Gentzen Sequent Calculus for Possibilistic Reasoning

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Abstract. Possibilistic logic is an important uncertainty reasoning mechanism based on Zadeh's possibility theory and classical logic. The deduction rules of possibilistic logic have been obtained from classical resolution rule by attaching possibility or necessity weight to ordinary clauses. However, since not all possibility-valued formulae can be converted into equivalent possibilistic clauses, the resolution rules are somewhat restricted. In this paper, we develop Gentzen sequent calculus for possibilistic reasoning to lift this restriction. This is done by first formulating possibilistic reasoning as a kind of modal logic. Then the Gentzen method for modal logics are generalized to it. Finally, some properties of possibilistic logic, such as Craig's interpolation lemma and Beth's definability theorem are also discussed in the context of Gentzen methods.

1 Introduction

Recently, many uncertainty reasoning mechanisms based on Zadeh's possibility theory[14] have grown rapidly. (See [4] for a survey of literature). Among them, possibilistic logic[5] is one of the most important approaches. Dubois and Prade successfully combine possibility theory and classical logic to create this logic and its deduction method. However, since their deduction method is based on the classical resolution rule[12], it suffers from the restriction of clausal form so that the method can not be applied to any well-formed formula. On the other hand, they and some others have pointed out that possibilistic logic can be viewed as a generalization of modal logics[5, 6, 7, 11], so it is possible to exploit the deduction method of modal logics to do possibilistic reasoning. In this paper, we will investigate an implementation of this idea to lift the clausal form restriction of possibilistic resolution. To do this, we first formulate possibilistic reasoning as a kind of modal logic with multi modal operators. Then, we generalize the sequent calculus of normal modal logics developed by Fitting[8] to our logic. Finally, some logical properties of our system is discussed.

1.1 Possibilistic resolution rule

Before going on, we must review the basic notions of possibilistic logic and its resolution rule. However, we can only sketch them briefly, and the readers can find the detail in [4, 5].

In possibilistic logic, two certainty measures, called possibility and necessity measures respectively, are assigned to the well-formed formulas of classical propositional logic. Specifically, let Π and N denote possibility and necessity measure respectively, then the following laws must be satisfied for all well-formed formulas f and g:

(i) $\Pi(f)$ and $N(f) \in [0,1]$
(ii) $N(f) = 1 - \Pi(\neg f)$,
(iii) $\Pi(\top) = 1$ and $\Pi(\bot) = 0$, and
(iv) $N(f \wedge g) = \min(N(f), N(g))$.

Thus, if f is a well-formed formula (wff) of propositional logic, then $(f(Nc))$ and $(f(\Pi c))$ where $c \in [0,1]$ are wffs of possibilistic logic. When f is a classical clause, the wffs are called possibilistic clauses. The intended meanings of the wffs are $N(f) \geq c$ and $\Pi(f) \geq c$ respectively. Then, the resolution rule of possibilistic is defined between two possibilistic clauses as follows:

$$\frac{(f\ w_1) \quad (g\ w_2)}{(R(f,g)\ w_1 * w_2)}$$

where $R(f,g)$ is the classical resolvent of f and g, and $*$ is defined by

$$\begin{aligned}
&(N\ c) * (N\ d) = (N\ min(c,d)) \\
&(N\ c) * (\Pi\ d) = \begin{cases} (\Pi\ d)\ if\ c+d > 1 \\ (\Pi\ 0)\ if\ c+d \leq 1 \end{cases} \\
&(\Pi\ c) * (\Pi\ d) = (\Pi\ 0)
\end{aligned}$$

By the axiom (iv) above, any wff of the form $(f(Nc))$ can be converted into an equivalent set of possibilistic clauses, so if only necessity measure N is concerned, the refutational completeness of this resolution rule can be proved[4]. However, if possibility-valued wffs are involved, then the completeness result does not hold in general. The following simple example will explain this point further.

