On Rough Quantifiers

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Abstract

Rough set theory is proposed as the theoretical foundation of information systems and has found rapidly growing applications in intelligent data analysis. An especially interesting problem is its relationship with other knowledge representation formalisms. The most well-known result is that it can provide a semantics for modal logics. Motivated by the strong analogy between quantifiers and modal operators, we would like to consider the generalization of universal and existential quantifiers in first order logic based on rough set notions. By viewing the set of variable assignments as the universe of the approximation space, we can present some quantificational logics based on rough set theory in a uniform way.
Keywords: Rough set, first order logic, rough quantifiers, generalized quantifiers, probability quantifiers, neighbourhood systems.

1 Introduction

The rough set theory is invented by Pawlak[Pawlak, 1982; 1991] to account for the definability of a concept in terms of some elementary ones in an approximation space. It captures and formalizes the basic phenomenon of information granulation. The finer the granulation is, the more concepts are definable in it. For those concepts not definable in an approximation space, the lower and upper approximations for them can be defined. These approximations construct a representation of the given concept in the approximation space. Pawlak claims that knowledge is deep-seated in the classificatory abilities of human beings and other species, so rough set theory is a framework for discussions about knowledge, in particular when imprecise knowledge is of primary concern([Pawlak, 1991], p.2). Thus the theory is in particular effective in extracting knowledge from data tables and it has been successfully and widely applied to domains such as intelligent data analysis (data mining and knowledge discovery in database), decision making, machine learning, pattern recognition and conflict analysis[Lin and Cercone, 1997]. An important common feature of these applications is the classification of objects and the representation of the classificatory knowledge in rough set notions.

On the other hand, philosophers and logicians have spent centuries of time in developing formal tools for the representation and reasoning of knowledge. The results are various of classical and nonclassical logics [Gabbay and Guenthner, 1983]. In particular, first order logic[Smullyan, 1968] and modal logics[Chellas, 1980] have been widely used as a formalism for knowledge representation in artificial intelligence and an analysis tool in computer science[Gabbay *et al.*, 1994; Abramsky *et al.*, 1992].

Since set theory and logic systems are strongly coupled in the development of modern logic, it is expected that the rough set-based knowledge representation framework and the logic-based one have close relationship. This expectation is verified almost from the very beginning stage of the invention of rough set theory. The most well-known relationship between rough set theory and logics is the connection of approximation space with possible world semantics for the modal epistemic logic $S5$. The connection is so obvious that it is difficult to point out who is the first one discovering it. The relationship between more general rough set model and modal logics have also been examined in [Yao and Lin, 1996; Yao, 1996; Yao *et al.*, 1997].

The analogy between quantifiers and modal operators has been noticed by Montague in 1960[Montague, 1960], and recently, the relationship is further investigated in detail[Benthem, 1997]. This suggests that the links between modal logics and rough set theory can also be carried through the quantification logics. In this paper, we would like to examine the results of applying rough set notions to quantifiers.

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In fact, different logical systems incorporating rough quantifiers and approximation operators have been developed [Krynicki, 1989; 1990; Szczerba, 1987; Bolc and Borowik, 1992; Rasiowa and Skowron, 1984; 1985]. The common semantic intuition behind these logics is to view the first order interpretation as an approximation space. These results all show the cross fertilization between rough set theory and logics, so it is worthwhile to investigate the further relationship between different generalizations of rough set models and logic systems. Based on this background, the purpose of this paper is twofold. The first is to survey and present the current results in a uniform way. The second is to fill some missing links between the existing general rough set models and logical formalisms and develop further some applied logics inspired by rough set theory.

In what follows, we will first review the rough set theory and some of its main generalizations. Then different logics of generalized quantifiers based on rough set theory are discussed. Finally, some concluding remarks are given.

