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A Bounded-curvature Shortest Path Generation Method for Car-like Mobile Robot Using Cubic Spiral

Tzu-Chen Liang
Institute of Information Science
Academia Sinica
Nankang, Taipei, Taiwan 115, R.O.C.

Jing-Sin Liu
Institute of Information Science
Academia Sinica
Nankang, Taipei, Taiwan 115, R.O.C.
Email: liu@iis.sinica.edu.tw

Abstract—**A trajectory generation method for car-like mobile robot based on cubic spirals and line segments is presented. The generated path is made up of portion of cubic spiral segments with zero curvature ends and upper bounded curvature and straight line segments. A numerically efficient process, which is resorted to minimization over the sum of length of each path segment of generated path via linear programming, is presented to generate a Cartesian shortest path linking start and destination configurations of car-like mobile robots through an intermediate configuration. The intermediate configuration is not necessarily selected from the symmetric means circle. The merits of our path generation method based on cubic spirals are: (i)The implementation is straightforward so that the generation of feasible paths with bounded curvature is efficient for real-time applications. (ii)Applicable to mobile robots with forward and backward driving abilities and only forward driving ability; (iii) Flexible to incorporate other constraints.**

I. Introduction

Path planning problem of autonomous mobile robots or vehicles (e.g. soccer robots) has been widely studied in recent years. The essential topic is to generate an acceptable path that optimizes a criterion to join two distinct configurations subject to constraints. It is challenging because nonholonomic robots can only move in the direction that they are facing. L. E. Dubins first discussed the shortest paths with bounded curvature synthesized by circular arc and straight line in 1957 [8]. J. A. Reeds and L. A. Shepp further studied the same path with cusp, i.e. the vehicle can drive both forward and backward, in 1990[9]. A complete characterization of path synthesized by arc of circle and straight line was addressed by P. Souères and J. Laumond [2]. These studies concentrated on the finding of the path with theoretical minimal length from a family of curves. They showed such kind of paths has at most two cusps, where the robot changes its moving direction. However, its non-continuous curvature results in a control difficulty. At the junction of a straight line and an arc, mobile robot needs to stop its wheel motion to make the perfect tracking achievable [4].

The necessity to generate paths with continuous curvature for the navigation of mobile robots or autonomous vehicles with good drive characteristics and small trajectory tracking errors was noted by [10], [11], [14], [15]. Cloithoid curve is used as a transition curve for smoothing a circular-arc line segment junction [3], [4], [16]. Scheuer and Laugier [3], [4] incorporated a new constraint that the derivative of curvature is bounded into the path planning problem to make the planned path smoother. Nelson [11] presented two types of paths, Cartesian quintics for lane changing maneuvers and polar splines for symmetric turns, which both can smoothly connect line segments with zero curvature. There are other curves which have been proposed for trajectory of mobile robots or autonomous vehicles, e.g. B-spline [18], quintic polynomials [15], quintic G^2-spline [17]. They are easy to compute but difficult to follow due to the complex curvature profiles.

Another parametric curve is polynomial spiral whose curvature is a polynomial of arc length [10]. Cubic spiral is a kind of trajectory whose direction function is cubic. A cubic spiral is cut at its two inflection points to obtain a curve with finite length and has zero curvature at both end-points. This portion of a cubic spiral can connect two configurations that are *symmetric*. Two configurations which are not symmetric can be joined through an intermediate configuration, which is called *symmetric mean*, by two cubic spirals [1]. Instead of path length, Kanayama and Hartman [1] used path curvature and derivative of path curvature as criteria of path optimization intended to maximize passenger comfort. Though the cubic spiral method can generate smoother path than other ones, it does not consider the bounded curvature constraint of car-like mobile robots [12], [19], a practical constraint in turning, or the avoidance of obstacles [13]. Furthermore, the "smoothest" criterion makes the path too long if two configurations are relatively far apart.

The problem of trajectory generation for car-like mobile robot investigated here builds on the work of [1]. Our aim is to compute the path with shortest length among a set of families of continuous and upper-bounded curvature paths made up of cubic spirals in connection with straight line segments at null curvature points.

