Nonmonotonic Reasoning Based on Incomplete Logic*

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ABSTRACT. What characterizes human reasoning is the ability of dealing with incomplete information. Incomplete logic is developed for modeling incomplete knowledge. The most distinctive feature of incomplete logic is its semantics. This is an alternative presentation of partial semantics. In this paper, we will introduce the general notion of incomplete logic(ICL), compare it with partial logic, and give the resolution method for it. We will also show how ICL can be applied to nonmonotonic reasoning. We define nonmonotonic derivation as monotonic derivation in ICL from the database and some consistent assumptions. The mechanism of ICL makes it easy to assert the consistency of an assumption without asserting the assumption itself.
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Introduction

Traditional deductive logics are always monotonic: Adding new premises will never invalidate old conclusions, i.e. the set of conclusions increases monotonically with the set of premises. However, in commonsense reasoning, we often draw conclusions that we retract later when more relevant facts are learned. The nonmonotonicity arises from incomplete information. The retracted conclusions are usually those made under incomplete information.

Many nonmonotonic formalisms have been proposed to cope with the problem of reasoning under incomplete information. The most well-known include modal nonmonotonic logics [15, 14, 9], autoepistemic logic[16, 10], default logic

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and its variants[20, 3, 11], circumscription[12, 13], and closed-world assumptions (CWA)[19]. One common feature of these formalisms is that their semantics all depend on complete models, so all classical tautologies are valid conclusions of these nonmonotonic reasoning mechanisms. That is, these formalisms are all conservative extensions of classical logic[1].

However, there have been various alternative frameworks where partiality is introduced to capture some linguistic phenomenon, such as vague predicates, presupposition failure, sortal incorrectness or the possibility of self-reference and the associated semantical paradoxes[1]. Though partial model theory is not aimed at nonmonotonic reasoning, however, since it is a generalization of standard model theory and the partial models may represent partial information about the world. it will be interesting to explore the possibility of applying partial semantics to nonmonotonic reasoning.

In this paper, we propose a kind of partial logic, called incomplete logic (ICL), to achieve the purpose. Traditionally, a partial model can be written as a 3-valued total function from the set of logical sentences to $\{0, \frac{1}{2}, 1\}$, or, equivalently, a partial function to $\{0, 1\}$, where the third truth value "$\frac{1}{2}$" is treated as a truth-value gap. In these models, the truth value of a compound sentence is completely determined by its component sentences, i.e., they are truth-functional. However, in ICL, we adopt an alternative presentation. An ICL-interpretation is still a total function from the set of logical sentences to $\{0, 1\}$. However, it is not truth-functional any more. The truth values of an atomic sentence and its negation are assigned independently with the constraint that at most one of them can be assigned the value 1. Furthermore, the value 0 denotes ignorance of truth instead of falsity now. The semantic shift, though not essential, facilitates the representation of some kind of logical equivalence that simplifies the clausal form of a sentence. We also introduce a novel implication connective that make it easy to formulate default rule in ICL.

In the next section, we will introduce the syntax and semantics of ICL. Then, the similarity and difference between ICL and traditional partial logics are considered. In section 4, we present a resolution method for ICL reasoning. In section 5, ICL is used to formulate nonmonotonic reasoning. Finally, the conclusion is given.

1 Incomplete Logic

In classical logic, the law of excluded middle and the law of contradiction are enforced, so in a complete model, one proposition or its negation is true exclusively. The standard models are adequate for modeling complete and consistent situation of knowledge. This is, however, just an ideal situation because human knowledge is usually incomplete and inconsistent, due to the

[1] Though, there are few exceptional cases. For example, in some Reiter's default theories, extensions don't exist, so it is unclear whether classical tautologies are derivable in this case.

restriction of the language for representation, the limitation of real knowledge, and the reality of our world, etc..

Here, we discard the consideration of inconsistent knowledge and restrict our attention to the reasoning of incomplete knowledge. To model incomplete knowledge, the shift of semantic paradigm to partial logic will be important. However, before presenting the semantics, let us first introduce the language.

The basic vocabulary of ICL language consists of a set of propositional variables, PV, a propositional constant $*$ (complete ignorance), the classical connectives $\neg$ (negation), $\wedge$ (and), $\vee$ (or), and an unary modal operator M. Let $\mathcal{L}_{\mathcal{M}}$ denote the smallest set containing PV and $*$ and being closed under these connectives. As usual, $A \supset B$ is considered as an abbreviation of $\neg A \vee B$. The elements of $\mathcal{L}_{\mathcal{M}}$ are called well-formed formulas (wffs) or sentences of ICL. We will use the capital letters A, B, C(possibly with indices) to denote wffs. We also define an implication connective "$\rightarrow$" such that $A \rightarrow B$ is taken as an abbreviation of $M\neg A \vee B$.

A formula has a modal depth k ($k \geq 0$) iff its modal operators are nested to a depth of k. For example, the modal depth of $M(\neg Mp \vee M(\neg Mp \vee q))$ is 3. A formula A is an objective formula if its modal depth is 0. We will use $\mathcal{L}$ to denote the set of all objective formulas not containing the constant $*$.

1.1 Semantics for Incomplete Logic

If we let a wff A have semantic value 1 when it is known to be true and have value 0 when it is not, then in an incomplete model, it is reasonable that A and $\neg A$ both have the value 0, i.e., the truth or falsity of A is yet unknown. This motivates the following definition of ICL-interpretation. Let p denote elements of PV and $\neg PV = \{\neg p : p \in PV\}$.

Definition 1 *(ICL-interpretation)*

An ICL-interpretation for $\mathcal{L}_{\mathcal{M}}$ is a mapping I from $PV \cup \neg PV$ to $\{0, 1\}$, such that

$\forall p \in PV, I(p) \times I(\neg p) = 0.$

We say that PV is the propositional domain of I.

An ICL-interpretation is extended to the whole domain $\mathcal{L}_{\mathcal{M}}$ in the following way.

Definition 2 *Let I be an ICL-interpretation. The value of a wff A in I, denoted by $I(A)$, is specified by the following rules:*

1. $I(*) = I(\neg *) = 0$
2. $I(A \wedge B) = 1$ *iff* $I(A) = 1$ *and* $I(B) = 1$
3. $I(A \vee B) = 1$ *iff* $I(A) = 1$ *or* $I(B) = 1$
4. $I(\neg(A \wedge B)) = 1$ *iff* $I(\neg A) = 1$ *or* $I(\neg B) = 1$

5. $I(\neg(A \vee B)) = 1$ *iff* $I(\neg A) = 1$ *and* $I(\neg B) = 1$

6. $I(MA) = 1$ *iff* $I(\neg A) = 0$

7. $I(\neg MA) = 1$ *iff* $I(\neg A) = 1$

8. $I(\neg\neg A) = 1$ *iff* $I(A) = 1$

The definition induces some asymmetry between the two values 0 and 1. If a wff A is known to be true, then it has the value 1 under the interpretation, so $\neg A$ is not known to be true (in fact, it is known to be false) and has the value 0. Nevertheless, if A has the value 0, then it is still possible that we do not know the truth of $\neg A$, so $\neg A$ does not necessarily have the value 1. Therefore, in ICL semantics, the two values are given an epistemic interpretation. Consequently, the semantics of disjunction means that $A \vee B$ is known iff A is known or B is known, i.e., the interpretation is intuitionistic. However, ICL is different with intuitionistic logic since it accepts double negation law.