2 Review of Rough Set Theory

2.1 Pawlak approximation space and generalizations

Let U be a set of objects (the universe)[1] and R be an equivalence relation on U, then for any $X \subseteq U$, we can associate two subsets with X,

$$\underline{R}X = \{x \in U \mid [x]_R \subseteq X\}$$

$$\overline{R}X = \{x \in U \mid [x]_R \cap X \neq \emptyset\},$$

where $[x]_R$ denotes the equivalence class containing x. $\underline{R}X$ and $\overline{R}X$ are called the R-lower and R-upper approximation of X respectively. From a practical viewpoint, R can be considered as an indiscernibility relation, so for a given concept X, we can only know that X contains at least all elements in $\underline{R}X$ and does not contain any element outside $\overline{R}X$. The pair $(\underline{R}X, \overline{R}X)$ is called the rough approximation of X and any such pair is called a rough set. The pair (U, R) defined as above is thus called a Pawlak approximation space(PAS).

For a given approximation space (U, R), the equivalence classes of R and the empty set are called the *elementary sets,* and the union of some elementary sets is called a *composed set.* The composed sets are those definable in the space. Note that both lower and upper approximations of a concept are composed sets, so they can be seen as the definable approximations of the concept.

A direct generalization of the above-mentioned idea is to relax the constraints on R. To allow R to be an arbitrary binary relation, we can get different useful generalizations of the Pawlak rough set model. For example, the case where R is a tolerance (reflexive and symmetry) relation has been considered in [Polkowski *et al.*, 1995]. To distinguish the Pawlak approximation space and the generalized one, we will refer the latter as relational approximation space(RAS). When (U, R) is an RAS, the lower and upper approximations of a set X are modified as

$$\underline{R}X = \{x \in U \mid R(x) \subseteq X\}$$

$$\overline{R}X = \{x \in U \mid R(x) \cap X \neq \emptyset\},$$

where $R(x) = \{y \in U \mid (x, y) \in R\}$.

Even further generalization of RAS is possible. The most well-known one is the neighborhood systems proposed by Lin[Lin and Yao, 1996]. A neighborhood system(NS) is a pair (U, N), where $N : U \rightarrow 2^{2^U}$ satisfies the following constraints:

1. $\emptyset \notin N(x)$ for all $x \in U$,
2. for all $x \in U$ and $X \subseteq Y \subseteq U$, if $X \in N(x)$, then $Y \in N(x)$.

The universe U is open if $U \in N(x)$ for all $x \in U$, or equivalently, $N(x) \neq \emptyset$ for all $x \in U$. The lower and upper approximations of a set X in an NS (U, N) is based on the definition of interior and closure in topology[Sierpenski and Krieger, 1956].

$$\underline{N}X = \{x \in U \mid \exists Y \in N(x), Y \subseteq X\}$$

$$\overline{N}X = \{x \in U \mid \forall Y \in N(x), Y \cap X \neq \emptyset\}.$$

Given an RAS (U, R), we can define an NS (U, N) by

$$N(x) = \{S \subseteq U \mid R(x) \subseteq S\},$$

so the latter is indeed a generalization of the former.

2.2 Probabilistic approximation space

For a PAS or RAS (U, R), and $X \subseteq U$, the accuracy of the approximation of X is defined by[Pawlak, 1982]

$$\rho(X) = \frac{|\underline{R}X|}{|\overline{R}X|}.$$

Furthermore, the rough membership function associated with X is defined by $\mu_X : U \rightarrow [0, 1]$

$$\mu_X(u) = \frac{|X \cap R(u)|}{|R(u)|}.$$

This provides a numeric characterization of rough sets. Based on the definition of rough membership function, a variable precision rough set model is proposed in [Ziarko, 1993; Katzberg and Ziarko, 1996]. For $0 \leq \alpha < \beta \leq 1$, the α and β-approximation of X is defined by

$$\underline{R_\alpha}X = \{u \in U \mid \mu_X(u) \geq 1 - \alpha\}$$

$$\overline{R_\beta}X = \{u \in U \mid \mu_X(u) > 1 - \beta\}.$$

Though the rough membership function and the accuracy of approximation are well-defined for finite universe U, it uses the cardinality which may be not

[1] In pawlak's original definition, U is assumed to be finite

finite in the infinite case. To cope with this situation, we extend the RAS to probabilistic approximation space(PRAS). A PRAS is just a triple (U, R, Pr), where (U, R) is still an RAS and Pr is a probability distribution on U. Then we can replace the definition of accuracy and rough membership function by the following equations:

$$\rho(X) = \frac{Pr(\underline{R}X)}{Pr(\overline{R}X)},$$

$$\mu_X(u) = \frac{Pr(X \cap R(u))}{Pr(R(u))}.$$

For convenience, $\mu_X(u) = 1$ if $Pr(R(u)) = 0$. When U is finite and Pr is a uniform distribution, the definitions just reduced to the original ones.