Example 1. Consider the assumption set $B = \{(p \wedge q(\Pi 0.7)), (p \wedge q \supset r\ (N\ 0.6))\}$ and the wff $f = (r\ (\Pi\ 0.7))$ in possibilistic logic. Then obviously, f should be derivable from B.[3] However, since $(p \wedge q\ (\Pi\ 0.7))$ is not a possibilistic clause, the resolution rule can not be applied. A possible approach to overcome the difficulty may be to infer $(p\ (\Pi\ 0.7))$ and $(q\ (\Pi\ 0.7))$ firstly. This indeed improves on some situations if the second sentence is either $(p \supset r\ (N\ 0.6))$ or $(q \supset r\ (N\ 0.6))$, but it is useless for the present example

To lift the restriction of clausal form, we must consider more general deduction methods. Gentzen sequent calculus is one possible choice. It is first developed by Gentzen for classical and intuitionistic logic[10], and then extended to modal logics by Fitting[8]. We will try to find the suitable Gentzen system for possibilistic reasoning in the following sections. But before doing this, we must reformulate possibilistic logic as a general modal logic.

[3] Replacing "$p \wedge q$" with a new propositional letter and using the given resolution rule will justify this.

2 Quantitative Modal Logic

Possibilistic reasoning is first formulated as modal logic in [7]. Then in [11], we call the resultant logic quantitative modal logic (QML) and thoroughly investigate the axiomatic systems for it. Here, for the presentation of the deduction method, we will only review its syntax and semantics briefly.

2.1 Syntax

QML is an extension of propositional logic with four classes of quantitative modal operators:$\langle c\rangle$, $\langle c\rangle^+$, $[c]$ and $[c]^+$ for all $c \in [0,1]$.

In addition to the syntactic rules for propositional logic, the following one is added to the formation rules of QML's wffs.

- if f is a wff, so are $\langle c\rangle f$, $\langle c\rangle^+ f$, $[c]f$ and $[c]^+ f$ for all $c \in [0,1]$.

The intended meaning of $\langle c\rangle f$, $\langle c\rangle^+ f$, $[c]f$ and $[c]^+ f$ is $\Pi(f) \geq c$, $\Pi(f) > c$, $N(f) \geq c$ and $N(f) > c$ respectively.

We usually use lower case letters (sometimes with indices) p, q, r to denote atomic formulas and f, g, h to denote wffs. We also assume all classical logical connectives are available, either as primitives or as abbreviations.

Note that we have enhanced the expressive power of possibilistic logic so that we can represent more complex wffs in QML. For example, we can represent higher order uncertainty in a wff like $\neg[0.7]^+(p \supset \langle 0.8\rangle q)$. The importance of higher order uncertainty is highlighted in [9]. Furthermore, the syntax make it easier to combine QML and other intensional logic. For instance, we can use $B\langle 0.6\rangle p$ and $\langle 0.6\rangle Bp$ to represent "the agent believes that the possibility of p is at least 0.6" and "the possibility of the agent believing p is at least 0.6" respectively.

Following the idea in [8], we may classify the non-literal wffs of QML into six categories according to the formula's main connective which is used to combine its immediate subformulas. We list the classification in Figure 1. We will abuse the notation and use $\alpha, \alpha_1, \cdots$, etc, to deonte wffs of the respective types and their immediate subformulae

2.2 Semantics

We now turn to the semantics of QML. Define a *possibility frame* $F = \langle W, R\rangle$, where W is a set of possible words and $R : W^2 \rightarrow [0,1]$ is a fuzzy accessibility relation on W. Let PV and FA denote the set of all propositional variables and the set of all wffs respectively. Then a model of QML is a triple $M = \langle W, R, TA\rangle$, where $\langle W, R\rangle$ is a possibility frame and $TA : W \times PV \rightarrow \{0,1\}$ is a truth value assignment for all worlds. A proposition p is said to be true at a world w iff $TA(w,p) = 1$. Given R, we can define a possibility distribution R_w for each $w \in W$ such that $R_w(s) = R(w,s)$ for all s in W. Similarly, we can also define TA_w for each $w \in W$ such that $TA_w(p) = TA(w,p)$ for all p in PV. Thus,

