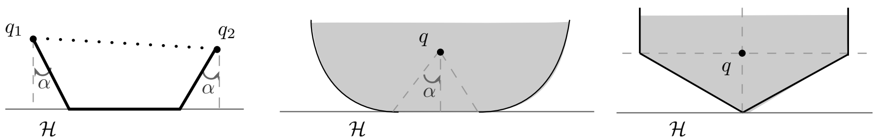

Fig. 1. (a) The only two possible paths for being a shortest time-path between q_1 and q_2 in L_2. (b)(c) The walking regions of a point $q \in \mathbb{R}^2$ under L_2 and L_1, respectively.

(b) The left discriminating parabola *is symmetric to the right discriminating parabola with respect to the line $x = x_q$.*

Proposition 2 ([3]). *For the L_1-metric, the walking region of a point $q = (x_q, y_q)$ with $y_q \geq 0$ is formed by the intersection of the following regions: (a) the vertical strip $x_q - y_q/\beta \leq x \leq x_q - y_q/\beta$, and (b) $y \geq \pm\beta(x - x_q)$, where $\beta = \frac{1}{2}\left(1 - \frac{1}{v_{\mathcal{H}}}\right)$.*

3 Hull-Structure under the General L_p-Metrics

In this section, we derive necessary properties to describe the structure of time-convex hulls under the general L_p metrics. First, we adopt the notion of *wavefront propagation* [2, 6], which is a well-established model used in path planning, and derive the behavior of a shortest time-path between any two points. Then we show how the corresponding walking regions are formed, followed by a description of the desired structural properties.

Wavefronts and Shortest Time-Paths. For any $q \in \mathbb{R}^2$, $t \geq 0$, and $p \geq 1$, the wavefront with *source* q and *radius* t under the L_p-metric is defined as

$$W_p(q, t) = \left\{ s\colon s \in \mathbb{R}^2, \hat{d}_p(q, s) = t \right\}.$$

Literally, $W_p(q, t)$ is the set of points whose time-distances to q are exactly t. Fig. 2 (a) shows the wavefronts, i.e., the "unit-circles" under the L_p metric, or, the p-circles, for different p with $0 < p \leq \infty$ when the highway is not used. The shortest time-path between q and any point $q' \in W_p(q, t)$ is the trace on which q' moves as t changes smoothly to zero, which is a straight-line joining q and q'.

When the highway $\mathcal{H}$ is present and the time-distance changes, deriving the behavior of a shortest time-path that uses $\mathcal{H}$ becomes tricky. Let $q_1, q_2 \in \mathbb{R}^2$, $q_1 \neq q_2$, be two points in the plane, and let $\hat{t}_{1/2}(q_1, q_2) \geq 0$ be the smallest real number such that

$$\text{Bisect}(q_1, q_2) \equiv W_p\big(q_1, \hat{t}_{1/2}(q_1, q_2)\big) \cap W_p\big(q_2, \hat{t}_{1/2}(q_1, q_2)\big) \neq \emptyset.$$

In other words, $\text{Bisect}(q_1, q_2)$ is the set of points at which $W_p(q_1, t)$ and $W_p(q_2, t)$ meet for the first time. The following lemma shows that $\text{Bisect}(q_1, q_2)$ characterizes the set of "*middle points*" of all shortest time-paths between q_1 and q_2.

Lemma 1. *For each $\mathcal{C} \in \text{STP}(q_1, q_2)$, we have $\mathcal{C} \cap \text{Bisect}(q_1, q_2) \neq \emptyset$. Moreover, for each $q \in \text{Bisect}(q_1, q_2)$, there exists $\mathcal{C}' \in \text{STP}(q_1, q_2)$ such that $q \in \mathcal{C}'$.*

Given the set $\text{Bisect}(q_1, q_2)$, a shortest time-path between q_1 and q_2 can be obtained by joining $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_1 \in \text{STP}(q_1, q)$ and $\mathcal{C}_2 \in \text{STP}(q, q_2)$ for some $q \in \text{Bisect}(q_1, q_2)$. By expanding the process in a recursive manner we get a set of middle points. Although the cardinality of the set we identified is countable while any continuous curve in the plane contains uncountably infinite