2.3 Multiple relations approximation space

An example of PAS is derivable from data table based knowledge representation systems (KRS). A KRS or data table is a pair $S = (U, A)$, where U is a nonempty, finite set (the universe) and A is a nonempty, finite set of primitive attributes. Every $a \in A$ is a total function $a : U \to V_a$, where V_a denotes possible values of a. An equivalence relation $IND(B)$ is associated with every subset of attributes $B \subseteq A$, and defined by

$$xIND(B)y \Leftrightarrow a(x) = a(y) \forall a \in B.$$

$IND(B)$ is called an indiscernibility relation. We will write $IND(a)$ instead of $IND(\{a\})$ for all $a \in A$. Obviously, $IND(B) = \bigcap_{a \in B} IND(a)$. Since $IND(B)$ is an equivalence relation, we can define $IND(B)$-lower and $IND(B)$-upper approximation of X for any $X \subseteq U$.

The definitions are used in the analysis of dependency between attributes in a data table. Let us say that attribute B_2 depends on B_1, denoted by $B_1 \Rightarrow B_2$, iff $IND(B_1) \subseteq IND(B_2)$, i.e., any two objects in U with same values in their attributes B_1 will have also same ones in B_2. It is easily to show that $B_1 \Rightarrow B_2$ iff $\underline{B_1}X = X$ for all X that is an equivalence class of $IND(B_2)$.

The data table example shows that there may be more than one indiscernibility relations in an approximation space, so motivates the definition of multiple relations approximation space(MRAS). An MRAS is a pair $(U, \{R_i \mid 1 \leq i \leq n\})$, where each R_i is a binary relation. In an MRAS, each primitive relation R_i is a relation expression, and if R and S are relational expressions, so are $R \cap S$ and $R \cup^* S$, where $R \cap S$ is the intersection of R and S and $R \cup^* S$ is the transitive closure of the union of R and S for all relations R and S. The rough approximation in an MRAS is the same as that in PAS or RAS except that it can now be defined with respect to any relational expressions.

3 First-order logic from the perspective of PAS

We start with the investigation of first-order logic(FOL) from the perspective of rough set theory. The notations used for FOL here are mostly drawn from [Gallier, 1986].

The alphabet of a first-order language consists of

- logical symbols: $\neg, \wedge, \vee, \supset, \forall, \exists$,
- individual variables: a countably infinite set $VAR = \{x_0, x_1, x_2, \ldots\}$,
- function symbols: a countable set $FS = \{f_0, f_1, f_2, \ldots\}$, where each f_i has an arity, denoted by $\tau(f_i)$, if $\tau(f_i) = 0$, then f_i is also called a constant symbol
- predicate symbols: a countable set $PS = \{P_0, P_1, P_2, \ldots\}$, where each P_i has an arity, denoted by $\tau(P_i)$, if $\tau(P_i) = 0$, then P_i is also called a propositional symbol

The set of terms is the smallest set containing VAR and satisfying that if $f \in FS$ with $\tau(f) = n$ and $t_1, \ldots, t_n$ are terms, then $ft_1 \ldots t_n$ is a term. If P is a predicate symbol with $\tau(P) = n$ and $t_1, \ldots, t_n$ are terms, then $Pt_1 \ldots t_n$ is an atomic formula. The set of wffs of FOL is the smallest set containing all atomic formulas and satisfying the following conditions:

- if φ is a wff and $x \in VAR$, then $\neg\varphi, \forall x\varphi, \exists x\varphi$ are wffs,
- if φ and ψ are wffs, then $\varphi \wedge \psi, \varphi \vee \psi, \varphi \supset \psi$ are wffs.

In a wff of the form $\forall x\varphi$ or $\exists x\varphi$, the variable x is said to be bound by the quantifier $\forall$ (or $\exists$), so it is called a bound variable (in the wff). A variable not bound in a wff is called a free variable in it. A wff not containing free variables is also called a sentence.