α wffs and their component formulas

α	α_1	α_2
$f \wedge g$	f	g
$\neg(f \vee g)$	$\neg f$	$\neg g$
$\neg(f \supset g)$	f	$\neg g$
$\neg\neg f$	f	f

β wffs and their component formalas

β	β_1	β_2
$f \vee g$	f	g
$\neg(f \wedge g)$	$\neg f$	$\neg g$
$f \supset g$	$\neg f$	g

ν and ν^+ wffs (with parameter c) and their component formulas

$\nu(c)$	$\nu_0(c)$	$\nu^+(c)$	$\nu_0^+(c)$
$[c]f$	f	$[c]^+f$	f
$\neg\langle 1-c\rangle^+ f$	$\neg f$	$\neg\langle 1-c\rangle f$	$\neg f$

π and π^+ wffs (with parameter c) and their component formulas

$\pi(c)$	$\pi_0(c)$	$\pi^+(c)$	$\pi_0^+(c)$
$\langle c\rangle f$	f	$\langle c\rangle^+ f$	f
$\neg[1-c]^+ f$	$\neg f$	$\neg[1-c]f$	$\neg f$

Fig. 1. classification of QML non-literal wffs

mathematically, a model can be equivalently written as $\langle W, \langle R_w, TA_w\rangle_{w\in W}\rangle$. Given a model $M = \langle W, R, TA\rangle$, we can define the satisfaction relation $\models_M \subseteq W \times FA$ as follows. First, let us define $N_w(f) = \inf\{1 - R(w,s) \mid s \models_M \neg f, s \in W\}$. Consequently, N_w is just the necessity measure induced by the possibility distribution R_w as defined by Dubois and Prade[5]. Then, $w \models_M [c]f$ (resp. $w \models_M [c]^+f$) iff $N_w(f) \geq c$ (resp. $> c$). The satisfaction relations for $\langle c\rangle f$ and $\langle c\rangle^+ f$ are defined analogously by replacing $N_w(f)$ with $\Pi_w(f) = 1 - N_w(\neg f)$. Here, for convenience, we define $\sup\emptyset = 0$ and $\inf\emptyset = 1$. Furthermore, the satisfaction of all other wffs are defined as usual in classical logic.

A wff f is said to be valid in $M = \langle W, R, TA\rangle$, writen $\models_M f$ iff for all $w \in W, w \models_M f$. If S is a set of wffs, then $\models_M S$ means that for all $f \in S$, $\models_M f$. If $\mathbf{C}$ is a class of models and S is a set of of wffs, then we write $S \models_{\mathbf{C}} f$ to mean that for all $M \in \mathbf{C}$, $\models_M S$ implies $\models_M f$. Note that "$\models_{\mathbf{C}}$" is defined "model by model". It can also be defined "world by world", i.e. for all $M \in \mathbf{C}$ and w of M, $w \models_M S$ implies $w \models_M f$. However, when S is finite, the alternative definition is equivalent to $\models_{\mathbf{C}} \bigwedge S \supset f$, so we would not consider it seperately.

A model is called serial iff $\forall w \in W, \sup_{s\in W} R(w,s) = 1$. As pointed out in [11], the consequence relation in serial models exactly reflects the required possibilistic reasoning, so in what follows, we will mainly focus on the development of deduction rules for it, and $S \models_{\mathbf{C}} f$ is abbreviated as $S \models f$ if $\mathbf{C}$ is the class of all serial models.

3 Gentzen Sequent Calculus

The Gentzen sequent calculus for possibilistic reasoning is a generalization of that for modal logics[8], and is an extension of classical logic. Therefore the system will contain all rules for classical logic. The principle of additional rules for possibilistic reasoning depends on the truth of wffs in different possible worlds. For instance, if $M = \langle W, R, TA \rangle$ is a model, and S is a set of wffs true in a world $w \in W$, then for all worlds w' such that $R(w,w') \geq c$, what wffs should be true in w'? Obviously, any wff f such that $N_w(f) > 1-c$ or $\Pi_w(\neg f) < c$ should satisfy the requirement. Thus, in general, we have the following definition.

