Nonlinear Model Predictive Control for Wheeled Mobile Robot in Dynamic Environment

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Abstract—This paper addresses the set-point regulation problem of a nonholonomic wheeled mobile robot with obstacle avoidance in a known dynamic environment populated with static and moving obstacles subject to robot kinematic and dynamic constraints by using the nonlinear model predictive control in polar coordinate. The terminal state penalty, terminal state constraints, and the input saturation constraints are taken into consideration in this optimization problem to guarantee the closed-loop regulation performance and stability. Simulation results are shown for illustrating the effectiveness of the control algorithm in steering a nonholonomic wheeled mobile robot.

I. INTRODUCTION

Autonomous wheeled mobile robot (WMR) navigation to the goal problem [1], [3], and nonholonomic motion planning in an environment populated with obstacles are concerned with obtaining controls and feasible trajectories which steer a nonholonomic wheeled robot between two given states in such a way as to avoid collisions with obstacles. In addition to the environment constraints, control constraints resulting from the physical limitations of the actuators driving the wheels should also be taken into account during the motion control and planning of WMRs [14]. This issue may lead some difficulties to design a time-invariant controller [13]. Many useful solutions such as potential field method were proposed as path planning method with obstacle avoidance strategy to deal with a variation of the regulation problems.

The Model Predictive Control (MPC) or equivalently the Receding Horizon Control (RHC) is a feedback control scheme that generates the control action based on a finite horizon open loop optimal control with the measured state as the initial state which can handle the state/input constraints directly. For this reason, MPC method has become very popular due to the excellent results in industrial process control applications [11]; moreover, many related theoretical and practical studies are also been proposed and discussed [4], [1]. The core of the MPC is composed of three components: The plant model, the performance index, and the receding horizon principle. According to these components, the MPC can be used to solve an on-line finite horizon optimization problem and to obtain a sequence of future control inputs which is based on current state. Note that only the first element of the control sequence is applied to the plant and the remaining control sequences are discarded. This on-line optimization procedure is solved repeatedly. In this paper, we formulate the navigation to goal of a nonholonomic WMR as a nonlinear constrained regulation problem in polar coordinate taking into account of the actuator saturation and environment (i.e. known static and moving obstacle avoidance) constraints. The objective of navigation is to determine feasible trajectories for the WMR to follow safely to reach a given goal in a known dynamic environment. Nonlinear model predictive control (NMPC) is applied to this regulation problem, and it has the advantage that a collision-free path is simultaneously generated in a dynamic environment.

The rest of this paper is organized as follows. In Section II the discrete kinematic model of a wheeled mobile robot will be constructed. Section III contains a brief discussion of NMPC algorithm. The modified quadratic cost functions are applied to implement the objective of static and dynamic obstacle avoidance. Simulation results are shown in the Section IV to illustrate the regulation performance on static/dynamic obstacle avoidance. Section V is the conclusion.

II. DISCRETE KINEMATIC MODEL OF WMR

To model the motion of a nonholonomic wheeled mobile robot (WMR), a simple unicycle model (shown in Figure. 1) is considered in this paper [1], [5], [13], where it is assumed that the motion of mobile robot cannot slip laterally so that the translational velocity is in the direction of heading, i.e. a pure rolling contact between the wheels and the ground:

\[
\begin{align*}
\dot{x} &= v \cos \theta \\
\dot{y} &= v \sin \theta \\
\dot{\theta} &= \omega
\end{align*}
\] (1)

The state \((x, y, \theta)\) describes the position and orientation of the center of the mobile robot with respect to the Cartesian coordinates of a global inertial frame and the control inputs denoted by \(u = (v, \omega)^T\) are linear and angular velocity, respectively. According to the kinematic model (1) we can find that the number of control variables is less than the number of state variables. Therefore, a standard nonholonomic constraint holds and the time-invariant controller cannot be used to control this system [3], [7], [8], [13]. (1) could be transformed from Cartesian coordinate \((x, y, \theta)\) into a polar state \((e, \phi, \alpha)\) via [2].
\[ e = \sqrt{x^2 + y^2} \]
\[ \phi = \alpha \tan(y, x) \]
\[ \alpha = \phi - \theta \]

where \( x = e \cos \phi \) and \( y = e \sin \phi \). The kinematic equation (1) can thus be transformed into polar coordinate

\[ \begin{align*}
\dot{e} &= v \cos \alpha \\
\dot{\phi} &= -v \sin \alpha / e \\
\dot{\alpha} &= \sin \alpha / e - \omega
\end{align*} \]

It is noted that the polar coordinate transformation can provide a smoother closed-loop trajectory and also can eliminate the steady-state error effectively [5].