The modal operator M capture the meta-linguistic notion of consistency or unprovability. A formula of the form MA is read "$\neg A$ is not known" or "A is consistent". According to our definition , there are two cases where $I(MA) = 1$: one is $I(\neg A) = 0$ and $I(A) = 1$, i.e., the classical situation, the other is $I(\neg A) = 0$ and $I(A) = 0$, i.e., the ignorance situation. Furthermore, the formula $A \rightarrow B$ means if A is known to be true, so is B under the interpretation.

The following definition is analogous to classical logic.

Definition 3 *Let I be an ICL-interpretation and suppose that S is a set of formulas. I is an ICL-model of S, denoted as $I \models_{ICL} S$ iff $\forall A \in S, I(A) = 1$. We say that S is ICL-unsatisfiable iff S has no ICL-model. And S is ICL-satisfiable iff S has at least one ICL-model. We say that A is ICL-valid iff A takes value 1 for all ICL-interpretations. Furthermore, A is an ICL-consequence of S, denoted by $S \models_{ICL} A$, iff any ICL-model of S is also one of A.*

2 Incomplete Logic and Partial Semantics

Partial semantics can be characterized by a 3-valued logical matrix[2]. The matrix is an algebra

$$\mathcal{V}_3 = \langle\{0, \frac{1}{2}, 1\}, \neg, \wedge, \vee, m\rangle,$$

with the designated value 1, where $\neg$ and m are unary functions and $\wedge$ and $\vee$ are binary ones defined as follows:

1. $\neg(x) = 1 - x$

2. $x \wedge y = \min(x, y)$

3. $x \vee y = \max(x, y)$

4. $m(x) = 1$ if $x \neq 0$ and $m(x) = 0$ if $x = 0$.

A *partial interpretation* is a homomorphism $I_3 : \mathcal{L}_{\mathcal{M}} \to \{0, \frac{1}{2}, 1\}$ with $I_3(*) = \frac{1}{2}$.

Note that the min and max function used above is according to the ordinary numerical ordering of rational numbers. If we define another reflexive *information ordering* on $\{0, \frac{1}{2}, 1\}$ as

$$\sqsubseteq = \{(\frac{1}{2}, 1), (\frac{1}{2}, 0), (0, 0), (1, 1), (\frac{1}{2}, \frac{1}{2})\},$$

then all classical operations $\neg$, $\wedge$ and $\vee$ are monotonic functions with respect to $\sqsubseteq$, whereas m is a nonmonotonic one. It is also a well-known result that the connectives of $\mathcal{L}_{\mathcal{M}}$ are expressively adequate with respect to all 3-valued functions([1], p.41). Recall that $A \to B =_{def} M\neg A \vee B$, then under the partial interpretation, the truth matrix of $\to$ is as follows:

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	1	1	1
1	0	$\frac{1}{2}$	1

,which is exactly the definition of implication in Słupecki's 3-valued logic([2], p.76).

Now, let X denote $\{(0, 1), (0, 0), (1, 0)\}$, and let us overload the above-defined operations on X as follows:

1. $\neg(x, y) = (y, x)$
2. $(x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge y_1, x_2 \vee y_2)$
3. $(x_1, x_2) \vee (y_1, y_2) = (x_1 \vee y_1, x_2 \wedge y_2)$
4. $m(x, y) = (1, 0)$ if $x \geq y$ and $= (0, 1)$, elsewhere.

Then, we have the following lemma.

Lemma 1 *1. The function $f : X \to \{0, \frac{1}{2}, 1\}$ defined as*

$$f(x, y) = \frac{x - y + 1}{2}$$

is an isomorphism from $\mathcal{X} = \langle X, \neg, \wedge, \vee, m\rangle$ to $\mathcal{V}_3$.

2. If I is an ICL-interpretation, then the function $H_I : \mathcal{L}_{\mathcal{M}} \to X$ defined as

$$H_I(A) = (I(A), I(\neg A))$$

is a homomorphism. Conversely, if H is a homomorphism from $\mathcal{L}_{\mathcal{M}}$ to $\mathcal{X}$, then $\pi_1 \circ H$ is an ICL-interpretation, where π_1 extracts the first component of an ordered pair.

According to the lemma, we can obtain the truth-preserving translation between partial and ICL interpretations.

Proposition 1 *Let f and H_I be defined as in the last lemma, and s be defined as $s(x) = 1$ if $x > 0$ and $= 0$ elsewhere.*

1. *If I is an ICL-interpretation, then $I_3 = f \circ H_I$ is a partial interpretation.*
2. *If I_3 is a partial interpretation, then I defined as $I(A) = s(2 \times I_3(A) - 1)$ is an ICL-interpretation.*

Furthermore, for any $A \in \mathcal{L}_{\mathcal{M}}$, $I(A) = 1$ iff $I_3(A) = 1$.

From the proposition, we can give the relation between ICL-satisfiability and logical consequence.

Corollary 1 1. *Let $S \subseteq \mathcal{L}$ and $A \in \mathcal{L}$, then $S \models A$ (A is a classical logical consequence of S) iff $S \cup \{\neg A\}$ is ICL-unsatisfiable.*

2. *Let $S \subseteq \mathcal{L}_{\mathcal{M}}$ and $A \in \mathcal{L}_{\mathcal{M}}$, then $S \models_{ICL} A$ iff $S \cup \{M\neg A\}$ is ICL-unsatisfiable.*

Proof.

1. It is sufficient to prove $S' = S \cup \{A\}$ is ICL-satisfiable iff it is classically satisfiable. The if part is easy since any interpretation is a special case of ICL-interpretations. On the other hand, if S' is satisfied by some ICL-interpretation I, then it is satisfied by a corresponding partial interpretation I_3. Let J be defined as

$$J(p) = \begin{cases} 1, & \text{if } I_3(p) = \frac{1}{2} \\ I_3(p), & \text{otherwise.} \end{cases}$$

Then since classical operations are all monotonic with respect to the information ordering, we have $I_3(B) \sqsubseteq J(B)$ and hence $J(B) = 1$ for all $B \in S'$, so J is an interpretation satisfying S'.

2. The result follows directly from the semantics of ICL.

■

This corollary implies that the resolution method provides in the next section can serve as a basis of both classical and ICL theorem provers. This is a side effect of ICL semantics. From this corollary, we can also have the complexity result of ICL.

Let the ICL-satisfiability problem, ICL-SAT for short, is to determine, given a formula F in $\mathcal{L}_{\mathcal{M}}$, whether it is ICL-satisfiable.