Definition 1. Define the world-alternative function K and strict world-alternative function K^+ as follows.
$K, K^+ : 2^{FA} \times [0,1] \rightarrow 2^{FA}$,

$$K(S,c) = \{\nu_0(c') \mid c' \geq c, \nu(c') \in S\} \cup \{\nu_0^+(c') \mid c' \geq c, \nu^+(c') \in S\}$$

and

$$K^+(S,c) = \{\nu_0(c') \mid c' > c, \nu(c') \in S\} \cup \{\nu_0^+(c') \mid c' \geq c, \nu^+(c') \in S\}.$$

Intuitively, $K(S,c)$ (resp. $K^+(S,c)$) contains all wffs f such that $N(f) \geq c$ (resp. $N(f) > c$) is derivable from S by inequality constraints. Thus, it is readily verified that if S is true in w and $R(w,w') \geq c$ (resp. $> c$), then $K^+(S, 1-c)$ (resp. $K(S,1-c)$) is true in w'.

Dually, we have the following definition for the falsity of a set S of wffs in a world w.

Definition 2. Define the dual world-alternative function $\hat{K}$ and strict dual world alternative function $\hat{K}^+$ as follows.
$\hat{K}, \hat{K}^+ : 2^{FA} \times [0,1] \rightarrow 2^{FA}$,

$$\hat{K}(S,c) = \{\pi_0(c') \mid c' \leq 1-c, \pi(c') \in S\} \cup \{\pi_0^+(c') \mid c' \leq 1-c, \pi^+(c') \in S\}$$

and

$$\hat{K}^+(S,c) = \{\pi_0(c') \mid c' \leq 1-c, \pi(c') \in S\} \cup \{\pi_0^+(c') \mid c' < 1-c, \pi^+(c') \in S\}.$$

Similarly, we can verify that if S is false in w and $R(w,w') \geq c$ (resp. $> c$), then $\hat{K}^+(S,1-c)$ (resp. $\hat{K}(S,1-c)$) is false in w'

We are now ready to present the Gentzen deduction system for possibilistic reasoning. First, let us introduce the notation of sequent. Let U, V be two sets of wffs, we use $U \longrightarrow V$ to denote a sequent. The intuitive meaning of the sequent $U \longrightarrow V$ is that if all elements of U are true, then at least one of the wffs in V is true.

Using the notations for the classification of the wffs, we can simplify the presentation of our proof methods. Therefore, we present the Gentzen sequent calculus for QML in Figure 2 by using these standard notations, where we use "U, f" to denote the union of a set of wffs U and a wff f. Every rule consists of one or two upper sequents called *premises* and of a lower sequent called the *conclusion*. Furthermore, the schema f appearing in the axioms mean any literals instead of any wffs.

1. Axioms:

$$U, f \longrightarrow V, f \quad U, \bot \longrightarrow V \quad U \longrightarrow V, \top \quad U \longrightarrow V, \pi(0) \quad U, f, \neg f \longrightarrow V$$
$$U, \nu^+(1) \longrightarrow V \quad U \longrightarrow V, \nu(0) \quad U \longrightarrow V, f, \neg f \quad U \longrightarrow V, \neg\bot \quad U, \pi^+(1) \longrightarrow V$$
$$U, \neg\top \longrightarrow V$$

2. Classical rules:

$$\alpha L: \ \frac{U, \alpha_1, \alpha_2 \longrightarrow V}{U, \alpha \longrightarrow V} \qquad \beta R: \ \frac{U \longrightarrow V, \beta_1, \beta_2}{U \longrightarrow V, \beta}$$

$$\beta L: \ \frac{U, \beta_1 \longrightarrow V \quad U, \beta_2 \longrightarrow V}{U, \beta \longrightarrow V} \qquad \alpha R: \ \frac{U \longrightarrow V, \alpha_1 \quad U \longrightarrow V, \alpha_2}{U \longrightarrow V, \alpha}$$

$$\neg^2 L: \ \frac{U, \neg\neg f \longrightarrow V}{U, f \longrightarrow V} \qquad \neg^2 R: \ \frac{U \longrightarrow V, \neg\neg f}{U \longrightarrow V, f}$$