In general, the NMPC algorithm is calculated in discrete time. To apply NMPC, it is convenient to consider the kinematic model (3) discretized in time. We consider discrete time and piecewise constant velocities, i.e.

\[ v(\tau) = v(t), \omega(\tau) = \omega(t) \quad \text{for} \quad \tau \in [t, t + T_s], t = kT_s \]

where \( T_s \) denotes sampling time and positive integer \( k \) is sampling instant. Utilizing the feed-forward difference method (Euler’s approximation) to (4), a discretized kinematic model at the \( k \)-th sampling time interval in polar coordinate can be obtained.

\[
\begin{bmatrix}
  e(k+1) \\ 
  \phi(k+1) \\ 
  \alpha(k+1)
\end{bmatrix} =
\begin{bmatrix}
  e(k) + T_s v(k) \cos \alpha(k) \\ 
  \phi(k) - T_s v(k) \sin \alpha(k)/e(k) \\ 
  \alpha(k) + T_s v(k) \sin \alpha(k)/e(k) - T_s \omega(k)
\end{bmatrix} \quad (5)
\]

We can rewrite the (5) into a more compact form as

\[ X(k+1) = f(X(k), u(k)) \]

![Figure 1. The Nonholonomic wheeled mobile robot (WMR) represented in polar coordinate.](image)

Note that kinematic model (5) is not defined at the singular point \( e(k) = 0 \). Without loss of generality, we can assign the origin as the reference set-point, and the corresponding error state can be written as \( X_e = X - X_{ref} \), i.e., the error between current state and the reference state (or reference trajectory). For navigation to the goal problem considered in the paper, the reference set-point is set to be the origin; namely, the error state is \( X_e = X \).

III. THE NMPC ALGORITHM FOR OBSTACLE AVOIDANCE

NMPC uses detailed nonlinear dynamical models to predict the performance of the process to be controlled over a sufficiently long time horizon \( N \). Given the initial state \( X_0 \) and a prediction horizon \( N \), NMPC algorithm is realized by formulating the problem as a regulation to set point, and the cost function can be defined by

\[ J = F(X_e(N)) + \sum_{j=0}^{N-1} l(X_e(j), u(j)) \]

where \( \ell = \ell_{reg} + \ell_{obst} \) with the first term for regulation purpose defined as

\[ \ell_{reg}(X(j), u(j)) = X_e(k + j)^T Q X_e(k + j) + u(k + j)^T R u(k + j) \]

with positive-definite weighting matrices \( Q \succ 0 \) and \( R \succ 0 \). The other term used for obstacle avoidance purpose is defined as

\[ \ell_{obst}(X(j), u(j)) = \frac{s}{d^2(k + j + 1) d^2(k + j + 1)} + \varepsilon \]

where the \( d^2(k + j) \) indicates the distance between the WMR and ith detected obstacle at time \( k \) with a weighting parameter \( s \). The \( \varepsilon \) is a small positive constant for non-singularity; furthermore, the terminal cost \( F(X_e(N)) \)

\[ F(X_e(N)) = X_e(k + N)^T P X_e(k + N) \]

is added to penalize the terminal state and to stabilize the closed-loop system. Note that the terminal cost can be approximated as the infinite horizon value function and the closed-loop stability can be guaranteed[6,1]. \( N \) is the prediction horizon or equivalently called optimization horizon [10]. \( Q \) and \( R \) are the weighting matrices which are used to weight the states and control efforts, respectively. According to above setting, the NMPC algorithm will solve the following optimal control problem \( \text{OP}_s \):

\[ \text{OP}_s : V^*_s = \min_{u} \int J(k, X_e, u) \]

Subject to

\[
\begin{align*}
  X(0) &= X_0 \\
  X(k + j + 1) &= f(X(k + j), u(k + j)), j = 0, \ldots, N \\
  v_{min} \leq v \leq v_{max} \\
  \omega_{min} \leq \omega \leq \omega_{max} \\
  X(k + N) &\in X_f
\end{align*}
\]

In (7), \( V^*_s \) denotes the optimal value function, \( u_{max} \) and \( u_{min} \) represents the upper bound and the lower bound of the control effort, respectively, and \( X_f \) is the terminal state constraint.