The semantics of FOL is based on first-order structure. An FOL structure is a pair (D, I), where D is a nonempty set, called the domain of the structure, and I is an interpretation function which assigns functions and relations on D to symbols in FS and PS as follows:

1. for each $f \in FS$ with $\tau(f) = n$, $I(f) : D^n \to D$ is an n-ary function, in particular, if $n = 0$, then $I(f)$ is just an element of D,
2. for each $P \in FS$ with $\tau(P) = n$, $I(P) : D^n \to \{0, 1\}$ is an n-ary relation, in particular, if $n = 0$, then $I(P) = 0$ or 1.

Given an FOL language and structure, a *variable assignment* is any function $v : VAR \to D$ from the set of variables to the domain of the structure. The set of all such functions is denoted by $U = [VAR \to D]$. Now, for each variable $x \in VAR$, we can define an equivalence relation $=_x$ on U by

$$u =_x v \Leftrightarrow (\forall y \neq x. u(y) = v(y)),$$

for all $u, v \in U$. Thus $(U, =_x)$ is a PAS for any $x \in VAR$. The relation $=_x$ has been used in the investigation of the relationship between quantifiers and modal operators in [Benthem, 1997].

The meaning of a wff is then a subset of U, that is, the set of variable assignments satisfying the wff. To define the meaning function, we first extend a variable assignment to a term assignment inductively. Let $v \in U$, then for any term t, if $t = x \in VAR$, then $v(t) = v(x)$, if $t = ft_1 \dots t_n$, then $v(t) = I(f)(v(t_1), \dots, v(t_n))$. Now, for a given structure $M = (D, I)$, we can define the denotation of a wff φ, $|\varphi|_M \subseteq U$ inductively as follows (we drop the subscript M in the definition):

1. $|Pt_1 \dots t_n| = \{v \in U \mid I(P)(v(t_1), \dots, v(t_n)) = 1\}$
2. $|\neg\varphi| = U - |\varphi|$
3. $|\varphi \wedge \psi| = |\varphi| \cap |\psi|$
4. $|\varphi \vee \psi| = |\varphi| \cup |\psi|$
5. $|\varphi \supset \psi| = (U - |\varphi|) \cup |\psi|$
6. $|\forall x\varphi| = \underline{=_x}|\varphi|$
7. $|\exists x\varphi| = \overline{=_x}|\varphi|$

Thus, the meaning of the wff $\forall x\varphi$ (resp. $\exists x\varphi$) is the lower (resp. upper) approximation of that of φ in the PAS $(U, =_x)$. A wff φ is said to be true in a structure M if $|\varphi|_M = U$, and it is valid if it is true in all structures.

4 Rough quantifiers and MRAS

We have seen that the quantified wffs can be given a rough set interpretation based on the equivalence relation $=_x$ defined in the space of all variable assignments. Let us now examine the relation $=_x$ more closely. If $\sim = D \times D$, then $\sim$ is obviously an equivalence relation on the domain D. The definition of $=_x$ is in fact equivalent to the following

$$u =_x v \Leftrightarrow u(x) \sim v(x) \wedge (\forall y \neq x. u(y) = v(y)),$$

so $=_x$ is only defined with respect to a special binary relation on D. A generalization of this is to consider any binary relation on D. Let R be a binary relation on D and $x \in VAR$, then we can define a binary relation R_x on $U = [VAR \rightarrow D]$ by

$$uR_x v \Leftrightarrow (u(x), v(x)) \in R \wedge (\forall y \neq x. u(y) = v(y)).$$

The idea has been exploited (at least implicitly) in [Szczerba, 1987; Krynicki, 1989; 1990] for the development of rough quantifiers. Here, we present a logic of rough quantifiers more general than that in [Szczerba, 1987] based on this idea.

The alphabet for the logic of rough quantifiers (LRQ) is that of FOL without the quantifier symbols $\forall$ and $\exists$, but with

- a finite set of primitive quantifiers $Q_0 = \{q_1, \dots, q_n, u, e\}$, where u and e denote the universal and null quantifiers respectively, and
- two binary operations $\cap$ and $\cup^*$ for forming quantifier expressions.