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The paper is organized as follows. In Section 2, the cubic spiral method is briefly reviewed, and the notations used in this paper are introduced. The numerical procedure to find a path of minimal length via change of potential intermediate orienations is presented in Section 3. The last section is the conclusion.

II. Review of Cubic Spiral Method

II.1 Basics of cubic spiral [1]

A triple $q \equiv (x, y, \theta)$ is to represent a vehicle configuration where (x,y) is the position and θ is the heading. For an arbitrary configuration q, $[q]$ denotes its position (x, y), and (q) its direction θ.

A directed curve Π with finite length ℓ is defined by a triple $\Pi \equiv (\ell, \kappa, q_0)$ where $\kappa : [0, \ell] \to \mathbb{R}$ is its curvature and $q_0 \equiv (x_0, y_0, \theta_0)$ is its initial configuration. A configuration $q(s) = (x(s), y(s), \theta(s))$ at arc length s is defined by

$$\begin{aligned} \theta(s) &= \theta_0 + \int_0^s \kappa(t)\,dt \\ x(s) &= x_0 + \int_0^s \cos\theta(t)\,dt \\ y(s) &= y_0 + \int_0^s \sin\theta(t)\,dt \end{aligned} \tag{1}$$

where s is defined as 0 at the initial configuration (x_0, y_0, θ_0).

By definition, cubic spiral is a set of trajectories that the direction function θ is cubic. The curvature of the portion of cubic spiral with two reflection points as end points is zero and has finite length. The curvature function of this portion of cubic spiral with length ℓ is represented as $\kappa(s) = As(\ell - s)$, where A is a nonzero constant to be determined. At the inflection points ($s = 0$ *and* $s = \ell$) the cubic spiral has zero curvature. The constant A of a cubic spiral joining two distinct configurations with relative angle $\alpha = \theta(\ell) - \theta(0)$ can be solved by the first equation of (3). The curvature function becomes (Lemma 2 , [1]),

$$\kappa(s) = \frac{6\alpha}{\ell^3} s(\ell - s) \tag{2}$$

For unit length cubic spiral, its *size* is given by (Lemma3, [1])

$$D(\alpha) \equiv 2\int_0^{1/2} \cos(\alpha(\frac{3}{2} - 2t^2)t)dt$$

The relation of ℓ and $d = size(q_1, q_2)$ by α can be evaluated by a pre-calculated $D(\alpha)$ vs. α table (Fig. 1) using the following equation (Proposition 8, [1]),

$$\ell = \frac{d}{D(\alpha)} \tag{3}$$

II.2 Cubic Spiral Path Planning Method

II.2.1 Concept of Symmetric Configurations

For a configuration pair (q_1, q_2), the *size* is the distance between the two points $[q_1]$ and $[q_2]$, and the *angle* is the difference between the two directions (q_1) and (q_2), i.e.

$$\begin{aligned} size(q_1, q_2) &\equiv d([q_1],[q_2]) \\ angle(q_1, q_2) &\equiv \Phi((q_2) - (q_1)) \end{aligned} \tag{4}$$

where the angle-normalizing function Φ is defined as

$$\Phi(\theta) \equiv \theta - 2\pi \left\lfloor \frac{\theta + \pi}{2\pi} \right\rfloor \tag{5}$$

For a vector $\vec{v} = (v_x, v_y)$,

$$angle(\vec{v}) = \operatorname{atan2}(v_y, v_x).$$

A configuration pair $[q_1, q_2]$ is said to be symmetric if

$$\tan(\frac{\theta_1 + \theta_2}{2}) = \frac{y_2 - y_1}{x_2 - x_1}, \text{ if } x_1 \neq x_2$$

$$\Phi(\frac{\theta_1 + \theta_2}{2}) = \pm\frac{\pi}{2}, \text{ if } x_1 = x_2$$

A *symmetric mean* q of any configuration pair $[q_1, q_2]$ is a configuration that both $[q_1, q]$ and $[q, q_2]$ are symmetric pairs. All symmetric means of a configuration pair $[q_1, q_2]$ forms a circle if $(q_1) \neq (q_2)$ or a line if $(q_1) = (q_2)$ (Proposition 3, [1]).