3. Common rules for possibilistic reasoning:

$$\pi L_1: \text{if } c > 0 \quad \frac{K^+(U, 1-c), \pi_0(c) \longrightarrow \hat{K}^+(V, 1-c)}{U, \pi(c) \longrightarrow V}$$

$$\pi L_2: \quad \frac{K(U, 1-c), \pi_0^+(c) \longrightarrow \hat{K}(V, 1-c)}{U, \pi^+(c) \longrightarrow V}$$

$$\nu R_1: \text{if } c < 1 \quad \frac{K^+(U, c) \longrightarrow \hat{K}^+(V, c), \nu_0^+(c)}{U \longrightarrow V, \nu^+(c)}$$

$$\nu R_2: \quad \frac{K(U, c) \longrightarrow \hat{K}(V, c), \nu_0(c)}{U \longrightarrow V, \nu(c)}$$

4. Rule D (for the serality of models):

$$\frac{K^+(U, 0) \longrightarrow \hat{K}^+(V, 0)}{U \longrightarrow V}$$

Fig. 2. Gentzen Sequent Calculus for QML

We define the *global assumption rule* with a set of wffs B as follows: if $f \in B$, then

$$\frac{U \longrightarrow V, \neg f}{U \longrightarrow V} \qquad\qquad \frac{U, f \longrightarrow V}{U \longrightarrow V}$$

3.1 Soundness and Completeness

Next, we will consider the soundness and completeness of Gentzen-type proof methods. First, we define the truth of a sequent $U \longrightarrow V$ in a world. Let $M = \langle W, R, TA \rangle$ be a model and $w \in W$, then $w \models_M U \longrightarrow V$ iff $w \models_M U$ implies there is at least a wff f in V such that $w \models_M f$. Then the validity of a sequent in a model and in a class of models is defined as that of a wff, and the notations $\models_M U \longrightarrow V$, $\models U \longrightarrow V$, and $B \models U \longrightarrow V$ are also used for sequents, where B is a set of wffs. We also use the convienience that $\longrightarrow V$ and $U \longrightarrow$ denote $\emptyset \longrightarrow V$ and $U \longrightarrow \emptyset$ respectively. A proof of a sequent $U \longrightarrow V$ from a set of wffs B is a sequence of sequents $\varphi_1, \varphi_2, \cdots, \varphi_n$ such that for each i, either φ_i is the instance of some axiom of the Gentzen system , or φ_i is the result of applying the rules in Figure 2 or the global assumption rule with B to the earlier sequents of this sequence. We write $B \vdash (U \longrightarrow V)$ to mean that there exists a proof of $U \longrightarrow V$ from B.

To give the soundness theorem of the Gentzen system, we need another definition. A sequent $U \longrightarrow V$ is called *short* iff $U \cup V$ is finite. We note that a short sequent $U \longrightarrow V$ is true in a world iff the wff "$\bigwedge U \supset \bigvee V$" is true in that world, where $\bigwedge U$ and $\bigvee V$ denote the conjunction of U and the disjunction of V respectively.

Theorem 3 Soundness. *If B is a finite set of wffs and $U \longrightarrow V$ is a short sequent, then $B \vdash (U \longrightarrow V)$ implies $B \models (U \longrightarrow V)$*

Proof. First, all axioms are valid. Secondly, we need to show that if the premise(s) is valid, then the conclusion, too. We consider πL_1 as an example. If its conclusion is false in some world w, then $w \models \pi(c)$. This guarantees the existence of another world w' such that $R(w, w') \geq c$ if the set of possible worlds W is finite[4], so by the remarks after Definition 1 and 2, the premise is false in w'.