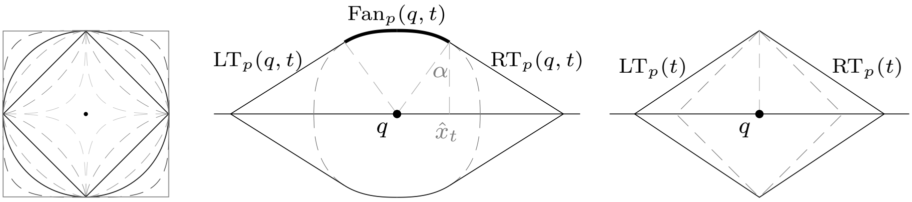

Fig. 2. (a) p-circles for different values of p: bold rhombus for $p = 1$, bold circle for $p = 2$, and bold square for $p = \infty$. (b) $W_p(q,t)$ for a point $q \in \mathcal{H}$, $v_{\mathcal{H}} < \infty$, and $p > 1$, where the angle α satisfies $\sin\alpha = \frac{\hat{x}_t}{\sqrt{\hat{x}_t^2 + y_p(\hat{x}_t)^2}}$. (c) $W_p(q,t)$ for $v_{\mathcal{H}} < \infty$ and $p = 1$.

points, it is not difficult to see that, the set of points we locate is *dense*[1] in the underlying curve, and therefore can serve as a representative.

To describe the shape of a wavefront when the highway may be used, we need the following lemma regarding the propagation of wavefronts.

Lemma 2. *Let $\mathcal{C}_p^\circ(q,t)$ denote the p-circle with center q and radius t. Then $W_p(q,t)$ is formed by the boundary of*

$$\mathcal{C}_p^\circ(q,t) \cup \bigcup_{s:\, s\in\mathbb{R}^2, \hat{d}_p(q,s) < d_p(q,s) \le t} W_p\Big(s, t - \hat{d}_p(q,s)\Big).$$

In the following we discuss the case when $1 < p < \infty$ and leave the discussion of shortest time-paths in L_∞ to the appendix for further reference. Let $\mathcal{H}$ be the highway placed on the x-axis with moving speed $v_{\mathcal{H}} > 1$. For any $t \ge 0$, any $0 \le x \le t$, and any $1 < p < \infty$, we use $y_p(x,t) = \left(t^p - |x|^p\right)^{1/p}$ to denote the y-coordinate of the specific point on the p-circle with x-coordinate x. Let $\hat{x}_t = t \cdot v_{\mathcal{H}}^{1/(1-p)}$. We have the following lemma regarding $W_p(q,t)$. Also refer to Fig. 2 (b) for an illustration.

Lemma 3. *For $1 < p < \infty$, $v_{\mathcal{H}} < \infty$, and a point $q \in \mathcal{H}$ which we assume to be $(0,0)$ for the ease of presentation, the upper-part of $W_p(q,t)$ that lies above $\mathcal{H}$ consists of the following three pieces:*

- Fan$_p(q,t)$*: the circular-sector of the p-circle with radius t, ranging from $(-\hat{x}_t, y_p(-\hat{x}_t,t))$ to $(\hat{x}_t, y_p(\hat{x}_t,t))$.*
- LT$_p(q,t)$, RT$_p(q,t)$*: two line segments joining $(-v_{\mathcal{H}}\cdot t, 0)$, $(-\hat{x}_t, y_p(-\hat{x}_t,t))$, and $(\hat{x}_t, y_p(\hat{x}_t,t))$, $(v_{\mathcal{H}}\cdot t, 0)$, respectively. Moreover,* LT$_p(q,t)$ *and* RT$_p(q,t)$ *are tangent to* Fan$_p(q,t)$.

The lower-part that lies below $\mathcal{H}$ follows symmetrically. For $v_{\mathcal{H}} = \infty$, the upper-part of $W_p(q,t)$ consists of a horizontal line $y = t$.