The definition of terms remains unchanged and the set of quantifier expressions Q is the smallest set containing Q_0 and satisfying that if $q, r \in Q$, then $q \cap r, q \cup^* r \in Q$. The set of wffs of LRQ is the smallest set containing all atomic formulas and satisfying the following condition:

- if φ is a wff, $q \in Q$, and $x \in VAR$, then $\underline{q}x\varphi, \overline{q}x\varphi$ are wffs,

and the formation rules for propositional connectives.

For the semantics of LRQ, an LRQ structure is a triple (D, I, E), where (D, I) is an FOL structure and E assigns binary relations on D to quantifier symbols in Q_0 such that $E(u) = D \times D$ and $E(e) = \{(a, a) \mid a \in D\}$. The assignment E is extended to all quantifier expressions in Q by $E(q \cap r) = E(q) \cap E(r)$ and $E(q \cup^* r) = E(q) \cup^* E(r)$. Then the denotation of the quantified wffs are defined by

$$|\underline{q}x\varphi| = \underline{E(q)_x}|\varphi|,$$

$$|\overline{q}x\varphi| = \overline{E(q)_x}|\varphi|.$$

According to the semantics, the wffs $\underline{u}x\varphi$ and $\overline{u}x\varphi$ correspond exactly to $\forall x\varphi$ and $\exists x\varphi$ in FOL respectively.

The LRQ proposed here is obviously more expressive than that in [Szczerba, 1987] since we allow more than one rough quantifiers in our language. However, there is another subtle difference between these two logics. This lies on the distinction of free and bound variables. In FOL, the value of a bound variable does not influence the denotation of the wff. In other words, if v is in $|\forall x\varphi|$, then so are any variable assignments different from v only in the value of x. In particular, if φ is a sentence in FOL, then $|\varphi|$ is either the universe U or the empty set. On the other hand, if φ contains some free variables, then $|\varphi|$ may be a proper subset of U. However, our rough quantifiers totally eliminate the difference between free and bound variables. The variable occurring in the wff $\underline{q}x\varphi$ may be free in the sense of FOL. Thus, for example, while $\underline{u}x(\underline{u}x\varphi) \equiv \underline{u}x\varphi$ is valid, it is not so for general quantifier expressions. In fact, the wffs $\underline{e}x\varphi$ and $\overline{e}x\varphi$ are equivalent to φ, so x is a free variable in them in the sense of FOL.

On the other hand, the rough quantifiers $\tilde{\forall}$ and $\tilde{\exists}$ introduced in [Szczerba, 1987] can be defined in our LRQ as follows:

$$\tilde{\forall}x\varphi = \underline{u}x\overline{q_1}x\varphi,$$

$$\tilde{\exists}x\varphi = \overline{u}x\underline{q_1}x\varphi,$$

for some fixed $q_1 \in Q_0$. So the variable x in $\tilde{\forall}x\varphi$ and $\tilde{\exists}x\varphi$ is indeed bound in the sense of FOL.

Whether the peculiarity of LRQ quantifiers is an advantage remains to be seen, however it indeed provides much flexibility in the representation of imprecise knowledge. In fact, LRQ also encompass another kind of approximation logics presented in [Chakraborty and Banerjee, 1993; Rasiowa and Skowron, 1984; 1985; Rasiowa, 1986; Bolc and Borowik, 1992]. For example, in the rough logic of [Chakraborty and Banerjee, 1993], there are wffs of the form $L\varphi$ as well as those of FOL. This kind of wff can be represented in LRQ as $\underline{q}x_1 \ldots \underline{q}x_n\varphi$ for some fixed quantifier expression q if $\{x_1, \ldots, x_n\}$ is the set of free variables occurring in φ.

The transition of the semantics from FOL to LRQ have an analogy on modal logic historically, that is, the development of Carnap's semantics to Kripke's one. In the Carnap's semantics, a wff of the form $\Box\varphi$ or $\Diamond\varphi$ is true in all possible worlds of a model if it is true in one. However, the accessibility relation provided in the Kripke model eliminates this property. The property then holds only when the accessibility relation is a universal one.

5 Generalized quantifiers and NS

The study of generalized quantifiers has become an important field of logic and linguistics in the last decade. Many fruitful theoretical results and applications to the analysis of natural language have been produced[Krynicki *et al.*, 1995]. It is impossible even to give a brief survey of the field here. However, we will consider a logic with generalized quantifiers proposed in [Schlechta, 1997] and discuss its relationship with NS.