Usually, we want to prove only a wff f from a finite global assumption set B, so it is useful to state a restricted version of the soundness theorem if we identify a wff f with a sequent $\longrightarrow \{f\}$.

Corollary 4. *$B \vdash f$ implies $B \models f$*

As for the completeness of Gentzen, it can be proven by the model existence theorem for QML[11]. However, since we use unsigned formulae contrast to the signed version of Fitting's methods[8], some tedious steps are needed. In particular, the rules $\neg^2 L$ and $\neg^2 R$ are needed for completeness. We believe that

[4] By a corollary in [11], i.e., the finite model property, we can assume that W is finite without loss of generality

the two rules can be removed without influencing completeness by using signed formulae.

Theorem 5 Completeness. *1. Let $U \longrightarrow V$ be a short sequent, then $B \models U \longrightarrow V$ implies $B \vdash U \longrightarrow V$.*
2. $B \models f$ iff $B \vdash f$

Now, we can return to Example 1 to see how Gentzen systems can derive the expected result. The deduction is shown in Figure 3, and the rules used from above are βL, πL_1 with $c = 0.7$, and global assumption rules twice respectively. Since right one of the two topmost sequents is an instance of axioms, and the other one can be derived from axioms by classical rules, this is indeed a proof in Gentzen system.

$$
\begin{array}{c}
\neg(p \wedge q), p \wedge q \longrightarrow r \qquad\qquad\qquad\qquad r, p \wedge q \longrightarrow r \\
\hline
p \wedge q \supset r, p \wedge q \longrightarrow r \\
\hline
[0.6](p \wedge q \supset r), \langle 0.7\rangle(p \wedge q) \longrightarrow \langle 0.7\rangle r \\
\hline
[0.6](p \wedge q \supset r) \longrightarrow \langle 0.7\rangle r \\
\hline
\langle 0.7\rangle r
\end{array}
$$

Fig. 3. A Gentzen deduction of Example 1

4 Interpolation Lemma and Definibility Theorem

In the sequel, we will explore some logical properties of QML by exploiting the soundness and completeness of the Gentzen system.

First, we discuss the interpolation lemma. The interpolation lemma of calssical logic is due to Craig[2, 3]. Here, it can be generalized to QML.

Definition 6. If $U \longrightarrow V$ is a sequent, then f is a interpolation formula for $U \longrightarrow V$ iff

(i) all propositional variables of f are common to U and V. In other word, $Pvar(f) \subseteq Pvar(U) \cap Pvar(V)$, where $Pvar(S) = \bigcup_{g \in S} Pvar(g)$ and $Pvar(g)$ is all propositional variables occurring in g for any wff g and set of wffs S.
(ii) $\models U \longrightarrow f$ and $\models f \longrightarrow V$.

We define an interpolation formula for $f \supset g$ as one for $f \longrightarrow g$, where f and g are two wffs. We will essentially follow Fitting's method[8] to prove the interpoaltion lemma.

Lemma 7 Craig's Interpolation Lemma. *If $\models f \supset g$, then $f \supset g$ has an interpolation formula.*

Proof. By the completeness of Gentzen system, $\models f \supset g$ implies $\models f \longrightarrow g$, and in turn implies $\vdash f \longrightarrow g$. we will try to show that

1. all axioms of Gentzen system has interpolation lemma, and
2. for all rules in Gentzen system, if the premise(s) has interpolation formula, then the conclusion has, too.

Then, $f \longrightarrow g$ has an interpolation formula, and it is also the interpolation formula of $f \supset g$ by definition.

We use $U \xrightarrow{h} V$ to denote that h is an interpolation formula of $U \longrightarrow V$.

