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A note on edge fault tolerance with respect to hypercubes

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Abstract

In the previous studies on k-edge fault tolerance with respect to hypercubes Q_n, matrices for generating linear k-$EFT(Q_n)$ graphs were used. Let $EFT_L(n,k)$ denote the set of matrices that generate linear k-$EFT(Q_n)$ graphs. A matrix in $EFT_L(n,k)$ with the smallest number of rows among all matrices in $EFT_L(n,k)$ is optimal. We use $eft_L(n,k)$ to denote the difference between the number of rows and the number of columns in any optimal $EFT_L(n,k)$ matrix. In terms of Hamming weight, in this work we present a necessary and sufficient condition for those matrices in $EFT_L(n,k)$ and another necessary and sufficient condition for those matrices in $EFT_L(n,k)$ of the form $\begin{bmatrix} I_n \\ D \end{bmatrix}$. We also prove that $eft_L(n,k+1) \geq eft_L(n,k)+1$ and that $eft_L(n,k+1) = eft_L(n,k)+1$ if k is even.

Keywords: Hypercube; Edge fault tolerance; Hamming weight

1. Introduction

In this work, any *graph* means an undirected graph in which multiple edges are allowed. Let $G = (V, E)$ be a graph where V is the vertex set of G and E is the edge set of G. For any vertex x of V, $\deg_G(x)$ denotes its degree in G. Let E' be a subset of E. We use $G - E'$ to denote the spanning subgraph of G with its edge set $E - E'$.

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We usually use a graph to represent the architecture of an interconnection network, where nodes represent processors and edges represent communication links between pairs of processors. Faults may occur in nodes/or edges of an interconnection network. We restrict faults to edges only in this work. Motivated by the study of computer and communication networks that can tolerate failure of their components, Harary and Hayes [1] formulated the concept of edge fault tolerance in graphs. Given a *target graph* $H = (V, E)$, let $G = (V, E^*)$ be a supergraph of H. G is said to be *k-edge-fault-tolerant* with respect to H, denoted by k-$EFT(H)$, if for any $F \subseteq E^*$ and $|F| = k$, $G - F$ contains a subgraph isomorphic to H. The graph G^* is said to be *optimal* if G^* contains the smallest number of edges among all k-$EFT(H)$ graphs. In this work, the target graphs are hypercubes. Edge-fault-tolerant graphs with respect to hypercubes have been studied in [1–6].

For interconnection networks proposed in the literature, the hypercubes Q_n are among the most popular topologies [7]. Let $u = u_{n-1}u_{n-2}\dots u_1u_0$ and $v = v_{n-1}v_{n-2}\dots v_1v_0$ be two n-bit strings. We use $u + v$ to denote the bitwise boolean sum of u and v. The *Hamming weight* of u, denoted by $w(u)$, is defined to be the number of i with $0 \le i \le n-1$ such that $u_i \ne 0$. The *Hamming distance* between u and v, denoted by $h(u, v)$, is the number of i with $0 \le i \le n-1$ such that $u_i \ne v_i$. Obviously, $h(u, v) = w(u + v)$. The *n-dimensional hypercube*, Q_n, consists of all the n-bit strings as its vertices and two vertices u and v are *adjacent* if and only if $h(u, v) = 1$.

Yamada et al. [6] used a vector-space approach to develop k-$EFT(Q_n)$ graphs for $k \ge 1$. Let B be any $m \times n$ matrix over $GF(2)$ and R be a proper subset of $\{1, 2, \dots, m\}$. We use $B(\bar{R})$ to denote the matrix obtained from B by deleting those rows with indices in R. Let k be any positive integer. Assume that B is an $m \times n$ matrix such that the rank of $B(\bar{R})$ is n for any $|R| \le k$. With this matrix B, we can build a graph $G_B = (V_n, E_B)$ where $V_n = V(Q_n)$ and any vertex $v \in V_n$ is joined to u if and only if $u = v + r^i$ where r^i is the i-th row vector of B. Obviously, the degree of any vertex v in G_B is m. We call the edge joining v to $v + r^i$ of class i. Suppose that E' is a subset of E_B with $|E'| \le k$. Let $R = \{i \mid e \text{ is an edge in } E' \text{ and } e \text{ is of class } i\}$. Obviously, $|R| \le k$. Hence the rank of $B(\bar{R})$ is n. Thus, we can choose n linearly independent rows from $B(\bar{R})$. Obviously, all the edges of classes in $B(\bar{R})$ induce a graph isomorphic to Q_n. Hence, G_B is a k-$EFT(Q_n)$. We call the corresponding graph G_B a *linear* k-$EFT(Q_n)$. Let $EFT_L(n, k)$ denote the set of matrices B such that G_B is a linear k-$EFT(Q_n)$. Obviously, the matrix $B \in EFT_L(n, k)$ with the smallest number of rows will derive a linear k-$EFT(Q_n)$ graph with the least number of edges among all linear k-$EFT(Q_n)$ graphs. Thus, we say a matrix $B \in EFT_L(n, k)$ with the smallest number of rows is an *optimum linear* k-$EFT(Q_n)$ and we use $eft_L(n, k)$ to denote $(m - n)$ where m is the number of rows in any optimum linear k-$EFT(Q_n)$.