II.2.2 Drawbacks

The path planning algorithm of [1] is to choose one best symmetric mean q from this circle as an intermediate configuration so as to minimize the sum of cost functions of the cubic spiral joining the symmetric pair (q_1, q) and the cubic spiral joining the symmetric pair (q, q_2).

There are two main drawbacks of this method that make it not fit practical use. First, the cost functions of Kanayama and Hartman [1] are either minimization of the integration of centripetal force or the change of centripetal force; the length of the path and maximal (or minimal) curvature along the path are not taken into consideration. Secondly, the method fails in the configuration pair that two configurations are originally symmetric. Though Kanayama and Hartman [1] declared that this kind of configuration pair can be joined by simple curve (specifically, one symmetric curve), but for some cases, for example $q_1 = [0,0,0]$ and $q_2 = [-a,0,0]$ configurations with the same horizontal heading but different positions, the simple curve may have infinite length.

III. Generation of continuous and bounded curvature trajectory using cubic spiral

The cubic spiral it is extensible by line segments: a path can be made up of connecting together cubic spirals and straight lines at zero curvature points and the assembled path still keeps the continuity of curvature. The path length is set as the criterion for optimization of paths [2],[7-9],[13] from family of continuous curvature curves assembled by at most two cubic spirals and two straight lines, while maximal (or minimal) value of curvature is set as constraint. The problem is to find a shortest path with bounded and continuous curvature joining a given ordered pair of configurations $[q_1, q_2]$.

III.1 Constraint of Maximal Curvature

In practice, the path of a wheeled mobile robot has its minimal radius of turning which is constrained by wheel arrangement [5]-[7]. We use a constant $\kappa_{\max}$ to describe the absolute value of maximal curvature of the planned path. Because curvature of a straight line is zero, this constraint only affects the cubic spiral segment of the path.

Consider a cubic spiral with angle α. Its curvature has the maximal (or minimal, if $\alpha < 0$) value $3|\alpha|D(\alpha)/2d$ at the middle point $s = \ell/2$.

Hence a constraint of curvature can be written as

$|\kappa(s)| \le \kappa_{\max} = 3|\alpha|D(\alpha)/2d$, or

$$d \ge d_{\min}(\alpha) = 3|\alpha|D(\alpha)/2\kappa_{\max} \tag{6}$$

III.2 Minimal Locomotion

Consider a mobile robot that can move forward and backward. We'll present the computation of the cubic spiral for minimal locomotion from q_1 through a specified intermediate direction θ_m to achieve a goal direction given by (q_2), with the intermediate and end positions unspecified.

We restrict the angle of a cubic spiral in the range $[-2\pi, 2\pi]$. Connecting an initial configuration q_1 which has direction θ_1 and the intermediate configuration with direction θ_m, there exist two cubic spirals that all have the maximal curvature. *Angles* of these two cubic spirals are,

$$\alpha_{c1}^{+} = \Phi(\theta_m - \theta_1), \alpha_{c1}^{-} = -\mathrm{sgn}(\alpha_{c1}^{+}) \cdot (2\pi) + \alpha_{c1}^{+} \tag{7}$$

If α_{c1}^{+} is positive, then α_{c1}^{-} must be negative, and vice versa. The above notation (14) assures $\alpha_{c1}^{+} \in [-\pi, \pi]$ and α_{c1}^{-} outside this range. Consider a mobile robot that can move forward and backward. To achieve each of both angles (7) by forward and backward motion, there are four cubic spirals can make it. These four types of motion are represented by the following notations,

$$(\alpha_{c1}^{+})^{+}, (\alpha_{c1}^{+})^{-}, (\alpha_{c1}^{-})^{+}, \text{and} (\alpha_{c1}^{-})^{-} \tag{8}$$

Positive sign outside the parenthesis means forward motion, and negative one means backward motion. Fig. 2 shows an example of the four trajectories with defining notations (8).