First, we show that all axioms have interpolation formulas by the following list.

$$U, f \xrightarrow{f} V, f \quad U, \perp \xrightarrow{\perp} V \quad U \xrightarrow{\top} V, \top \quad U \xrightarrow{\top} V, \pi(0) \; U, f, \neg f \xrightarrow{\perp} V$$
$$U, \nu^+(1) \xrightarrow{\perp} V \; U \xrightarrow{\top} V, \nu(0) \; U \xrightarrow{\top} V, f, \neg f \; U \xrightarrow{\top} V, \neg\perp \; U, \pi^+(1) \xrightarrow{\perp} V$$
$$U, \neg\top \xrightarrow{\perp} V$$

Second, we check that each sequent rule transform the interpolation formula(s) of the premise sequent(s) into an interpolation formula of the conclusion sequent.

1. if $U, \alpha_1, \alpha_2 \xrightarrow{h} V$, then $U, \alpha \xrightarrow{h} V$.
2. if $U \xrightarrow{h} V, \beta_1, \beta_2$, then $U \xrightarrow{h} V, \beta$.
3. if $U \xrightarrow{h_1} V, \alpha_1$ and $U \xrightarrow{h_2} V, \alpha_2$, then $U \xrightarrow{h_1 \wedge h_2} V, \alpha$.
4. if $U, \beta_1 \xrightarrow{h_1} V$ and $U, \beta_2 \xrightarrow{h_2} V$, then $U, \beta \xrightarrow{h_1 \vee h_2} V$.
5. if $U, \neg\neg f \xrightarrow{h} V$, then $U, f \xrightarrow{h} V$.
6. if $U \xrightarrow{h} V, \neg\neg f$, then $U \xrightarrow{h} V, f$.
7. if $K^+(U, 1-c), \pi_0(c) \xrightarrow{h} \hat{K}^+(V, 1-c)$, then $U, \pi(c) \xrightarrow{\langle c\rangle h} V$.
8. if $K(U, 1-c), \pi_0^+(c) \xrightarrow{h} \hat{K}(V, 1-c)$, then $U, \pi^+(c) \xrightarrow{\langle c\rangle^+ h} V$.
9. if $K^+(U, c) \xrightarrow{h} \hat{K}^+(V, c), \nu_0^+(c)$, then $U \xrightarrow{[c]^+ h} V, \nu^+(c)$.
10. if $K(U, c) \xrightarrow{h} \hat{K}(V, c), \nu_0(c)$, then $U \xrightarrow{[c]h} V, \nu(c)$.
11. if $K^+(U, 0) \xrightarrow{h} \hat{K}^+(V, 0)$, then $U \xrightarrow{[0]^+ h} V$ and $U \xrightarrow{\langle 1\rangle h} V$.

Next, we will consider the QML version of Beth's definability theorem which is proposed in [1] for classical logic and generalized by Fitting[8] for modal logic.

Definition 8. 1. Let p be a propositional variable and f(p) be a wff containing p. We say there is an explicit definition of p from f(p) in the QML if there is a formula g, in which p does not occur, such that $\models f(p) \supset (p \equiv g)$
2. f(p) is said to define p implicitly in the QML if for each propositional variable q that does not occur in f(p), $\models f(p) \wedge f(q) \supset (p \equiv q)$

Theorem 9 Beth's Definability Theorem. *If f(p) defines p implicitly in QML, then there is also an explicit definition of p from f(p) in QML.*

Proof. By using Craig's interpolation lemma of QML, the proof is same as that for modal logics[8].

5 Conclusion

We have accomplished two points in this paper. First, possibilistic reasoning is reformulated as a kind of modal logic so that the expressive power is enhanced. In the resultant language, QML, we can represent both high-order uncertainty and the interaction between weighted wffs and classical wffs. Secondly, we develop Gentzen deduction method for QML so that the restriction to clausal form of possibilistic resolution is lifted. In summary, we have a sound and complete deduction system for general possibilistic reasoning as long as the global assumption set is finite. Since the problem encountered in the real world is usually finite, the system reported in this paper should be practical. The only remained problem is the efficiency of the method. It is generally believed that resolution is a more efficient deduction method in automated theorem proving. However, we believe, by careful control strategies, Gentzen system can also have good performance. The method developed by Wallen[13] is a good example. We are now considering the application of his method to our system.

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