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[1] *Dense* is a concept used in classical analysis to indicate that any element of one set can be approximated to any degree by elements of a subset being dense within.

For each $1 \leq p < \infty$ and $1 < v_{\mathcal{H}} \leq \infty$, we define the real number $\alpha(p, v_{\mathcal{H}})$ as follows. If $p = 1$ or $v_{\mathcal{H}} = \infty$, then $\alpha(p, v_{\mathcal{H}})$ is defined to be zero. Otherwise, $\alpha(p, v_{\mathcal{H}})$ is defined to be

$$\arcsin \frac{v_{\mathcal{H}}^{1/(1-p)}}{\sqrt{v_{\mathcal{H}}^{2/(1-p)} + \left(1 - v_{\mathcal{H}}^{p/(1-p)}\right)^{2/p}}}.$$

Note that, when $p = 2$, this is exactly $\arcsin(1/v_{\mathcal{H}})$. For brevity, we simply use α when there is no ambiguity. The behavior of a shortest time-path that takes the advantage of traversal along the highway is characterized by the following lemma.

Lemma 4. *For any point $q = (x_q, y_q)$, $1 \leq p < \infty$, and $1 < v_{\mathcal{H}} \leq \infty$, if a shortest time-path starting from q uses the highway $\mathcal{H}$, then it must enter the highway with an incidence angle α.*

Walking Regions. For any point $q = (x_q, y_q) \in \mathbb{R}^2$ with $y_q \geq 0$, let $q_{\mathcal{H}}^{+}$ and $q_{\mathcal{H}}^{-}$ be two points located at $(x_q \pm y_q \tan\alpha, 0)$, respectively. By Lemma 4, we know that, $q_{\mathcal{H}}^{+}$ and $q_{\mathcal{H}}^{-}$ are exactly the points at which any shortest time-path from q will enter the highway if needed. This gives the walking region for any point. Let $\alpha = \pi/4$ when $p = \infty$. The following lemma is an updated version of Proposition 1 for general p with $1 \leq p \leq \infty$.

Lemma 5. *For any p with $1 \leq p \leq \infty$ and any point $q = (x_q, y_q)$ with $y_q \geq 0$,* $\mathrm{WR}_p(q)$ *is characterized by the following two curves: (a)* right discriminating curve, *which is the curve $q' = (x', y')$ satisfying $x' \geq x_q + y_q \tan\alpha$ and*

$$\left|\overline{qq'}\right|_p = \left|\overline{qq_{\mathcal{H}}^{+}}\right|_p + \left|\overline{q'q_{\mathcal{H}}'^{-}}\right|_p + \frac{1}{v_{\mathcal{H}}}\left|\overline{q_{\mathcal{H}}^{+}q_{\mathcal{H}}'^{-}}\right|_p.$$

(b) The left discriminating curve *is symmetric with respect to the line $x = x_q$.*

For any point q, let $\mathrm{WR}_\ell(q)$ and $\mathrm{WR}_r(q)$ denote the left- and right- discriminating curves of $\mathrm{WR}(q)$, respectively. We have the following *dominance property* of the walking regions.

Lemma 6. *Let $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$ be two points such that $x_1 \leq x_2$. If $y_1 \geq y_2$, then $\mathrm{WR}_\ell(q_2)$ lies to the right of $\mathrm{WR}_\ell(q_1)$. Similarly, if $y_1 \leq y_2$, then $\mathrm{WR}_r(q_1)$ lies to the left of $\mathrm{WR}_r(q_2)$.*