For each of four cubic spirals (8), we define:

(i)four intermediate configurations $q_m^{++}, q_m^{+-}, q_m^{-+}, and\ q_m^{--}$ and vectors

$$\vec{v}_{c1}^{++} \equiv [q_m^{++}] - [q_1], \vec{v}_{c1}^{+-} \equiv [q_m^{+-}] - [q_1]$$
$$\vec{v}_{c1}^{-+} \equiv [q_m^{-+}] - [q_1], \vec{v}_{c1}^{--} \equiv [q_m^{--}] - [q_1] \tag{9}$$

where the first superscript denotes the range of the angle and the second superscript represents the motion direction.

(ii)four distances for each path as the length of each of the above vectors

$$d_{c1}^{++} \equiv d_{\min}(\alpha_{c1}{}^{+}), d_{c1}^{+-} \equiv -d_{\min}(\alpha_{c1}{}^{+})$$
$$d_{c1}^{-+} \equiv d_{\min}(\alpha_{c1}{}^{-}), d_{c1}^{--} \equiv -d_{\min}(\alpha_{c1}{}^{-}) \tag{10}$$

where $d_{\min}(\bullet)$ is always positive. Two of the above four d_{c1} corresponding to backward motion are negative, where negative distance value denotes backward motion of vehicle. Similarly for the angle $\alpha_{c2} = \Phi(\theta_2 - \theta_m)$, there are four paths for each q_m. As a result, for a given intermediate direction (θ_m), there are 16 different paths with maximal curvature whose end direction is $(q_2) = \theta_2$. Only two of them are selected from the symmetric mean circle.

Let the minimal locomotion vector of two connecting cubic spirals with initial heading θ_1 through a given intermediate orientation θ_m to a desired orientation θ_2 be denoted the vector $\vec{v}_c^{ij,kl}$. It is defined by the addition of minimal locomotion vector in each cubic spiral segment

$$\vec{v}_c^{ij,kl} = \vec{v}_{c1}^{ij} + \vec{v}_{c2}^{kl} = d_{c1}^{ij}\vec{n}_{c1} + d_{c2}^{kl}\vec{n}_{c2},\ \ i,j,k,l \in \{+,-\} \tag{11}$$

where the corresponding minimal distance is d_{c1}^{ij}, d_{c2}^{kl} and $\vec{n}_{c1}, \vec{n}_{c2}$ is unit vector of motion direction.

III.3 Assemble a Path

Following [1], we adopt two cubic spirals to form parts of a path. There are three zero curvature points at a two-cubic-spiral path. These points can be extended by straight lines to enhance reachability of path. In combination with two cubic spirals, five segments are used to assemble a path to connect a given configuration pair. The five directions are denoted by $\theta_1, \theta_{c1}, \theta_m, \theta_{c2}, \theta_2$ with corresponding vectors $\vec{v}_1, \vec{v}_{c1}, \vec{v}_m, \vec{v}_{c2}, \vec{v}_2$ where

$$\theta_{c1} \equiv \Phi((\theta_1 + \theta_m)/2), \theta_{c2} \equiv \Phi((\theta_m + \theta_2)/2)$$

The directions of the five vectors are known, and only the lengths are to be determined. Define the unit vectors of the five vectors as $\vec{n}_1, \vec{n}_{c1}, \vec{n}_m, \vec{n}_{c2}, \vec{n}_2$ where positive direction of these five unit vectors represents forward motion direction. A feasible combination of these vectors satisfy the following equation,

$$[q_2]=[q_1]+\vec{v}_1+\vec{v}_{c1}+\vec{v}_m+\vec{v}_{c2}+\vec{v}_2 \tag{12}$$

which can be rewritten more explicitly as

$$[q_2]=[q_1]+d_1\vec{n}_1+d_{c1}\vec{n}_{c1}+d_m\vec{n}_m+d_{c2}\vec{n}_{c2}+d_2\vec{n}_2 \tag{13}$$

where d_1, d_{c1}, d_m, d_{c2}, and d_2 are length to be decided.

By the definition of (17), the constraints imposed on the two cubic spiral segments of a path are

$$\begin{cases} d_{c1} > d_{c1}^{ij} & if\ d_{c1}^{ij} > 0, \\ d_{c1} < d_{c1}^{ij} \end{cases} \begin{cases} d_{c2} > d_{c2}^{kl} & if\ d_{c2}^{kl} > 0, \\ d_{c2} < d_{c2}^{kl} \end{cases} \tag{14}$$

while the other coefficients $d_1, d_m, and\ d_2$ are free. It is noted that positive (negative) value of each d denotes forward (backward) motion.