The language of the generalized quantifier logic (GQL) include two generalized quantifiers ∇ and $\clubsuit$ as well as the alphabet of FOL. In addition, the following formation rule is added to those of FOL

- if φ is a wff, then $\nabla x\varphi$ and $\clubsuit x\varphi$ are wffs,too.

The intuitive meaning of $\nabla x\varphi$ is "most x satisfy φ" and $\clubsuit x\varphi$ means that "for at least a few x, φ holds".

A GQL structure is then a triple (D, I, N), where (D, I) is an FOL structure, and (D, N) is an NS satisfying the following conditions:

1. Openess: D is open,
2. Uniformity: N is uniform in the sense that $N(a) = N(b)$ for all $a, b \in D$, so N can be identified with a subset of 2^D, and we will view N in such way from now on,
3. Coherence: if $A, B \in N$, then $A \cap B \neq \emptyset$.

Given a GQL structure (D, I, N) and a variable $x \in VAR$, we can induce an NS (U, N_x), where $U = [VAR \to D]$ and $N_x : U \to 2^{2^U}$ is defined by

$$N_x(v) = \{S \subseteq U \mid (\{u(x) \mid u \in S\} \in N) \wedge (\forall u \in S u =_x v)\}$$

for all $v \in U$. Then the following two rules are added to the denotation function for FOL

8. $|\nabla x\varphi| = \underline{N_x}|\varphi|$,
9. $|\clubsuit x\varphi| = \overline{N_x}|\varphi|$.

This shows that NS can be used to define some useful generalized quantifiers.

6 Probability quantifiers and PRAS

In [Halpern, 1990], two kinds of probability on first-order logics are distinguished. One puts a probability on possible worlds and results in a kind of probabilistic modalities. The other approach puts a probability on the domain and results in probability quantifiers. In this section, we will consider the relationship between this kind of probabilistic logic and PRAS.

The alphabet of probability quantifier logic (PQL) consists of a class of probability quantifiers $(> r)$ for all $r \in [0, 1]$ as well as that of FOL. For simplicity, we assume that the set of variables VAR is finite. Let VAR^+ denote the set of finite nonempty sequences of variables. The wffs of PQL will include $(> r)\mathbf{x}\varphi$ if φ is a wff and $\mathbf{x} \in VAR^+$.

A PQL structure is a triple (D, I, Pr), where (D, I) is just an FOL structure, and Pr is a probability distribution on D. The probability measure Pr can be extended to a product measure Pr^n on the domain D^n for any $n > 1$ by measure theoretic techniques[Halpern, 1990]. For any sequence of variables $\mathbf{x}$, we can define the equivalence relation $=_\mathbf{x}$ on $U = [VAR \to D]$ by

$$u =_\mathbf{x} v \Leftrightarrow \forall y \notin \mathbf{x} u(y) = v(y).$$

If the cardinality of VAR is n, then $(U, =_\mathbf{x}, Pr^n)$ forms a PRAS for any $\mathbf{x} \in VAR^+$, and the denotation of the wff $(> r)\mathbf{x}\varphi$ is defined by

$$|(> r)\mathbf{x}\varphi| = \overline{(=_\mathbf{x})_{1-r}}|\varphi|.$$

We must remarks that PQL is in fact only a fragment of the logic developed in [Halpern, 1990] since we do not allow arithmetic expressions formed by probability terms. However, it is still sufficiently expressive to represent statistical knowledge such as "The probability that a randomly chosen bird flies is greater than 0.9". For example, $(> 0.5)xyLike(x, y)$ means that if we pick a pair randomly from the domain, the probability that x likes y is greater than 0.5. Moreover, we consider only the probability quantifiers based on equivalence relations of the form $=_\mathbf{x}$. However, it is not difficult to combine the idea of rough quantifiers discussed above, and have a logic of rough probability quantifiers.

7 Concluding Remarks

Though all logics discussed in this paper are by themselves a topic of research, we only touch upon the basic aspects of them and concentrate on their relationship with rough set theory. Therefore, there remain many research problems deserving further study. Among them, one of the most important ones is the study of the logical properties of the rough quantifiers introduced in Sec. 4.

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