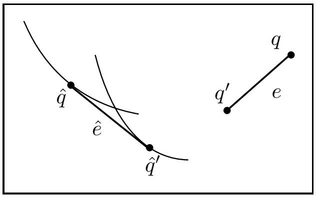

Fig. 3. The left boundary of the walking region for the edge $e = (q', q)$

Lemma 6 suggests that, to describe the leftmost and the rightmost boundaries of the walking-regions for a set of points, it suffices to consider the *extreme points*. Let $e = \overline{q_1 q_2}$ be a line segment between two points q_1 and q_2, where q_1 lies to the left of q_2. If e has non-positive slope, then the left-boundary of the walking region for e is dominated by $\mathrm{WR}_\ell(q_1)$. Otherwise, we have to consider $\bigcup_{q \in e} \mathrm{WR}_\ell(q)$. By parameterizing each point of e, it is not difficult to see that the left-boundary consists of $\mathrm{WR}_\ell(q_1)$, $\mathrm{WR}_\ell(q_2)$, and their common tangent line. See also Fig. 3 for an illustration.

Closure and Time-Convex Hull of a Point Set. By Lemma 1, to obtain the union of possible shortest time-paths, it suffices to consider the set of all possible bisecting sets that arise inside the recursion. We begin with the closure between pairs of points.

Lemma 7. *Let $q_1, q_2 \in \mathbb{R}^2$ be two points. When the highway is not used, the set of all shortest time-paths between q_1 and q_2 is:*
- *The smallest bounding rectangle of $\{q_1, q_2\}$, when $p = 1$.*
- *The straight line segment $\overline{q_1 q_2}$ joining q_1 and q_2, when $1 < p < \infty$.*
- *The smallest bounding parallelogram whose slopes of the four sides are ± 1, i.e., a rectangle rotated by $45°$, that contains q_1 and q_2.*

Lemma 7 suggests that when the highway is not used and when $1 < p < \infty$, the closure, or, convex hull, of a point set $\mathbf{S}$ with respect to the L_p-metric is identical to that in L_2, while in L_1 and L_∞ the convex hulls are given by the bounding rectangles and bounding square-parallelograms.

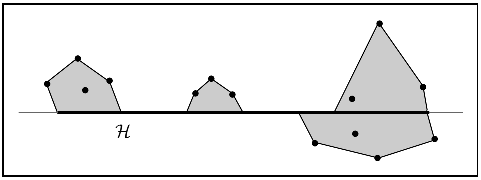

Fig. 4. Time-convex hull for a set of points under the L_p metric where $1 < p < \infty$

When the highway may be used, the structure of the time-convex hull under the general L_p-metric consists of a set of clusters arranged in a way such that the following holds: (1) Any shortest time-path between intra-cluster pair of points must use the highway. (2) If any shortest time-path between two points does not use the highway, then the two points must belong to the same cluster. Fig. 4 and Fig. 5 illustrate examples of the time-convex hull for the L_p metrics with $1 < p < \infty$ and $p = \infty$, respectively. Note that, the shape of the closure for each cluster does depend on p and $v_{\mathcal{H}}$, as they determine the incidence angle α.

4 Constructing the Time-Convex Hull

In this section, we present our algorithmic results for this problem. First, we show that, although our definition of convexity simplifies the structures of the resulting convex hulls, e.g., in L_1 and L_∞, the problem of deciding if any given pair of points belongs to the same cluster already requires $\Omega(n \log n)$ time. Then we present our optimal $\mathcal{O}(n \log n)$ algorithm.

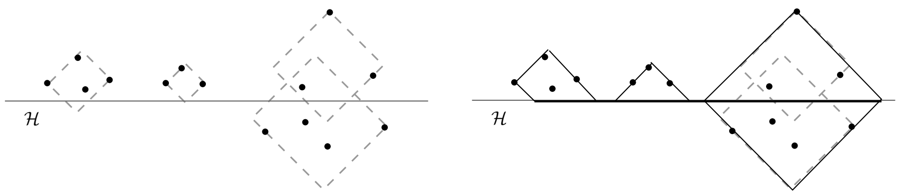

Fig. 5. (a) The closure of each cluster under the L_∞ metric when the highway $\mathcal{H}$ is not considered. (b) The closure, i.e., the time-convex hull, for the L_∞ metric.