For environment constraints, we suggest to approximate the static obstacles by circles. It is defined for each (regularly- or irregularly-shaped) obstacle a bounding circle, though in certain cases an ellipse instead of a circle is a better approximation. Suppose the obstacle information (the enclosing circle in case of static obstacle and the trajectory in
case of moving obstacle) is given, then an additional obstacle avoidance constraint [13] is incorporated into \( \mathcal{OP}_N \) (7) as follows:

\[
\begin{bmatrix}
    x(k) \\
    y(k) \\
    x^{\prime}(k) \\
    y^{\prime}(k)
\end{bmatrix} \geq D
\]  

(9)

This hard constraint (9) indicates the distance between wheeled mobile robot and the obstacle point of the detected obstacle should be larger than a threshold \( D \) (safety constant) to account for the WMR size. This setting is a simplification to and a conservative approximation to general shape robot that will ensure that the obstacles are avoided.

For state \( X \) at time \( k \), the problem \( \mathcal{OP}_N \) is solved to obtain a control \( u^* = [u^*(0), u^*(1), \ldots, u^*(N-1)] \) and a corresponding whole trajectory \( X^* = [X^*_0, X^*_1, \ldots, X^*_N] \) and the associated optimal (minimal) value function can be obtained as

\[
V^*_N(X) = V^*_N(X, u^*)
\]

Only the first element of the optimal control sequence is applied to the system. Thus, an implicit time invariant model predictive control law \( \kappa_N(\cdot) \) can be represented as

\[
\kappa_N(X_e) = u^*(0)
\]

(10)

The whole process is repeated with new updates on the robot and WMR position. Solving this on-line optimization problem repeatedly, the current state \( X(k) \) will converge to the reference set point \( X^{\text{ref}} \).

IV. STABILITY ANALYSIS

In the control system, the stability analysis is essential index to determine the closed-loop system properties. In this section, we based on the Lyapunov direct method [9] which is summarized by [10],[11] to derive the closed-loop stability properties of NMPC regulation problem under previous setting. For a closed terminal constraint set \( X_f \subset X \), we assume the following

**Assumption 1:** for each state \( X_e \subset X_f \) there exists an admissible terminal control inputs \( \kappa_f(X_e) \subset U \), such that \( f(X_e, \kappa_f(X_e)) \subset X_f \), \( \forall X_e \subset X_f \); also, the terminal constraint \( X_f \) is positively invariant under \( \kappa_f(\cdot) \)

**Assumption 2:**

\[
F(X_e^+) - F(X_e) + \ell(X_e, \kappa_f(X_e)) \leq 0, \quad \forall X_e \in X_f
\]

\( F(\cdot) \) is a local Lyapunov function.

**Theorem (Stability)** The system (5) is asymptotic stable, if terminal control efforts exists \( \kappa_f(X_e) \subset U \) so that the assumption 1 and assumption 2 holds.

**Proof:**

The conditions on terminal state region \( X_f \), terminal cost \( F(\cdot) \), and local terminal stabilizing control law \( \kappa_f(\cdot) \) ensures

\[
V^*_N(X_e^+) - V^*_N(X_e^-) \leq 0
\]

(11)

By principle of optimality:

\[
V^*_N(X_e) = \ell(X_e, \kappa_N(X_e)) + V^*_N(X_e^-)
\]

We can equivalently rewrite the equation (9) into

\[
V^*_N(X_e^+, \kappa_N(X_e)) - V^*_N(X_e, \kappa_N(X_e)) + \ell(X_e, \kappa_N(X_e)) \leq 0
\]

(12)

Recall the \( X_e^+ = f(X_e, u) \). In the Lyapunov direct method approach, the calculation of \( V^*_N(X_e^+, \kappa_N(X_e)) \) is avoided by computation of an upper bound \( V^*_N(X_e^+, \tilde{u}) \) with a feasible control \( \tilde{u}(X_e) \) for the optimal control problem \( \mathcal{OP}_N \).