III.4 Criterion of Minimal Path Length

The objective of this subsection is to formulate a minimal length solution of the family of paths composed of straight lines and cubic spirals. Assume θ_m and i, j, k, and l have been selected already. The cost criterion for this family of paths can be defined as the sum of the length of each of the five segments

$$\text{cost}(\Pi)=|d_1|+\frac{|d_{c1}|}{D(\alpha_{c1}^i)}+|d_m|+\frac{|d_{c2}|}{D(\alpha_{c2}^k)}+|d_2| \tag{15}$$

Because d_{c1}^{ij} and d_{c2}^{kl} have been chosen, $D(\alpha_{c1}^i)$ and $D(\alpha_{c2}^k)$ are constants and d_1, d_{c1}, d_m, d_{c2}, and d_2 in (13) are variables to be determined that minimize the cost criterion (15) subject to the constraint (14).

The cost of minimal locomotion for chosen d_{c1}^{ij} and d_{c2}^{kl} is the constant

$$\text{cost}(\Pi_{\theta_m}^{ij,kl})=\frac{|d_{c1}^{ij}|}{D(\alpha_{c1}^i)}+\frac{|d_{c2}^{kl}|}{D(\alpha_{c2}^k)}$$

and cannot be further reduced. We define

$$d_{c1}^{*}=d_{c1}-d_{c1}^{ij},\ d_{c2}^{*}=d_{c2}-d_{c2}^{kl} \tag{16}$$

Then to minimize (15) is equivalent to minimize the reduced cost

$$\begin{aligned}\text{cost}^*(\Pi) &= \text{cost}(\Pi)-\text{cost}(\Pi_{\theta_m}^{ij,kl}) \\ &= |d_1|+\frac{|d_{c1}^*|}{D(\alpha_{c1}^i)}+|d_m|+\frac{|d_{c2}^*|}{D(\alpha_{c2}^k)}+|d_2|\end{aligned} \tag{17}$$

And the constraints (14) can be rewritten as

$$\begin{cases} d_{c1}^* > 0 & if\ d_{c1}^{ij} > 0, \\ d_{c1}^* < 0 \end{cases} \begin{cases} d_{c2}^* > 0 & if\ d_{c2}^{kl} > 0, \\ d_{c2}^* < 0 \end{cases} \tag{18}$$

III.5 Vector Choice

Now the condition (13) for path assembling can be rewritten as

$$[q_2]=[q_1]+d_1\vec{n}_1+(d_{c1}^{ij}\vec{n}_{c1}+d_{c1}^*\vec{n}_{c1})+d_m\vec{n}_m+(d_{c2}^{kl}\vec{n}_{c2}+d_{c2}^*\vec{n}_{c2})+d_2\vec{n}_2 \tag{19}$$

Define the vector

$$\vec{v}_{goal}=[q_2]-[q_1]-(d_{c1}^{ij}\vec{n}_{c1}+d_{c2}^{kl}\vec{n}_{c2}) \tag{20}$$

where the minimal locomotion vector $\vec{v}_c^{ij,kl}$ in (18) written explicitly as

$$\vec{v}_c^{ij,kl}=d_{c1}^{ij}\vec{n}_{c1}+d_{c2}^{kl}\vec{n}_{c2} \tag{21}$$

which can be written equivalently as

$$\vec{v}_{goal}=d_1\vec{n}_1+d_{c1}^*\vec{n}_{c1}+d_m\vec{n}_m+d_{c2}^*\vec{n}_{c2}+d_2\vec{n}_2 \tag{22}$$

where the coefficients d_1, d_m, and d_2 are free, and the signs of d_{c1}^* and d_{c2}^* are pre-decided by the constraints (18). To solve the coefficients from (22) for a given $\vec{v}_{goal}$, we define

$$\vec{n}_\Delta^+=\vec{n}_\Delta \text{ and } \vec{n}_\Delta^-=-\vec{n}_\Delta, \Delta\in\{1, c1, m, c2, 2\} \tag{23}$$