Theorem 1 *ICL-SAT is NP-complete.*

Proof. We need to prove that

1. ICL-SAT $\in$ NP.

2. SAT is polynomially reducible to ICL-SAT.

It is easily seen that (2) follows immediately from the preceding corollary. To prove (1), we present a nondeterministic algorithm for ICL-SAT in Figure 1.

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1. input $F \in \mathcal{L}_{\mathcal{M}}$

2. for each $p \in PV$ occurring in F do

   2.1 choose in a nondeterministic way

   Case1: $F := F|_{p:=0,\neg p:=0}$
   Case2: $F := F|_{p:=1,\neg p:=0}$
   Case3: $F := F|_{p:=0,\neg p:=1}$

   2.2 simplify the resulting formula

   2.3 if the result is 1 then accept and halt

3. halt with rejection

Figure 1: **Nondeterministic Algorithm** ICL-SAT

Let $|F| = n$, it is clear that each step of the algorithm takes time at most a polynomial of n. ■

Thus, technically, ICL is equivalent to partial logic. However, they are quite different in the philosophy of model theory. Let us explain the idea in the following sense.

Definition 4 *Formulas A and B are said to be logically equivalent, written as $A \equiv B$ iff $I(A) = I(B)$ for any ICL-interpretation I.*

Then, obviously, the translations between ICL and partial interpretations do not preserve logical equivalence. In fact, we can prove that it is impossible to have nontrivial equivalence-preserving translations. Let $\mathcal{V} = \langle V, \neg, \wedge, \vee, m\rangle$ be an Ω-algebra with unary operations $\neg$ and m and binary operations $\wedge$ and $\vee$, then an *embedding* of ICL into $\mathcal{V}$ is a homomorphism h from $\mathcal{X}$ to $\mathcal{V}$ such that if $A \equiv B$, then $h \circ H_I(A) = h \circ H_I(B)$, for any ICL-interpretation I. An embedding h is called *trivial* if $h(x) = h(y)$ for all $x, y \in X$.

Proposition 2 *Let $\mathcal{V}$ be any Ω algebra defined as above, then there does not exist any nontrivial embedding of ICL into $\mathcal{V}$.*

Proof. Assume h is an embedding. First, since h is a homomorphism, we have $\neg h((0,0)) = h(\neg(0,0)) = h((0,0))$. Second, it can be shown that in ICL, the following equivalences hold,

$$\neg p \equiv \neg Mp$$

$$\neg\neg Mp \equiv Mp.$$

Let I be any ICL-interpretation such that $I(p) = I(\neg p) = 0$, then $h \circ H_I(\neg Mp) = h((0,1)) = h \circ H_I(\neg p) = h((0,0))$ and $h \circ H_I(Mp) = h((1,0)) = h \circ H_I(\neg\neg Mp) = \neg h \circ H_I(\neg Mp) = \neg h((0,1)) = \neg h((0,0)) = h((0,0))$, so h is trivial. ■

This shows an essential difference between ICL and partial interpretations. Because, in partial interpretations, just as in the classical ones, the truth value 0 denotes falsity, $\neg Mp$ (i.e., "p is inconsistent") is false when the truth value of $\neg p$ is unknown, so $\neg Mp$ and $\neg p$ have unequal values in this case. However, in ICL, the binary values 1 and 0 are given an epistemic interpretation consistently, so $\neg p$ is not known iff $\neg Mp$ is not, either. In summary, a model in classical and partial logics is a reality model, while that in ICL is an epistemic one.

3 Resolution Method for Incomplete Logic

To provide a computational implementation of ICL, we present the resolution method for it in this section. First, as in classical logic, it is necessary to determine the conjunctive normal form of an ICL wff.

Definition 5 *Formulas A and B are said to be classically equivalent, written $A \overset{cl}{\equiv} B$ iff $A \equiv B$ and $\neg A \equiv \neg B$.*

Definition 6 1. *An ICL-literal is p, $\neg p$, Mp, or $M\neg p$, where $p \in PV$.*

2. *A formula is in conjunctive normal form if it is a conjunction of disjunctions of ICL-literals.*

3. *A formula B is a conjunctive normal form of a formula A iff B is in conjunctive normal form and $A \equiv B$.*

Definition 7 *The set of unit formulas, denoted by $\mathcal{U}$, is defined as:*

1. *For any ICL-literal l, $l \in \mathcal{U}$.*

2. *If $F \in \mathcal{U}$ then $MF \in \mathcal{U}$, and $\neg F \in \mathcal{U}$.*

Lemma 2 *Every unit formula A is logically equivalent to an ICL-literal l.*

Proof. For reducing the number of M and $\neg$ of A, it is enough to consider the following five cases:

1. $A = MMA'$

2. $A = \neg MA'$

3. $A = \neg\neg A'$

4. $A = M\neg MA'$

5. $A = M\neg\neg A'$.

where A and A' are unit formulas.

It is easy to check that

$$
\begin{aligned}
&MMA \equiv MA \\
&\neg MA \equiv \neg A \\
&\neg\neg A \equiv A \\
&M\neg MA \equiv \neg A \\
&M\neg\neg A \equiv MA.
\end{aligned}
$$

So we can construct the procedure to reduce the number of M and $\neg$ of A as follows:

1. If A is of the form MMA' then replace A by MA'.
2. If A is of the form $\neg MA'$ then replace A by $\neg A'$.
3. If A is of the form $\neg\neg A'$ then replace A by A'.
4. If A is of the form $M\neg MA'$ then replace A by $\neg A'$.
5. If A is of the form $M\neg\neg A'$ then replace A by MA'.
6. Repeat above replacement step as long as possible.

In every replacement step, the resulting formula is logically equivalent to the initial one. And since in every replacement step, the number of M and $\neg$ decreases, we can reduce A to the form p, $\neg p$, Mp, or $M\neg p$. ■

Theorem 2 *(Replacement Theorem) Suppose $B \overset{cl}{\equiv} C$, and that the formula A' results from the formula A by replacing one or more occurrence of B within A by C, then $A \equiv A'$.*

Proof. We skip the proof because that it is similarly to the proof of replacement theorem in classical logic. ■

Theorem 3 *Every ICL-formula A can be effectively transformed into its logically equivalent conjunctive normal form.*

Proof. The transformation proceeds in three basic steps.

Step 1. Move M and $\neg$ inward. Replace

$M(B \vee C)$ by $MB \vee MC$
$M(B \wedge C)$ by $MB \wedge MC$
$\neg(B \vee C)$ by $\neg B \wedge \neg C$
$\neg(B \wedge C)$ by $\neg B \vee \neg C$
until all occurrences of $\neg$ and M immediately precede unit formulas.

Step 2. Distribute $\wedge$ over $\vee$. In the formula obtained in Step 2, replace as long as possible

$$(B \wedge C) \vee D \text{ by } (B \vee D) \wedge (C \vee D)$$
$$B \vee (C \wedge D) \text{ by } (B \vee C) \wedge (B \vee D)$$

and denote the resulting formula by A'.

Step 3. Transform every unit formula of maximal length within A' to an logically equivalent ICL-literal.