4.1 Problem Complexity

We make a reduction from the *minimum gap problem*, which is a classical problem known to have the problem complexity of $\Theta(n \log n)$. Given n real numbers $a_1, a_2, \ldots, a_n$ and a target gap $\epsilon > 0$, the minimum gap problem is to decide if there exist some i, j, $1 \leq i, j \leq n$, such that $|a_i - a_j| \leq \epsilon$.

For any $y \geq 0$, consider the point $q(y) = (0, y)$. Let $\mathcal{C}_{q(y)}\colon \big(x, f_{q(y)}(x)\big)$ denote the right discriminating curve of $\mathrm{WR}_p(q(y))$. For any $\epsilon > 0$, let $y_0(p, \epsilon)$ denote the specific real number such that $f_{q(y_0(p,\epsilon))}(\epsilon) = y_0(p, \epsilon)$. Our reduction is done as follows. Given a real number $p \geq 1$ and an instance $\mathcal{I}$ of minimum gap, we create a set $\mathbf{S}$ consisting of n points $q_1, q_2, \ldots, q_n$, where $q_i = (a_i, y_0(p, \epsilon))$ for $1 \leq i \leq n$. The following lemma shows the correctness of this reduction and establishes the $\Omega(n \log n)$ lower bound.

Lemma 8. *$y_0(p, \epsilon)$ is well-defined for all p with $1 \leq p \leq \infty$ and all $\epsilon > 0$. Furthermore, the answer to the minimum gap problem on $\mathcal{I}$ is "yes" if and only if the number of clusters in the time-convex hull of $\mathbf{S}$ is less than n.*

Corollary 1. *Given a set of points $\mathbf{S}$ in the plane, a real number p with $1 \leq p \leq \infty$, and a highway $\mathcal{H}$ placed on the x-axis, the problem of deciding if any given pair of points belongs to the same cluster requires $\Omega(n \log n)$ time.*

4.2 An Optimal Algorithm

In this section, we present our algorithm for constructing the time-convex hull for a given point set $\mathbf{S}$ under a given metric L_p with $p \geq 1$. The main approach is to insert the points incrementally into the partially-constructed clusters in ascending order of their x-coordinates. In order to prevent a situation that leads to an undesirably complicated query encountered in the previous work by Aloupis et al. [3], we exploit the symmetric property of cluster-merging conditions and reduce the sub-problem to the following geometric query.

Definition 4 (One-Sided Segment Sweeping Query). *Given a set of points $\mathbf{S}$ in the plane, for any line segment L of finite slope, the* one-sided segment sweeping query*, denoted $\mathcal{Q}(\mathrm{L})$, asks if $\mathbf{S} \cap \mathrm{L}^+$ is empty, where L^+ is the intersection of the half-plane above L and the vertical strip defined by the end-points of L. That is, we ask if there exists any point $p \in \mathbf{S}$ such that p lies above L.*

In the following, we first describe the algorithm and our idea in more detail, assuming the one-sided segment sweeping query is available. Then we show how this query can be answered efficiently.

The given set $\mathbf{S}$ of points is partitioned into two subsets, one containing those points lying above $\mathcal{H}$ and the other containing the remaining. We compute the time-convex hull for the two subsets separately, followed by using a linear scan on the clusters created on both sides to obtain the closure for the entire point set. Below we describe how the time-convex hull for each of the two subsets can be computed.

Let $q_1, q_2, \ldots, q_n$ be the set of points sorted in ascending order of their x-coordinates with ties broken by their y-coordinates. During the execution of the algorithm, we maintain the set of clusters the algorithm has created so far, which we further denote by $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$. For ease of presentation, we denote the left- and right-boundary of the walking region of C_i by $\mathrm{WR}_\ell(C_i)$ and $\mathrm{WR}_r(C_i)$, respectively. Furthermore, we use $q \in \mathrm{WR}_\ell(C_i)$ or $q \in \mathrm{WR}_r(C_i)$ to indicate that point q lies to the right of $\mathrm{WR}_\ell(C_i)$ or to the left of $\mathrm{WR}_r(C_i)$, respectively.