Define a vectors set $\mathbb{N}$ so that its corresponding set $\mathbb{C}$ of coefficients are nonnegative,

$$\mathbb{N}\equiv\{\vec{n}_1^+,\vec{n}_1^-,\vec{n}_m^+,\vec{n}_m^-,\vec{n}_2^+,\vec{n}_2^-,\vec{n}_{c1}^j,\vec{n}_{c2}^l\}, j,l=+,-$$

$$\mathbb{C}\equiv\{d_1^+,d_1^-,d_m^+,d_m^-,d_2^+,d_2^-,d_{c1}^j,d_{c2}^l\}$$

$$d_\Delta=d_\Delta^+-d_\Delta^-,\ \Delta=1,m,2,\ d_{c1}^j=|d_{c1}^*|, d_{c2}^l=|d_{c2}^*|$$

Then (22) can be transformed into

$$\vec{v}_{goal}=d_1^+\vec{n}_1^++d_1^-\vec{n}_1^-+d_{c1}^j\vec{n}_{c1}^j+d_m^+\vec{n}_m^++d_m^-\vec{n}_m^-+d_{c2}^l\vec{n}_{c2}^l+d_2^+\vec{n}_2^++d_2^-\vec{n}_2^- \tag{24}$$

with all coefficients nonnegative. The criterion (17) can be rewritten by

$$\text{cost}^*(\Pi)=d_1^++d_1^-+\frac{d_{c1}^j}{D(\alpha_{c1}^i)}+d_m^++d_m^-+\frac{d_{c2}^l}{D(\alpha_{c2}^k)}+d_2^++d_2^- \tag{25}$$

Now the path length minimization can be formulated as a canonical form of linear programming problem, and only two elements of the coefficient set $\mathbb{C}$ are needed, while others are zero. Assume a pair of linearly independent vectors $\vec{n}_a$ and $\vec{n}_b\in\mathbb{N}$ are chosen with corresponding nonzero components d_a and $d_b\in\mathbb{C}$, so that (24) can be partitioned as

$$
\begin{aligned}
&\vec{v}_{goal} = \vec{v}_a + \vec{v}_b = d_a\vec{n}_a + d_b\vec{n}_b \\
&d_a = \left|\vec{v}_{goal}\right| \frac{-\sin\theta^-}{\sin(\theta^+ - \theta^-)}, d_b = \left|\vec{v}_{goal}\right| \frac{\sin\theta^+}{\sin(\theta^+ - \theta^-)} \\
&\theta^+ = \Phi(angle(\vec{n}_a) - angle(\vec{v}_{goal})), \theta^- = \Phi(angle(\vec{n}_b) - angle(\vec{v}_{goal}))
\end{aligned} \tag{26}
$$

Without loss of generality, the choice of $\vec{n}_a$ and $\vec{n}_b$ is defined so that θ^+ is positive and θ^- is negative. Then, by (26), the coefficients d_1, d_{c1}, d_m, d_{c2} and d_2 are solved to minimize (25).

The procedure to find a shortest path joining two configurations of mobile robot is to search from all possible combination of two cubic spirals and straight lines, and find the one that minimizes the cost (17). Fig. 3 is an example to demonstrate the synthesis procedure of a shortest path whenever θ_m is selected and thus i, j, k, and l have been decided. First the minimal locomotion vector $\vec{v}_{c1}^{ij} + \vec{v}_{c2}^{kl}$ is computed to define the $\vec{v}_{goal}$. Then two vectors to be extended are selected from eight candidates. In the case shown here, $\vec{v}_{c2}^{l}$ and $\vec{v}_1^{+}$ are chosen. The result path (plotted in solid line) is composed of one straight line and two cubic spirals.

It is noted here that the selection of intermediate θ_m direction is not necessarily from symmetric means circle (as [1] did), whose configuration (direction and orientation) is completely specified. Examples of shortest paths planned by the algorithm are shown in Fig. 4 for mobile robot capable of forward and backward motion and forward motion only.

