Randomized Algorithms

close-book final exam

June 9, 2004

Instruction Please write down your name and ID on each piece of your answer sheets. Cheating will be most seriously punished. Any dishonest attempt in this exam implies an 'F' as your final grade.

You may use anything of our lectures as a subroutine to your answers, unless you are explicitly asked to explain something we explained in class.

Problem 1 (20 points)

- (5 points) State Loomis' Theorem.
- (5 points) State Yao's Minimax Principle.
- (10 points) Explain why Loomis' Theorem implies Yao's Minimax Principle.

Problem 2 (20 points)

- (5 points) State Adleman's Theorem on derandomizing randomized circuits.
- (15 points) Prove the above theorem.

Problem 3 (20 points)

Give and justify an expected $O(n)$-time algorithm by random sampling for selecting the $n^{1/8}$-th smallest number out of n distinct numbers. (Giving the well known deterministic $O(n)$-time algorithm for selecting a number is not a feasible solution to this problem. An ideal solution to this problem would be an algorithm modified from the randomized median-selection algorithm explained in class.)

Problem 4 (20 points)

Let C_x be the cycle $(x_1, x_2, \ldots, x_n, x_1)$. Let C_y be the cycle $(y_1, y_2, \ldots, y_n, y_1)$. Let graph G consist of C_x and C_y with x_1 and y_1 identified. That is, G has $2n - 1$ nodes forming two simple length-n cycles C_x and C_y joined at $x_1 = y_1$.

- (10 points) What is the hitting time $H(x_i, x_j)$ in G?
- (10 points) What is the hitting time $H(x_i, y_j)$ in G?

Problem 5 (25 points)

We proved in class that the node with rank k has expected depth $H_k + H_{n-k+1} - 1$ in a random TREAP of n nodes. We used Mulmuley's Games A and B. Recall that $A(n) = H_n$ and $B(n, m) = H_m + H_n - H_{m+n}$. Yi-Wei showed us a very elegant proof for the first equality.

- (15 points) Prove that the node with rank k also has expected subtree size $H_k + H_{n-k+1} - 1$.

  Hint: you may resort to Mulmuley's games. Alternatively (or equivalently) you might want to prove and use an observation that the node with rank i is an ancestor of the node with rank j with probability exactly $1/(|i - j| + 1)$.

- (10 points) Let us introduce another game, called game C. It has n regular cards and m stop cards. The counting is just like game A, but as soon as a stop card appears the counting is over. You are asked to prove that $C(n, m) = H_{m+n} - H_m$. (For example, it should be pretty easy to prove the equality for $m = 0$ or $m = 1$.)

Problem 6 (15 points)

Let G be an n-node m edge graph. In the very last lecture, we get stuck at a step for proving

$$\Pr[e \in \hat{T}] \leq \frac{n}{r},$$

where e is a randomly chosen edge in the input graph G, and $\hat{T}$ is the minimum spanning tree of the union of $\{e\}$, a fixed spanning tree T_0 of G, and a size-r random sample R of G's edges. The above inequality can, as a matter of fact, be proved very easily. You are asked to give it a try possibly with the following hint:

> Instead of considering adding e to R, try to think of e as a randomly chosen edge in $\hat{R} = R \cup \{e\}$.

(This is exactly what Timothy Chan meant by "Backward Analysis".)