Performing step 1 and step 2 of above algorithm, we construct a sequence of formula, $A_0, A_1, \ldots, A_n$, such that $A_0 = A$, $A_n = A'$, A' is in the form of a conjunction of disjunctions of unit formulas, and, for each $0 \leq i < n$, A_{i+1} is the result of replacing a subformula of A_i by a classically equivalent subformula. Thus, by the Replacement Theorem, $A_i \overset{cl}{\equiv} A_{i+1}$, for each $0 \leq i < n$, and hence, by the transitivity of the equivalent relation, $A_0 \overset{cl}{\equiv} A_n$. Finally, because all maximal length unit formulas occur positively in A_n, the transformation step 3 preserves logical equivalence. ■

Example 1 Consider the formula $A = M(p \wedge \neg(q \wedge Mr)) \vee \neg p$. After step 1, we have $A_1 = (Mp \wedge (M\neg q \vee M\neg Mr)) \vee \neg p$. Next, distributing $\wedge$ over $\vee$, we get $A_2 = (Mp \vee \neg p) \wedge (Mp \vee M\neg q \vee M\neg Mr)$. Finally we transform every unit formula within A_2 to an equivalent ICL-literal as $A_3 = (Mp \vee \neg p) \wedge (Mp \vee M\neg q \vee \neg r)$, that is the desired conjunctive normal form of A. ■

As in classical logic, we could define a disjunction of ICL-literals as a clause.

Definition 8 *An ICL-clause is a formula C of the form $l_1 \vee l_2 \vee \cdots \vee l_n (n \geq 0)$, where $l_1, l_2, \ldots, l_n$ are ICL-literals. When $n = 0$, C is called the empty clause and written as $\Box$.*

It is convenient to write an ICL-clause as a set of ICL-literals.

An ICL-clause set is a set of ICL-clauses, possibly containing the empty clause, possibly empty itself, and possibly infinite. Every ICL-formula in conjunctive normal form corresponds to an ICL-clause set in a natural way, and every finite ICL-clause set not containing the empty ICL-clause and not itself empty corresponds to one or more ICL-formulas in conjunctive normal form. The empty ICL-clause set is to be distinguished from the empty ICL-clause. We use $\emptyset$ to denote the empty ICL-clause set.

As in classical logic, the idea of resolution principle is to extract additional implicit information from a clauses set, beyond what it states explicitly.

Example 2 Consider the ICL-clause set $\{\{p, M\neg q, Mr\}, \{q, \neg t\}\}$. In order for an ICL-interpretation I to verify this clause set, it is necessary that $I(p) = 1$,

$I(M\neg q) = 1$ or $I(Mr) = 1$, as well as $I(q) = 1$ or $I(\neg t) = 1$. Since it is impossible that $I(M\neg q) = 1$ and $I(q) = 1$. we have $I(p) = 1$ or $I(Mr) = 1$, or $I(\neg t) = 1$. It follows that the ICL-clause set $\{\{p, M\neg q, Mr\}, \{q, \neg t\}\}$ is equivalent to $\{\{p, M\neg q, Mr\}, \{q, \neg t\}, \{p, Mr, \neg t\}\}$. Formally, $\{p, Mr, \neg t\}$ is said to be a ICL-resolvent of the other two ICL-clauses. ■

More generally, we have the following definition.

Definition 9 *Let C_1 and C_2 be ICL-clauses and $p \in PV$. Then D is a resolvent of C_1 and C_2 iff*

1. *$D = (C_1 - \{p\}) \cup (C_2 - \{\neg p\})$ for some $p \in C_1$, $\neg p \in C_2$, or*
2. *$D = (C_1 - \{p\}) \cup (C_2 - \{M\neg p\})$ for some $p \in C_1$, $M\neg p \in C_2$, or*
3. *$D = (C_1 - \{\neg p\}) \cup (C_2 - \{Mp\})$ for some $\neg p \in C_1$, $Mp \in C_2$.*

It is noted that two ICL-clauses may have more than one ICL-resolvent; for example, $\{Mp, Mq\}$ and $\{\neg p, \neg q\}$ have the ICL-resolvents $\{Mq, \neg q\}$ and $\{Mp, \neg p\}$. Also, two nonempty ICL-clause may have the empty ICL-clause as an ICL-resolvent.

Lemma 3 *(The Soundness Result) Let S be an ICL-clause set and let D be a resolvent of two ICL-clauses in S.Then S and $S \cup \{D\}$ are equivalent.*

Proof. Let I be an ICL-interpretation appropriate to S. If $I \models_{ICL} S \cup \{D\}$ then clearly $I \models_{ICL} S$. Now suppose $I \models_{ICL} S$, consider the following three cases:

case 1 if $D = (C_1 - \{p\}) \cup (C_2 - \{\neg p\})$,

case 2 if $D = (C_1 - \{p\}) \cup (C_2 - \{M\neg p\})$,

case 3 if $D = (C_1 - \{\neg p\}) \cup (C_2 - \{Mp\})$,

where $p \in PV$, and C_1 and C_2 are clauses in S. Then since it is impossible that $I(p) = I(\neg p) = 1$ or $I(p) = I(M\neg p) = 1$ or $I(\neg p) = I(Mp) = 1$, and it is assumed that $I \models_{ICL} C_1$ and $I \models_{ICL} C_2$, we can derive $I \models_{ICL} D$. ■

By the soundness result, the operation of adding to an ICL-clause set all the resolvents of ICL-clauses in that set results is an equivalent ICL-clause set. So we define,

$$R(S) = S \cup \{D : D \text{ is a resolvent of two ICL-clauses in } S\}.$$

We also have $S \equiv R(S)$. Now let

$$\begin{array}{ll} R^0(S) = S, & \\ R^{i+1}(S) = R(R^i(S)) & \text{for each } i \geq 0, \text{ and let} \\ R^*(S) = \cup\{R^i(S) : i \geq 0\} & \end{array}$$

In other words, $R^*(S)$ is the closure of S under the operation R of adding all resolvents of ICL-clauses already presented.

Theorem 4 *(Resolution Theorem) An ICL-clause set S is ICL-unsatisfiable iff $\Box \in R^*(S)$.*

Proof. ($\Leftarrow$) If $\Box \in R^*(S)$, then S is ICL-unsatisfiable, since $S \equiv R^*(S)$.

($\Rightarrow$) Assume that S is ICL-unsatisfiable. By the Compactness Theorem of ICL in [8], some finite subset of S is ICL-unsatisfiable.

Let PV be ordered as $\{p_i : i < \lambda\}$, where $\sup \lambda$ is its ordinality. For each integer $n > 0$, let $\mathcal{C}_n (n \geq 0)$ be the set of all ICL-clauses that can be constructed using $p_1, p_2, \ldots, p_n$ alone, and $\mathcal{C}_0 = \{\Box\}$. Then there is an $(n \geq 0)$ such that $S \cap \mathcal{C}_n$, and hence $R^*(S) \cap \mathcal{C}_n$, is ICL-unsatisfiable. We prove the following claim by induction on k.