Assume that the optimal control sequence \( u^* = [u^*(0), u^*(1), \ldots, u^*(N-1)] \) is determined by solving the optimal control problem \( \mathcal{OP}_N \) and the first element of this optimal control sequence \( u = \kappa_N(X_e) u^*(0) \) is employed as the optimal NMPC control law which can be used to steer the initial state \( X_e \) to the successor state \( X_e^+ \). Now, we wish to determine a feasible control sequence \( \tilde{u}(X_e) \) for \( X_e^+ \) and for an upper bound of \( V^*_N(X_e^+) \). In order to obtain a feasible control for \( \mathcal{OP}_N(X_e^+) \), we add one further element \( u^* \) to optimal control sequence \( [u^*(1), \ldots, u^*(N-1), u^*] \); this sequence will be feasible for \( \mathcal{OP}_N(X_e^+) \) if \( u^* = \kappa_f(X_e) X_f \) and the local terminal control law \( \kappa_f(\cdot) \) have the following properties:

\[
X_f \subset X, \kappa_f(X_e) \subset U, \quad \forall X_e \subset X_f
\]

For this reason, the terminal state region \( X_f \) is invariant when the control law is \( \kappa_f(\cdot) \). If these condition satisfied, the control sequence

\[
\tilde{u}^* := [u^*(1), \ldots, u^*(N-1), \kappa_f(X_e+N)]
\]

is feasible for \( \mathcal{OP}_N(X_e^+) \). The state trajectory resulting from initial state \( X_e^+ = X_e(1) \) and control sequence

\[
\tilde{u}^* := [u^*(1), \ldots, u^*(N-1), \kappa_f(X_e+N)]
\]

is

\[
X_e^+ = [X_e^+(1), \ldots, X_e^+(N-1), f(X_e+N)]
\]

The corresponding cost is

\[
V^*_N(X_e^+, \tilde{u}) = V^*_N(X_e) + \ell(X_e, u(0)) + F(f(X_e+N)) - F(X_e)+ \ell(X_e+N)
\]

(13)

This cost is an upper bound for \( V^*_N(X_e^+, \kappa_N(X_e^+)) \) and
satisfies \( V_{\kappa}(X_{E},\tilde{u}) \leq V_{\kappa}(X_{E}) - \ell(X_{E},\tilde{u}(0)) \) if
\[
F\left(f\left(X_{E},\kappa_{f}(X_{E})\right)\right) - F(X_{E}) + \ell(X_{E},\kappa_{f}(X_{E})) \leq 0
\] (14)
if inequality (14) holds, then
\[
V_{\kappa}(X_{E},\kappa_{f}(X_{E})) - V_{\kappa}(X_{E},\kappa_{f}(X_{E})) + \ell(X_{E},\kappa_{f}(X_{E})) \leq 0
\]
holds for all \( X_{E} \in X_{s} \) which is sufficient to ensure that the state of closed-loop system \( X_{E}^{*} = f(X_{E},\kappa(X_{E})) \) converges to zero as \( k \to \infty \) if its initial state lies in \( X_{s} \).