Actually the concept of linear k-$EFT(Q_n)$ had already been used before the formulation proposed by Yamada et al. [6]. Bruck et al. [2] used this approach to construct the 1-$EFT(Q_n)$. Assume that B is a matrix in $EFT_L(n, k)$. Obviously, the rank of B is n. By changing coordinates, we can transform B into $\begin{bmatrix} I_n \\ D \end{bmatrix} = \begin{bmatrix} I_n \\ d^1, d^2, \dots, d^n \end{bmatrix}$. Shih and Batcher [3] proved that any such B in $EFT_L(n, k)$ satisfies the following two conditions: (1) $w(d^j) \ge k$ for every $1 \le j \le n$; i.e., the Hamming weight of each d^i is at least k; and (2) $w(d^i + d^j) \ge k - 1$ for every $1 \le i < j \le n$; i.e., the Hamming distance $h(d^i, d^j)$ between d^i and d^j is at least $k - 1$. With this observation, they employed an ad hoc program to verify edge fault tolerance and thus generate optimal linear k-EFT graphs for $k = 2, 3$ and $n \le 26$. Sung et al. [4] show that the above two conditions are actually the necessary and sufficient conditions for matrix $B \in EFT_L(n, k)$ with $k = 2$. They also show that $eft_L(n, 3) = eft_L(n, 2) + 1$ and present a construction scheme for the optimal linear k-$EFT(Q_n)$ for $k = 2, 3$. They also conjectured that $eft_L(n, 5) = eft_L(n, 4) + 1$. In the following section, we extend the idea in [6] to present some necessary and sufficient conditions for matrix B to be in $EFT_L(n,k)$. We also prove that $eft_L(n,k+1) \geq eft_L(n,k)+1$ and $eft_L(n,k+1) = eft_L(n,k)+1$ if k is even.

2. Edge-fault-tolerant graphs for hypercubes

Let B be any $m \times n$ matrix and R be any proper subset of $\{1, 2, \ldots, m\}$. Using column vectors, we can write B as $[c^1, c^2, \ldots, c^n]$ and $B(\bar{R})$ as $[c_R^1, c_R^2, \ldots, c_R^n]$. Let h be a column/or row vector. We use h^{tr} to denote the transpose of h.

Theorem 1. *B is a matrix in $EFT_L(n,k)$ if and only if the Hamming weight of any column vector, that is, a summation of t different columns of B with $1 \leq t \leq n$, is greater than k, i.e., $w(c^{i_1}+c^{i_2}+\cdots+c^{i_t}) > k$ for any $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ and $1 \leq t \leq n$.*

Proof. Suppose that B is a matrix such that the Hamming weight of some column vector, that is, a summation of t different columns of B with $1 \leq t \leq n$, is at most k. In other words, there exist some $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ and $1 \leq t \leq n$ satisfying $w(c^{i_1} + c^{i_2} + \cdots + c^{i_t}) \leq k$. Let $h = (h_1, h_2, \ldots, h_m)^{\text{tr}} = c^{i_1} + c^{i_2} + \cdots + c^{i_t}$. Let $R = \{i \mid h_i = 1\}$. Obviously, $|R| = w(h) \leq k$ and $c_R^{i_1} + c_R^{i_2} + \cdots + c_R^{i_t} = 0$. Therefore, $\{c_R^{i_1}, c_R^{i_2}, \ldots, c_R^{i_t}\}$ is linearly dependent. Hence, the rank of $B(\bar{R})$ is less than n. Therefore, $B \notin EFT_L(n,k)$.

On the other hand, suppose that B is a matrix such that the Hamming weight of any column vector, that is, a summation of t different columns of B with $1 \leq t \leq n$, is greater than k. Let R be any subset of $\{1, 2, \ldots, m\}$ with $|R| \leq k$. Obviously, any nontrivial linear combination of at most n different columns of $B(\bar{R})$ is not zero. Hence the rank of $B(\bar{R})$ is n. Therefore, $B \in EFT_L(n,k)$. □

Theorem 2. $eft_L(n,k+1) \geq eft_L(n,k)+1$. *Moreover,* $eft_L(n,k+1) = eft_L(n,k)+1$ *if k is even.*

Proof. Let B^* be any matrix in $EFT_L(n,k+1)$. Let B be any matrix obtained by deleting any row from B^*. Obviously, B is a matrix in $EFT_L(n,k)$. Hence, $eft_L(n,k+1) \geq eft_L(n,k)+1$.