> <u>**Claim.**</u> For each $k = n, n-1, \ldots, 0$, and each ICL-interpretation I on propositional domain $\{p_1, p_2, \ldots, p_k\}$, there is some ICL-clause C in $R^*(S) \cap \mathcal{C}_k$, such that $I \not\models_{ICL} C$.

That is, for every ICL-interpretation I assigns a truth-value to each of the first k propositional variables, there is some ICL-clause C in $R^*(S) \cap \mathcal{C}_k$ that $I \not\models_{ICL} C$. Since in the case $k = 0$, there is only $\Box$ and for every ICL-interpretation I, $I \not\models_{ICL} \Box$. It follows that $\Box \in R^*(S)$.

For $k = n$, the Claim is immediate, since $R^*(S) \cap \mathcal{C}_n$ is ICL-unsatisfiable.

Now Suppose that the Claim holds for $k+1 \leq n$ but fails for k. Then there is some ICL-interpretation I on propositional domain $\{p_1, p_2, \ldots, p_k\}$, such that $I \models_{ICL} R^*(S) \cap \mathcal{C}_k$.

Let I_1, I_2 and I_3 have the same truth-value as I on propositional domain $\{p_1, p_2, \ldots, p_k\}$ but

$I_1 \not\models_{ICL} p_{k+1}$, $I_1 \not\models_{ICL} \neg p_{k+1}$, and
$I_2 \models_{ICL} p_{k+1}$, $I_2 \not\models_{ICL} \neg p_{k+1}$, and
$I_3 \not\models_{ICL} p_{k+1}$, $I_3 \not\models_{ICL} \neg p_{k+1}$.

By the induction hypothesis there are ICL-clauses C_1, C_2, and $C_3 \in R^*(S) \cap \mathcal{C}_{k+1}$ such that $I_1 \not\models_{ICL} C_1$, $I_2 \not\models_{ICL} C_2$ and $I_3 \not\models_{ICL} C_3$. Then it must be that C_1 contains p_{k+1} or $\neg p_{k+1}$, C_2 contains $M\neg p_{k+1}$ or $\neg p_{k+1}$ and C_3 contains Mp_{k+1} or p_{k+1}. (Under any other circumstances $I_1 \models_{ICL} C_1$ or $I_2 \models_{ICL} C_2$ or $I_3 \models_{ICL} C_3$ since $I \models_{ICL} R^*(S) \cap \mathcal{C}_k$.) Consider the general case:

$C_1 = C_1' \cup \{p_{k+1}\} \cup \{\neg p_{k+1}\}$
$C_2 = C_2' \cup \{\neg p_{k+1}\} \cup \{M\neg p_{k+1}\}$
$C_3 = C_3' \cup \{p_{k+1}\} \cup \{Mp_{k+1}\}$ where $C_1', C_2', C_3' \in R^*(S) \cap \mathcal{C}_k$.

Then from C_1 and C_2 we have $D_1 = C_1' \cup C_2' \cup \{\neg p_{k+1}\}$.
From D_1 and C_3 we have $D_2 = C_1' \cup C_2' \cup C_3' \cup \{Mp_{k+1}\}$.
From D_1 and D_2 we have $D = C_1' \cup C_2' \cup C_3'$.
Then $D \in R^*(S) \cap \mathcal{C}_k$: D is in $\mathcal{C}_k$ because all occurrences of p_{k+1}, $\neg p_{k+1}$ or Mp_{k+1} are discarded from C_1 and C_2 in forming D, and $D \in R^*(S)$ because it is an ICL-resolvent of ICL-clauses in $R^*(S)$. Moreover, $I \not\models_{ICL} D$; for if

$I \models_{ICL} D$ and $D = C_1' \cup C_2' \cup C_3'$, then $I \models_{ICL} C_1'$ or $I \models_{ICL} C_2'$ or $I \models_{ICL} C_3'$, and then $I_1 \models_{ICL} C_1$ or $I_2 \models_{ICL} C_2$ or $I_3 \models_{ICL} C_3$. So it contradicts that $I \models_{ICL} R^*(S) \cap \mathcal{C}_k$. This complete the induction and the proof of the theorem. ■

4 Nonmonotonic Reasoning with ICL

To represent knowledge in ICL, we define the notion of knowledge frame. The notion is analogous to level default theory introduced by Brewka[3], which is in turn a generalization of Poole's system for hypothetical reasoning[17]. A *frame* of knowledge is a tuple $\langle \mathcal{T}_1, \cdots, \mathcal{T}_n \rangle$ of subsets of $\mathcal{L}_{\mathcal{M}}$, where each $\mathcal{T}_i$ is a set of possible assumptions. From the general notion, we can define the concept of *preferred subtheories.*

Definition 10 *A set of formulas $\mathcal{S}$ is a preferred subtheory of a frame $\langle \mathcal{T}_1, \cdots, \mathcal{T}_n \rangle$, iff $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n$ such that for all $1 \leq k \leq n$, $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_k$ is a maximal consistent subset of $\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_k$ in ICL. A preferred subtheory is also called an N-extension.*

4.1 Formulating default reasoning

Using the notions of knowledge frames and N-extensions, we can formulate ICL-based default reasoning easily. In classical default logic, a default theory $\Delta = \langle \mathcal{W}, \mathcal{D} \rangle$ consists of a set of defaults $\mathcal{D}$ and a knowledge base $\mathcal{W} \subseteq \mathcal{L}$ which states the facts of the world. A default is written as $\frac{A:B_1, B_2, \ldots, B_n}{C}$, where $A, B_1, \cdots, B_n, C \in \mathcal{L}$. A is called the prerequisite, C the consequent, and B_i's the justifications of the default. The intended meaning of a default is as follows[3, 20]:

If A can be shown and $\neg B_1, \cdots, \neg B_n$ can not be shown, then derive C.

For a given default theory $\Delta = \langle \mathcal{W}, \mathcal{D} \rangle$, we can construct a corresponding frame $\langle \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \rangle$ in the following way.

1. $\mathcal{T}_1 = \mathcal{W}$
2. $\mathcal{T}_2 = \{A \wedge MB_1 \wedge \cdots \wedge MB_n \rightarrow C \mid \frac{A:B_1, B_2, \ldots, B_n}{C} \in \mathcal{D}\}$.
3. $\mathcal{T}_3 = \{MB_1, \cdots, MB_n \mid \frac{A:B_1, B_2, \ldots, B_n}{C} \in \mathcal{D}\}$

Let $Tr(\Delta)$ denote the resultant frame and $Tr(\mathcal{W})$, $Tr_d(\mathcal{D})$, and $Tr_J(\mathcal{D})$ denote $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ respectively.

This shows a translation from classical default logic to ICL frame. However, the translation is not faithful. In fact, due to the difference between the base logics of these two formalisms, the same formula may have different meaning in classical logic and ICL. For example, $p \vee \neg p$ is valid in all classical default theories, while it is in general not derivable from N-extensions. The obvious advantage of ICL frames is that the computing of N-extensions does not resort to the fixed-point definition as in classical default theories. Though the structure of ICL frames is essentially the same as that of level default theory, the difference of base logics results in some interesting properties. We will discuss some of them in the following.