In iteration i, $1 \leq i \leq n$, the algorithm inserts q_i into $\mathcal{C}$ and checks if a new cluster has to be created or if existing clusters have to be merged. This is done in the following two steps.

(a) *Point inclusion test.* In this step, we check if there exists any j, $1 \leq j \leq k$, such that $q_i \in \mathrm{WR}_r(C_j)$. If not, then a new cluster C_{k+1} consisting of the point q_i is created and we enter the next iteration. Otherwise, the smallest index j such that $q_i \in \mathrm{WR}_r(C_j)$ is located. The clusters $C_j, C_{j+1}, \ldots, C_k$ and the point q_i are merged into one cluster, which will in turn replace $C_j, C_{j+1}, \ldots, C_k$. Let $\mathbf{E}$ be the set of newly created edges on the upper-hull of this cluster whose slopes are positive. Then we proceed to step (b).

(b) *Edge inclusion test.* Let k be the number of clusters, and x_0 be the x-coordinate of the leftmost point in C_k. Pick an arbitrary edge $e \in \mathbf{E}$, let $\hat{e}$ denote the line segment appeared on $\mathrm{WR}_\ell(e)$ to which e corresponds, and let $\hat{e}(x_0)$ be the intersection of $\hat{e}$ with the half-plane $x \leq x_0$. Then we invoke the one-sided segment sweeping query $\mathcal{Q}(\hat{e}(x_0))$. If no point lies above $\hat{e}(x_0)$, then e is removed from $\mathbf{E}$. Otherwise, C_k is merged with C_{k-1}. Let e' be the newly created bridge edge between C_k and C_{k-1}. If e' has positive slope, then it is added to the set $\mathbf{E}$. This procedure is repeated until the set $\mathbf{E}$ becomes empty.

An approach has been proposed to resolve the *point inclusion test* efficiently, e.g., Yu and Lee [11], and Aloupis et al. [3]. Below we state the lemma directly and leave the technical details to the appendix for further reference.

Lemma 9 ([3,11]). *For each iteration, say i, the smallest index j, $1 \leq j \leq k$, such that $q_i \in \mathrm{WR}_r(C_j)$ can be located in amortized constant time.*

To see that our algorithm gives the correct clustering, it suffices to argue the following two conditions: (1) Each cluster-merge our algorithm performs is valid. (2) At the end of each iteration, no more clusters have to be merged.

Apparently these conditions hold at the end of the first iteration, when q_1 is processed. For each of the succeeding iterations, say i, if no clusters are merged in step (a), then the conditions hold trivially. Otherwise, the validity of the cluster-merging operations is guaranteed by Lemma 9 and the fact that if any point lies above $\hat{e}$, then it belongs to the walking region of e, meaning that the last cluster, C_k, has to be merged again. See also Fig. 6 for an illustration.

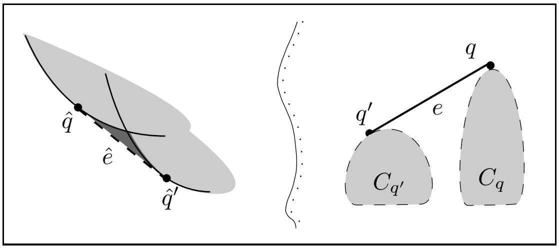

Fig. 6. When two clusters $C_{q'}$ and C_q are merged and new hull edge e is created, it suffices to check the new walking region e corresponds to, i.e., the dark-gray area

To see that the second condition holds, let $e = (q', q) \in \mathbf{E}$ be a newly created hull edge, and let $C_{q'}$ and C_q be the two corresponding clusters that were merged. By our assumption that the clusters are correctly created before q_i arrives, we know that the walking-regions of $C_{q'}$ and C_q contain only points that do belong to them, i.e., the light-gray area in the left-hand side of Fig. 6 contains only points from $C_{q'}$ or C_q. Therefore, when $C_{q'}$ and C_q are merged and e is created, it suffices to check for the existence of points other than C_k inside the new walking region e corresponds to, which is exactly the dark-gray area in Fig. 6. Furthermore, by the dominance property stated in Lemma 6, it suffices to check those edges with positive slopes. This shows that at the end of each iteration when $\mathbf{E}$ becomes empty, no more clusters need to be merged. We have the following theorem.