In our case, the terminal cost \( F(\cdot) \) is chosen as a Lyapunov function that satisfies
\[
\frac{\sin(\alpha_{y})}{e_{N}}v_{j}^{2} + (q - 1)[e_{N}^{2} + \alpha_{y}^{2} + \phi_{y}^{2}] + r_{v_{j}}^{2}
\]
if inequality (14) holds, then
\[
\frac{\sin(\alpha_{y})}{e_{N}}v_{j}^{2} + (\alpha_{y} + T_{s}\left(\frac{\sin(\alpha_{y})v_{j}}{e_{N}} - \omega_{y}\right)^{2} + r_{o_{j}}^{2})
\]
+ \[
\sqrt{e_{N}^{2} - 2(\cos(\phi_{x})x_{y}^{2} + \sin(\phi_{x})y_{y}^{2})e_{N} + (x_{y}^{2}) + (y_{y}^{2})^{2} + \varepsilon}
\]
\leq 0
(15)
For symbolic representation simplicity, let the sampling time \( T_{s} = 1 \) and choose the admissible feedback terminal controller
\[
u_{j} = \begin{bmatrix} v_{j} & \omega_{y} \end{bmatrix}^{T}
\]
as
\[
u_{j} = -\kappa\phi_{x}, \kappa \geq 0
\]
\[
\omega_{y} = \lambda(\phi_{x} - \alpha_{y}), \lambda \geq 0
\]
The stability condition can be rewritten as follows
\[
F\left(X_{E}^{*}\right) - F(X_{E}) + \ell(X_{E},\kappa_{f}(X_{E})) \leq 0
\]
\[
= 2\kappa_{x}^{2}\sin^{2}(\alpha_{x}) - 2\kappa_{x}(\lambda - 1)\phi_{x}, \sin(\alpha_{x}) + (r\kappa_{x}^{2} + \cos(\alpha_{x})(\kappa_{x}\cos(\alpha_{x}) - 2)\kappa_{x} + q)e_{N}^{2}
\]
\[
+ (q + \lambda(\lambda + 1))\phi_{x}^{2} + ((r + 1)\lambda^{2} + q)\phi_{y}^{2}
\]
\[
+ 2\alpha_{y}(\kappa_{x}^{2} - 1)\sin(\alpha_{y}) + \lambda(\lambda + 1)\phi_{y}^{2} + 2\alpha_{y}(\kappa_{x}^{2} - 1)\sin(\alpha_{y}) - \lambda(\lambda + 1)\phi_{y}^{2})
\]
\[
+ \sqrt{e_{N}^{2} - 2(\cos(\phi_{x})x_{y}^{2} + \sin(\phi_{x})y_{y}^{2})e_{N} + (x_{y}^{2}) + (y_{y}^{2})^{2} + \varepsilon}
\]
\leq 0
(17)
The closed-loop stability can be guaranteed by choosing a proper \( q, r, \kappa, \) and \( \lambda \) to satisfy the stability constraint (17).

V. SIMULATIONS
In this section, based on the Kuhun et al. [5] robot simulation conditions, we verify the effectiveness of NMPC algorithm on regulation problem. The control input constraints are chosen as follows:
\[
u_{\min} = \begin{bmatrix} -0.47 & m/s \n -3.77 & rad/s \end{bmatrix}, \quad \nu_{\max} = \begin{bmatrix} 0.47 & m/s \n 3.77 & rad/s \end{bmatrix}
\]
The sampling time is chosen as \( T_{s} = 0.1s \), the terminal controller gain are assigned as \( \kappa = \lambda \). The weighting matrices \( Q = P = \text{diag}(0.1,0.1,0.1) \) and \( R = \text{diag}(0.1,0.1) \) are chosen to satisfy the stability condition. We are given the initial configuration \( X = (-10,10,\pi/2) \) and the reference set-point is origin, i.e., \( X^{ref} = (0,0,0) \). In the following, we will show three navigation simulations that the trajectory between the initial configuration and set-point configuration is to be planned to illustrate the regulation performance of WMR under NMPC algorithm. The first regulation case is considered as the wheeled mobile robot in a free environment, the second is the environment consisting of irregularly shaped static obstacles which are modeled by outer bounding circles, and the last one is a dynamic environment consisting of moving as well as static obstacles.

A. Free Environment
In order to verify the regulation performance in free environment, WMR starts from six different initial positions given by
\[
(0,1,\pi/2),(1,0,\pi/2),(0,-1,\pi/2)
\]
\[
(0.7,0.7,\pi/2),(0.7,-0.7,\pi/2),(0.7,-0.7,\pi/2)
\]
We choose the prediction horizon \( N = 10 \) and the closed-loop trajectory results are shown in Figure. 2. It clearly demonstrates that the WMR moves toward to the goal and finally settles at the origin.

Figure. 2. Traversal to goal in free environment: The closed-loop trajectory of WMR with six different initial positions, \( N = 10 \).

B. Simple Static Environment
We first consider the environment consisting of only static obstacles. In general, the shape of obstacles in real environment is irregular and difficult to describe precisely. In our simulation, the irregular shapes of obstacles can be approximated as a disc with a radius \( r_{c} \), and we may assign a larger enclosed circle with radius \( D = r_{c} + k_{p} \) for each obstacle where \( 0 < k \leq 1 \) is an appropriate obstacle avoidance factor to describe the forbidden region. If the WMR detects the safety region, then it will activate the obstacle avoidance...
mechanism.