First, in level default theory, a default "Typically, A implies B" is represented as $A \supset B$. However, because $A \supset B$ is equivalent to $\neg B \supset \neg A$, there are some undesirable side effects. An example due to Nic Wilson([3], p.69) is "People on this planet typically do not know about nonmonotonic reasoning". From this, by the classical representation, we will conclude that those who know about nonmonotonic reasoning are typically not people on this planet. This is indeed undesirable. The problem is called contraposition problem. The solution proposed by Poole is to introduce an artificial variable d for each default $\frac{A:B}{B}$, represent the default as $d \supset (A \supset B)$ and add a constraint $\neg B \supset \neg d$ to the theory. However, in the ICL frames, a normal default is represented as $\{A \wedge MB \to B\}$, and since $\{A \wedge MB \to B, \neg B\} \not\vdash_{ICL} \neg A$, so ICL frames solve contraposition problem naturally.

An argument against this type of representation may be that

$$\{A, A \wedge MB \to C, \neg C\} \vdash_{ICL} \neg B.$$

This does not cause any problem if the default is semi-normal, i.e., B is of the form $C \wedge D$. However, for defaults that are not semi-normal, we may consider the following example.

Example 3 Consider the following facts and default.

1. *Typically, a bird can fly unless it has no wings.*
2. *Tweety is a bird.*
3. *Tweety can not fly*

This can be represented as an ICL frame $\langle \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \rangle$, where $\mathcal{T}_1 = \{B, \neg F\}$, $\mathcal{T}_2 = \{B \wedge MW \to F\}$, and $\mathcal{T}_3 = \{MW\}$, and from the only N-extension of this frame, we can conclude that Tweety has no wings. This seems a too strong result at the first glance, however it is in fact reasonable since we do not mention other possible exceptional cases of non-flying birds in our knowledge base. The problem can be clarified further if we replace the clause "unless it has no wings" by "unless it is abnormal" and represent the default by $B \wedge M\neg Ab \to F$ with the Ab predicate proposed by McCarthy[13], then is is very natural to conclude that Tweety is abnormal (or atypical) knowing that it is a non-flying bird. ■

Second, in [3], a prioritized default logic(PDL) is proposed.

Definition 11 *1. Let $\mathcal{W}$ be a set of wffs in $\mathcal{L}$ and $\mathcal{D}_i$ be a set of classical defaults for $1 \le i \le k$, then $\Delta = \langle \mathcal{W}, \mathcal{D}_1, \ldots, \mathcal{D}_k \rangle$ is called a prioritized default theory(PDT).*

2. *$\mathcal{E}$ is a PDT-extension of Δ iff there exist sets of wffs $\mathcal{E}_1, \ldots, \mathcal{E}_k$ such that $\mathcal{E}_i$ is an extension of $\langle \mathcal{E}_{i-1}, \mathcal{D}_i \rangle$ for $1 \leq i \leq k$ and $\mathcal{E} = \mathcal{E}_k$, where for convenience, $\mathcal{E}_0 = \mathcal{W}$.*

It is also shown that if all defaults in a PDT are supernormal, i.e., of the form $\frac{:B}{B}$, then the PDT can be translated into a level default theory. However, it is unclear how to represent a general PDT in a level default theory. Since a general default can be represented as a ICL wff naturally as above. A PDT $\Delta = \langle \mathcal{W}, \mathcal{D}_1, \ldots, \mathcal{D}_k \rangle$ can be represented as an ICL frame $\langle Tr(\mathcal{W}), Tr_d(\mathcal{D}_1), Tr_J(\mathcal{D}_1), \ldots, Tr_d(\mathcal{D}_k), Tr_J(\mathcal{D}_k) \rangle$.

Thus, though there is some essential difference between the base logics, the ICL frames are more close to default theories in spirit than level default theories. In general, the structure of level default theories allow stronger interactions between defaults, however, the use of material implication to modeling default rules produces some undesirable effects. On the other hand, classical default theories do not have the facilities of rule interaction, so some potential conflicts between defaults can not be detected.

Example 4 Consider the following default theory, $\langle \{\neg p \vee \neg q\}, \{\frac{:r}{p}, \frac{:\neg r}{q}\} \rangle$. It has no extensions according to the classical fixed point definition of default reasoning. However, since the justifications of the two defaults are inconsistent, only one of them should be applied, and we expect that two extensions should be produced, one contains $\{p, \neg q\}$ and the other contains $\{q, \neg p\}$. It can be shown that the ICL representation indeed produce the expected results. ■

Example 5 *[3]* The following default theory is given by Poole,

$$\mathcal{W} = \{Broken(Leftarm) \vee Broken(Rightarm)\}$$

$$\mathcal{D} = \{\frac{: Usable(x) \wedge \neg Broken(x)}{Usable(x)}\}.$$

The classical method will produce an extension containing both $Usable(Leftarm)$ and $Usable(Rightarm)$. The ICL representation of the theory is as follows (all wffs are simplified into equivalent clausal forms and the meaning of the propositional symbols is obvious):

1. $\mathcal{T}_1 = \{B_l \vee B_r\}$,
2. $\mathcal{T}_2 = \{\neg U_l \vee B_l \vee U_l, \neg U_r \vee B_r \vee U_r\}$,
3. $\mathcal{T}_3 = \{M\neg B_l, M\neg B_r, MU_l, MU_r\}$.

Obviously, the N-extensions of the ICL frame produce desired results. ■

In fact, we can prove that conflict defaults will not be applied simultaneously in a more precise sense.

Definition 12 *Let* $S, \{A_1, \ldots, A_n\} \subseteq \mathcal{L}_{\mathcal{M}}$, *then we say that* $A_1, \ldots, A_n$ *are strongly inconsistent under* S *iff* $S \vdash_{ICL} \neg A_1 \vee \ldots \vee \neg A_n$.

Definition 13 *Assume* $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \rangle$ *is an ICL frame that is the translation of a classical default theory, and* $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ *is an N-extension of the frame. Let* d *denote a default* $A \wedge MB_1 \wedge \cdots \wedge MB_n \rightarrow C$.

1. *Then the generating defaults of* $\mathcal{S}$, *is defined by*

$$GD(\mathcal{S}, \mathcal{T}) = \{d \in \mathcal{S}_2 : \mathcal{S} \vdash_{ICL} A \text{ and } MB_i \in \mathcal{S}_3, \forall i = 1, \ldots, n\}.$$

2. *Let* $con(d) = \{C\}$, *then the set of consequents of a set of defaults* $\mathcal{D}$ *is defined by*

$$Con(\mathcal{D}) = \bigcup_{d_i \in \mathcal{D}} con(d_i).$$

3. *Let* $just(d) = \{B_i : 1 \leq i \leq n\}$, *then the set of justifications of a set of defaults* $\mathcal{D}$ *is defined by*

$$Just(\mathcal{D}) = \bigcup_{d_i \in \mathcal{D}} just(d_i).$$

Theorem 5 *Let* $\mathcal{T}$ *and* $\mathcal{S}$ *be defined as above, then* $Just(GD(\mathcal{S}, \mathcal{T}))$ *is not strongly inconsistent under* $Th_{ICL}(\mathcal{S}_1 \cup Con(GD(\mathcal{S}, \mathcal{T}))$.