IV. Conclusion

For generation of bounded curvature path linking two configurations of wheeled mobile robots, this paper considers the family of curvature-constrained curves constituted by line segments and cubic spirals segments with ends at inflection points. The reachability is enlarged due to the addition of line segments as parts of the cubic spiral path. A shortest path planning method searches a minimal length path from all feasible paths generated by linear programming optimization over the length of each path segment via the change of intermediate configuration. The method is applicable to generate paths of mobile robots with or without backward motion capability. As for practical use, the implementation is straightforward for real-time applications and flexible to incorporate obstacle avoidance , for navigating a mobile robot in a constrained workspace [20], e.g. in robot soccer game.

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Fig. 1 Distance function of cubic spirals. The dashed vertical line corresponds to $|\alpha| = 180^o$; left side of this line is α^+ region and right side is α^- region [1].

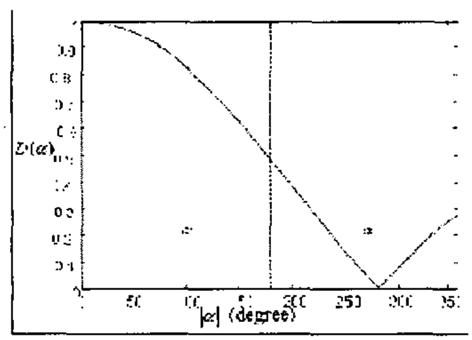
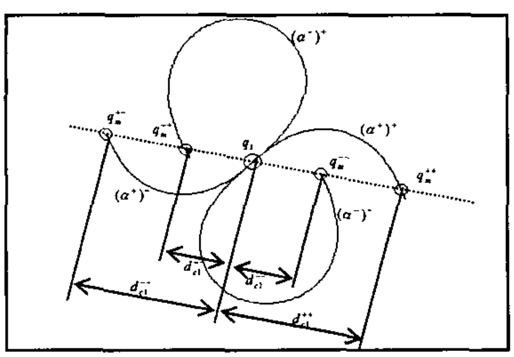


Fig.2 There are four cubic spirals from an initial configuration q_1 to reach an intermediate direction (q_m) with maximal (or minimal) curvature value at their middle points.



[20]T.C. Liang and J.S. Liu, "Collision-free path planning of mobile robots using cubic spirals," *2004 IEEE Int. Conference on Robotics and Biomimetics*, Shenyang, China, August, 2004.

Fig. 3 This figure shows how to synthesize a shortest path if θ_m is given and thus i, j, k, and l have been decided. First the minimal locomotion vector $\vec{v}_{c1}^{ij} + \vec{v}_{c2}^{kl}$ is computed to define the $\vec{v}_{goal}$. Then two vectors to be extended are selected from eight candidates. In this case, $\vec{v}_{c2}^{l}$ and $\vec{v}_1^{+}$ are chosen. The path (solid line) is composed of one straight line and two cubic spirals.

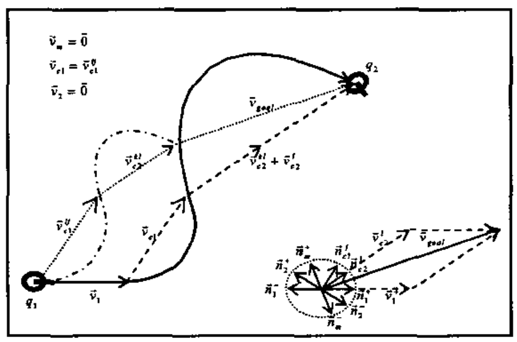
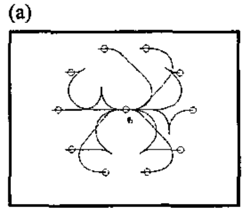
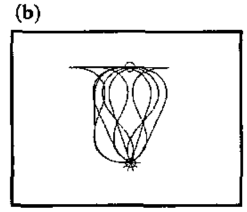


Fig.4 Example of generated shortest paths. (a)Each path has the same (q_2) but different $[q_2]$. (b) Each path has the same $[q_2]$ but different (q_2). (c)Forward motion only for (a) (d) forward motion only for (b)

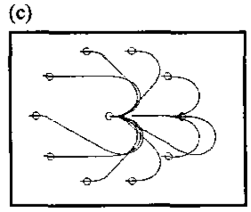
(a)



(b)



(c)



(d)