Theorem 1. *Provided that the one-sided segment sweeping query can be answered in $Q(n)$ time using $P(n)$ preprocessing time and $S(n)$ storage, the time-convex hull for a given set $\mathbf{S}$ of n points under the given L_p-metric can be computed in $\mathcal{O}(n \log n + nQ(n) + P(n))$ time using $\mathcal{O}(n + S(n))$ space.*

Regarding the One-Sided Segment Sweeping Query. Below we sketch how this query can be answered efficiently in logarithmic time. Let $\mathbf{S}$ be the set of points, L be the line segment of interest, and $\mathcal{I}_{\mathrm{L}}$ be the vertical strip defined by the two end-points of L. We have the following observation, which relates the query $\mathcal{Q}(\mathrm{L})$ to the problem of computing the upper-hull of $\mathbf{S} \cap \mathcal{I}_{\mathrm{L}}$.

Lemma 10. *Let $\mathcal{I}$ be an interval, $\mathcal{C}\colon \mathcal{I} \to \mathbb{R}$ be a convex function, i.e., we have $\mathcal{C}\left(\frac{1}{2}(x_1 + x_2)\right) \geq \frac{1}{2}(\mathcal{C}(x_1) + \mathcal{C}(x_2))$ $\forall x_1, x_2 \in \mathcal{I}$, that is differentiable almost everywhere,* L *be a segment with slope θ_L, $-\infty < \theta_L < \infty$, and $q = (x_q, \mathcal{C}(x_q))$ be a point on the curve $\mathcal{C}$ such that*

$$\lim_{x \to x_q^-} \frac{\mathrm{d}\mathcal{C}(x)}{\mathrm{d}x} \geq \theta_L \geq \lim_{x \to x_q^+} \frac{\mathrm{d}\mathcal{C}(x)}{\mathrm{d}x}.$$

If q lies under $\overleftrightarrow{\mathrm{L}}$, then the curve $\mathcal{C}$ never intersects L.

To help compute the upper-hull of $\mathbf{S} \cap \mathcal{I}_{\mathrm{L}}$, we use a data structure due to Guibas et al [7]. For a given simple path $\mathcal{P}$ of n points with an x-sorted ordering of the points, with $\mathcal{O}(n \log \log n)$ preprocessing time and space, the upper-hull of any subpath $p \in \mathcal{P}$ can be assembled efficiently in $\mathcal{O}(\log n)$ time, represented by a balanced search tree that allows binary search on the hull edges. Note that, $q_1, q_2, \ldots, q_n$ is exactly a simple path by definition. The subpath to which $\mathcal{I}_{\mathrm{L}}$

corresponds can be located in $\mathcal{O}(\log n)$ time. In $\mathcal{O}(\log n)$ time we can obtain the corresponding upper-hull and test the condition specified in Lemma 10. We conclude with the following lemma.

Lemma 11. *The* one-sided segment sweeping query *can be answered in* $\mathcal{O}(\log n)$ *time, where* n *is the number of points, using* $\mathcal{O}(n \log n)$ *preprocessing time and* $\mathcal{O}(n \log \log n)$ *space.*

5 Conclusion

We conclude with a brief discussion as well as an overview on future work. In this paper, we give an optimal algorithm for the time-convex hull in the presence of a straight-line highway under the general L_p-metric where $1 \leq p \leq \infty$. The structural properties we provide involve non-trivial geometric arguments. We believe that our algorithm and the approach we use can serve as a base to the scenarios for which we have a more complicated transportation infrastructure, e.g., modern city-metros represented by line-segments of different moving speeds. Furthermore, we believe that approaches supporting dynamic settings to a certain degree, e.g., point insertions/deletions, or, dynamic speed transitions, are also a nice direction to explore.

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