Proof. By the definitions above,

$$Th_{ICL}(\mathcal{S}_1 \cup Con(GD(\mathcal{S}, \mathcal{T})) \subseteq Th_{ICL}(\mathcal{S})$$

and

$$\{MA \mid A \in Just(GD(\mathcal{S}, \mathcal{T}))\} \subseteq \mathcal{S}_3,$$

so if the theorem does not hold, then

$$\mathcal{S} \vdash_{ICL} \bigvee_{MA \in \mathcal{S}_3} \neg A,$$

and $\mathcal{S}$ will be inconsistent by the resolution rule of ICL. ■

The theorem illustrates a main characterization of ICL-based default reasoning. In Reiter's default logic, for each $d \in GD(\mathcal{S}, \mathcal{T})$, $Just(d)$ must be consistent with $\mathcal{S}_1 \cup Con(GD(\mathcal{S}, \mathcal{T}))$, however, the set $Just(GD(\mathcal{S}, \mathcal{T}))$ itself may be inconsistent with $\mathcal{S}_1 \cup Con(GD(\mathcal{S}, \mathcal{T}))$. This is called weak regularity condition in [4]. On the other hand, some variants of default logic(e.g. [7]) require the strong regularity condition, i.e., $\mathcal{S}_1 \cup Con(GD(\mathcal{S}, \mathcal{T})) \cup Just(GD(\mathcal{S}, \mathcal{T}))$ must be consistent. The result of Theorem 5 shows that the requirement of ICL-based default reasoning is close to but weaker than strong regularity condition. Since strong regularity condition requires the joint consistency in the classical sense, two defaults $\frac{p}{q}$ and $\frac{\neg p}{r}$ can not be applied simultaneously even though no information about p is available, while in ICL, the two defaults can be applied simultaneously if $p \vee \neg p$ is not derivable, i.e., if we are ignorant about whether p is true.

To sum up, level default theory improves the classical one in many aspects, in particular, in the handling of interactions between defaults. However, material implication is not very suitable for the representation of defaults, so it calls for a change of the base logic. Here, we propose ICL as the alternative base logic and show that the ICL frames indeed combine the advantages of level and classical default theories.

Finally, we remark that other alternative base logics are also possible. For example, Clarke and Gabbay[6] propose a similar system based on intuitionistic logic(INL). Their system is a kind of two level default theories. The big difference between ICL and INL is that the former accepts the double negation law, whereas the latter rejects it. Thus, it will be interesting to compare their theories and ICL frames. The comparison, though, will be left for the further research.

4.2 Formulating Circumscriptive reasoning

Though it is shown above that ICL based default reasoning has some advantages over the classical one, we sometimes hope that the ignorance of atomic sentences can be minimized. The minimization can be achieved by circumscription.

Suppose p and q are two propositional symbols, then let $p \leq q$ denote $Mp \wedge M\neg p \rightarrow Mq \wedge M\neg q$. Let $\mathbf{p}=(p_1, \ldots, p_n)$ and $\mathbf{q}=(q_1, \ldots, q_n)$ be two tuples of propositional symbols, then $\mathbf{p} \leq \mathbf{q}$ is an abbreviation of $\bigwedge_{1 \leq i \leq n} p_i \leq q_i$. If $A = A(\mathbf{p}) \in \mathcal{L}_{\mathcal{M}}$, where $\mathbf{p}=(p_1, \ldots, p_n)$ is the tuple of propositional symbols occurring in A, then the circumscription of A with respect to ignorance is defined by

$$CI(A) = A \wedge \forall \mathbf{x}(A(\mathbf{x}) \wedge \mathbf{x} \leq \mathbf{p} \rightarrow \mathbf{p} \leq \mathbf{x}).$$

Note $CI(A)$ is not a wff in $\mathcal{L}_{\mathcal{M}}$. In fact, it contains quantifier on propositional variables, so it is a second order incomplete logic wffs. However, its semantics is clear enough so that we can define the models of $CI(A)$.

Let us now consider the minimal model semantics of the circumscription. An ICL-interpretation I is less than or equal to another I', denoted by $I \leq I'$, iff $I'_3(p) \sqsubseteq I_3(p)$, where I_3 and I'_3 are the corresponding partial interpretations of I and I' respectively, and $\sqsubseteq$ is the information ordering defined in section 2. Thus, the more informative an ICL-interpretation is, the more preferred it is. We say that I is a minimal ICL-model of A iff I is a ICL-model of A and for any I' such that $I' < I$, I' is not a ICL-model of A. Then the minimal entailment relation $\models_{ICM}$ may be defined as $A \models_{ICM} B$ iff all minimal ICL-models of A is also ICL-models of B.

Theorem 6 *For all $A, B \in \mathcal{L}_{\mathcal{M}}$, we have $A \models_{ICM} B$ iff $CI(A) \models_{ICL} B$.*

Proof. The proof follows from the definitions just as in classical circumscription. ■

The strategy of defining minimal models are similar to that proposed in [18], where a paraconsistent model is considered and the minimal models are those containing the least contradiction sentences.

In addition to second order ICL, we have also another formulation of circumscribing ignorance based on ICL frames. Let us define a frame $\mathcal{C}(\mathcal{T})$ as

$$\langle \mathcal{T}, \{p \vee \neg p : p \in PV\} \rangle,$$

for any subset $\mathcal{T} \subseteq \mathcal{L}_{\mathcal{M}}$. Then it is easy to show the following theorem.

Theorem 7 *For any ICL-interpretation I, I is a minimal ICL-models of* $\mathcal{T}$ *iff it is a ICL-model of some N-extension of* $\mathcal{C}(\mathcal{T})$.

Example 6 Let us consider the linguistic processing context. If we have a background theory $\mathcal{T}$ which contains the occurrence of the propositional symbol p. Suppose that p denotes the well-known sentence "The present king of France is bald.", and $\mathcal{T}$ provides "France is a nation", "The king of a nation is a person", and "Baldness is a property appropriate for persons", but not the knowledge of whether France has a king now, so the wff $Mp \wedge M\neg p$ is not in our knowledge base, and we can derive $p \vee \neg p$ from $\mathcal{C}(\mathcal{T})$, i.e., p is a normal sentence. However, if it is later known that there does not exist the present king of France, then p is a presupposition failure sentence, so we can add the wff $Mp \wedge M\neg p$ to $\mathcal{T}$, and $p \vee \neg p$ is not derivable any more. ■

Plainly, we can also circumscribe the ordinary predicates, in addition to ignorance, in the ICL frames by adding $\neg p$ to the second level theory for any propositional symbol p that we want to circumscribe. However, because the definition is very straightforward, we will not go through the further details here.