Figure 3 shows closed-loop trajectories of WMR in a map without and with obstacles under the prediction horizon $N = 20$. In the free environment, the trajectory is a nearly straight line except the portions of starting and ending configuration. When the WMR detected the obstacles in its way to the origin with a sufficient prediction horizon length so that NMPC has perfect information of predicted static and dynamic obstacles and WMR movement, the WMR keeps a prespecified safety distance away from the obstacles, detours its path by changing its heading and then approaches the origin.

Figure 3. The closed-loop trajectory of WMR, where the fine-line is the trajectory in a free environment and the bold-line is the trajectory of WMR in an environment with three irregularly-shaped obstacles. The outer circle of each obstacle is the safety margin with $k = 0.5$.

Figure 4. The controller effort of WMR, where the fine-line is the idealized trajectory (the trajectory without obstacle environment) and the bold-line is the controller effort of WMR in obstacle environment.

Figure 4 reveals the controller effort (linear and angular velocities) of the controlled WMR in free and static obstacle environments. It is seen that in both cases controller saturation constraints are not violated. According to Figure 5, NMPC controlled WMR with the prediction horizon $N = 30$ turns earlier and smoother than that with $N = 20$. It implies that to ensure a smoother navigation to the goal a larger prediction horizon is necessary for WMR to encode more environment information by detecting obstacles earlier in an obstacles obstructed environment.

C. Dynamic Environment

The moving obstacles will cause changes in the environment over time. In this part of simulation, two patrolling WMRs which follow the priori designated paths are added into the previous static environment. The simulation conditions are the same as the previous subsection. This formulation can be regarded as a single player evasion games. The controlled WMR should avoid both the moving WMRs with prescribed trajectories and the static obstacles in the environment while move toward to the origin. For notational simplicity, the NMPC controlled WMR is denoted as WMR$_c$, and the other moving WMRs are denoted as WMR$_1$ and WMR$_2$, respectively; The kinematic equation of WMR$_1$ and WMR$_2$ is governed by equation (1). We assume that their initial positions and moving trajectories are known. The trajectory of moving WMR$_1$ is a circular-like motion path which is originated from $(x,y) = (-8,6)$ and the trajectory of WMR$_2$ is a reciprocating motion between two static obstacles initiated from the position $(x,y) = (-4,5)$.

Their corresponding inputs are given as follows:

$$WMR_1 : (v, \omega) = \begin{bmatrix} -0.2 m/s, & -0.1 rad/s \end{bmatrix}^T$$

$$WMR_2 : (v, \omega) = \begin{bmatrix} \cos(0.8t), & 0 rad/s \end{bmatrix}^T$$

where $t = kT_s$ with the sampling time $T_s$. Figure 6 illustrates the scenes of the motion in different time steps, which there are two moving WMRs as moving obstacles (bold-line trajectories) to disturb the regulation performance of WMR$_c$, whose closed-loop trajectory is the fine-line trajectory. It is
seen that the mobile robot WMRc controlled by NMPC with $N=10$ implements a very simple control heuristics of lookahead to predict the possible collisions with obstacles that change positions. The resulting control is to detour its forward path to avoid both the two moving obstacles and static obstacles simultaneously on its way toward the goal in a dynamic map.

VI. CONCLUSION

To safely navigate a WMR to a goal in the presence of both static and moving obstacles, this paper applied the Nonlinear Model Predictive Control (NMPC) to an input-constrained discrete-time nonholonomic WMR model in the polar

![Figure 6. The Cartesian closed-loop trajectory of a NMPC-controlled WMR in the dynamic environment consisting of two patrolling WMRs and three static obstacles. Prediction horizon, $N=10$. The closed-loop stability, static as well as dynamic obstacle avoidance capabilities have been verified and demonstrated by simulations showing that for each robot position at sampling instant NMPC implicitly encodes the changes on obstacles positions to adapt to the environmental change. For the future work, to be implementable in real time computational complexity analysis is needed for applicable to either offline or online navigation. Furthermore, we will focus on the robustness analysis of more complicated WMR model and the pursuit-evasion games of multiple WMRs by using NMPC approach.](image)

REFERENCES