Assume that k is an even integer. Let $B = (b_{i,j})$ be any $m \times n$ matrix in $EFT_L(n,k)$. Form a new matrix $B' = (b'_{ij})$ from B by adding a new row $(b'_{m+1,1}, b'_{m+1,2}, \ldots, b'_{m+1,n})$ where $b'_{m+1,j} = \sum_{i=1}^{m} b_{i,j}$. In other words, the new row is the even parity check row of B. Hence, the Hamming weight of any column in B' is even. Thus, the Hamming weight of any linear combination of column vectors of B' is even. Let $h = (h_1, h_2, \ldots, h_m, h_{m+1})^{\text{tr}}$ be a summation of t columns of B' with $1 \leq t \leq n$. We set $h' = (h_1, h_2, \ldots, h_m)^{\text{tr}}$. Obviously, $w(h) \geq w(h')$. By Theorem 1, $w(h') > k$. Since both $w(h)$ and k are even integers, $w(h) > k+1$. Thus, $B' \in EFT_L(n,k+1)$ follows from Theorem 1. Therefore, $eft_L(n,k+1) = eft_L(n,k)+1$ if k is even. □

Since the rank of any matrix is an invariant on changing coordinates, we can find an optimal linear k-$EFT(Q_n)$ among all the matrices of the form $\begin{bmatrix} I_n \\ D \end{bmatrix}$.

Theorem 3. *Let*

$$B = \begin{bmatrix} I_n \\ D \end{bmatrix} = \begin{bmatrix} I_n \\ d^1, d^2, \ldots, d^n \end{bmatrix} = [c^1, c^2, \ldots, c^n] = \begin{bmatrix} r^1 \\ r^2 \\ \vdots \\ r^m \end{bmatrix}.$$

Then, B is a matrix in $EFT_L(n,k)$ if and only if the Hamming weight of any column vector, that is, a summation of t different columns of D with $1 \le t \le k$, is greater than $k-t$.

Proof. We note that $w(c^{i_1}+c^{i_2}+\cdots+c^{i_j}) = w(d^{i_1}+d^{i_2}+\cdots+d^{i_j})+j$ for any $1 \le j \le n$ and $1 \le i_1 < i_2 < \cdots < i_j \le n$.

Assume that the Hamming weight of some column vector, that is, a summation of t different columns of D with $1 \le t \le k$, is at most $k-t$. Thus, there exist some $1 \le i_1 < i_2 < \cdots < i_t \le n$ and $1 \le t \le k$ satisfying $w(c^{i_1}+c^{i_2}+\cdots+c^{i_j}) \le k$. By Theorem 1, $B \notin EFT_L(n,k)$.

On the other hand, suppose that the Hamming weight of any column vector, that is, a summation of t different columns of D with $1 \le t \le k$, is greater than $k-t$. Let R be any subset of $\{1,2,\ldots,m\}$ with $|R| \le k$. We set $I_R = \{t \in R \mid t \le n\}$ and $|I_R| = s$. Obviously, $s \le \min\{k,n\}$. Let $R^* = R - I_R$.

Suppose that $s = 0$. Obviously, $r^1, r^2, \ldots, r^n$ form n independent rows in $B(\bar{R})$. Thus, the rank of $B(\bar{R})$ is n. Suppose that $0 < s \le n$. Let B^* be the submatrix of B formed by those columns with their indices not in I_R. By our assumption, any column vector that is a summation of t different columns of B with $1 \le t \le s$ is greater than $k-t$. Hence, any column vector that is a summation of t different columns of $B(\bar{R})$ with $1 \le t \le s$ and with indices not in I_R is greater than 0. Hence, any nontrivial linear combination of at most s different columns of $B^*(\bar{R}^*)$ is not a zero vector. Therefore, the rank of $B(\bar{R}^*)$ is s. Since the row rank of any matrix equals its column rank, we can find s independent rows, $r^{i_1}, r^{i_2}, \ldots, r^{i_s}$, that span all the row vectors of $B^*(\bar{R}^*)$. Obviously, the rows of $\{r^{i_1}, r^{i_2}, \ldots, r^{i_s}\} \cup \{r^t \mid t \notin R \text{ and } 1 \le t \le n\}$ in $B(\bar{R})$ form n independent row vectors. Thus, the rank of $B(\bar{R})$ is n. Therefore, B is in $EFT_L(n,k)$. $\square$

Obviously, I_n is an optimal linear 0-$EFT(Q_n)$. Using Theorem 2, we obtain an optimal linear 1-$EFT(Q_n)$. With Theorem 3, any $B_{m\times n}$ in $EFT_L(n,2)$ of the form $\begin{bmatrix} I_n \\ D \end{bmatrix}$ satisfies $\binom{m-n}{2} + \binom{m-n}{3} + \cdots + \binom{m-n}{m-n} \ge n$. Suppose that m and n satisfy the above inequality. We can choose any n different $1 \times (m-n)$ columns with their Hamming weight at least 2 to form the matrix D. Again, by Theorem 3, the matrix B is in $EFT_L(n,2)$. Hence, $eft_L(n,2)$ is the smallest integer r that satisfies $\binom{r}{2} + \binom{r}{3} + \cdots + \binom{r}{r} \ge n$. By Theorem 2, we obtain an optimal linear 3-$EFT(Q_n)$. However, we have difficulty in constructing the optimal linear k-$EFT(Q_n)$ with $k \ge 4$.

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