4.3 Formulating autoepistemic reasoning

In addition to formulations of default and circumscriptive reasonings, it is also possible to formulate another important nonmonotonic reasoning mechanism – autoepistemic reasoning in ICL.

The classical autoepistemic logic (AEL) proposed by Moore[16] is to give a formal account of an ideal agent reasoning about his own beliefs. The language is the same as classical modal logic[5], with the modal operator L interpreted as "the agent knows". Given a subset $\mathcal{S}$ of AEL wffs, the AEL expansions are defined as the fixed points of the following operator:

$$\Gamma_1(\mathcal{T}) = Th(\mathcal{S} \cup \{LA : A \in \mathcal{T}\} \cup \{\neg LA : A \notin \mathcal{T}\}),$$

where Th is the classical logical closure operation.

Now, if we consider the operator M in ICL and let LA denote the abbreviation of $\neg M \neg A$, then due to the semantics of ICL, $LA \equiv A$, so for any $\mathcal{S} \subseteq \mathcal{L}_{\mathcal{M}}$, we can define AE expansions of $\mathcal{S}$ in ICL as the fixed points of the following operator:

$$\Gamma_2(\mathcal{T}) = Th_{ICL}(\mathcal{S} \cup \{M\neg A : A \notin \mathcal{T}\}).$$

The definition is analogous to the corresponding one in McDermott and Doyle's nonmonotonic logic (NML)[15] except the difference on the base logics. Since there are some notorious anomalies in NML (see Examples 4.13–4.15 of [11]), it is doubtful whether our mechanism suffers the same defects. Fortunately, these anomalies are avoided by the change of the base logic just as in the INL-based nonmonotonic logic[6]. The reason has been explained in Examples 4.130–4.132 of [11].

The common problem to our mechanism and the INL-based nonmonotonic logic is that the theory $\{Mp \rightarrow p\}$ has an AE expansion including $\neg p$. This seems counter-intuitive at the first glance. However, we think that the problem is due to the misreading of $\{Mp \rightarrow p\}$ as " p is believed by default". In fact, our formulation of default reasoning in section 4.1 illustrates the point. The default $\frac{:p}{p}$ is formulated as a frame $\langle\{Mp \rightarrow p\}, \{Mp\}\rangle$ instead of a single theory $\{Mp \rightarrow p\}$. The correct reading of $\{Mp \rightarrow p\}$ should be "p is believed or $\neg p$ is believed" and no preference is imposed on the two beliefs. However, the frame $\langle\{Mp \rightarrow p\}, \{Mp\}\rangle$ means that we tend to believing p if no contrary information is available.

Interestingly, the AE reasoning in ICL can also be formulated as ICL frames. For $\mathcal{S} \subseteq \mathcal{L}_{\mathcal{M}}$, define an ICL frame

$$\mathcal{A}(\mathcal{S}) = \langle \mathcal{S}, \{MA : A \in \mathcal{L}_{\mathcal{M}}\}\rangle.$$

Then we have the following result.

Theorem 8 *Let $\mathcal{S}$ be a consistent subset of $\mathcal{L}_{\mathcal{M}}$. Then, an ICL theory $\mathcal{T}_1$ is an AE expansion of $\mathcal{S}$ iff there exists an N-extension of $\mathcal{A}(\mathcal{S})$, $\mathcal{T}_2$, such that $\mathcal{T}_1 = Th_{ICL}(\mathcal{T}_2)$.*

Proof.

1. If
$$\mathcal{T}_1 = Th_{ICL}(\mathcal{S} \cup \{M\neg A : A \notin \mathcal{T}_1\}),$$
then $\mathcal{T}_1$ will be ICL-consistent since $\mathcal{S}$ is, so $\mathcal{T}_2 = \mathcal{S} \cup \{MA : MA \in \mathcal{T}_1\}$ is. too. Now, if MA is consistent with $\mathcal{T}_2$, then $\neg A \notin Th_{ICL}(\mathcal{T}_2) = \mathcal{T}_1$, so $M\neg\neg A \in \mathcal{T}_2$. Thus $MA \in \mathcal{T}_1$ and $MA \in \mathcal{T}_2$ Therefore, $\mathcal{T}_2$ is an N-extension of $\mathcal{A}(\mathcal{S})$

2. If $\mathcal{T}_2 = \mathcal{S} \cup \mathcal{S}_1$ is an N-extension of $\mathcal{A}(\mathcal{S})$, where $\mathcal{S}_1$ is a maximal subset of $\{MA : A \in \mathcal{L}_{\mathcal{M}}\}$ such that $\mathcal{T}_2$ is consistent, then we can prove $Th_{ICL}(\mathcal{T}_2) = \mathcal{T}_1 = \Gamma_2(\mathcal{T}_1)$ in the following way.

First, for each $MA \in \mathcal{S}_1$, $\neg A \notin \mathcal{T}_1$ since $\mathcal{T}_2$ is consistent, so there exists an equivalent wff, i.e., $M\neg\neg A$ in $\{M\neg A : A \notin \mathcal{T}_1\}$. Thus,

$$\mathcal{T}_1 = Th_{ICL}(\mathcal{S} \cup \mathcal{S}_1) \subseteq Th_{ICL}(\mathcal{S} \cup \{M\neg A : A \notin \mathcal{T}_1\}) = \Gamma_2(\mathcal{T}_1).$$

Second, if $A \notin \mathcal{T}_1$, then $M\neg A$ is consistent with $\mathcal{T}_1$, so $M\neg A \in \mathcal{S}_1$ by the maximality of N-extensions. Thus $\{M\neg A : A \notin \mathcal{T}_1\} \subseteq \mathcal{S}_1$, and consequently, $\Gamma_2(\mathcal{T}_1) \subseteq \mathcal{T}_1$.

■

According to the definition of $\mathcal{A}(\mathcal{S})$, it is easy to see that contrary to circumscription, AE operation in ICL maximizes the ignorance of a theory.

Conclusion

Recently, how to formalize human's common-sense reasoning has become more and more important in A.I. research. There have been many nonmonotonic reasoning formalisms proposed to modeling reasoning with incomplete information.

In this paper, we attempt to combine a version of partial logic – ICL and hypothetical reasoning framework. We present the relationship between ICL and traditional partial logics and a resolution proof method for ICL. Then, it is shown that three important types of nonmonotonic reasoning can be formulated in ICL. The extension of ICL to first order version and search of other proof methods, e.g., tableau methods, Gentzen sequent calculi, and natural deduction systems, are the first possible further research directions.

Second, we will seek the proof procedures for the nonmonotonic formalisms based on ICL. In particular, when the frames contain infinite subsets of wffs, for example, in the case of AEL reasoning, how to check maximal consistency will be the difficulty to be overcome.

Finally, we will consider frames based on other nonclassical logics. For example, the logics with supervaluation semantics[21] is the next one we would like to investigate.

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