Hybrid Logic for Possibilistic Reasoning

Churn-Jung Liau
Institute of Information Science
Academia Sinica,Taipei, Taiwan
E-mail: liaucj@iis.sinica.edu.tw

Abstract

Hybrid logic is an extension of modal logic with nominals for referring to the possible worlds. A nominal is an atomic formula which is true at exactly *one* point in any model. By using hybrid logic, the labelled deduction mechanism can be internalized to the object language, so the proof methods for the logic can be developed without resort to meta-linguistic devices. In this paper, we present two hybrid logics for possibilistic reasoning. The first is based on the graded modal formulation of possibilistic logic, so its proof method is essentially a modification of the tableau system for hybrid multi-modal logic. The second is a kind of fuzzy modal logic and has a many-valued semantics. We present its proof method by using a tableau system for annotated formulas.

Key words: Possibilistic logic, Hybrid language, Graded modal logic, Fuzzy modal logic, Tableau method.

1 Introduction

Possibilistic logic[5, 4, 3] is a logic of partial ignorance and/or inconsistency based on possibility theory[15]. Though the original possibilistic logic is not presented in the modal logic form, it has strong modal flavor so that a modal formulation is possible. It has been shown that possibilistic reasoning can be formulated in a kind of graded modal logic[12, 6]. This kind of graded modal logic can do not only possibilistic but also similarity-based reasoning[6], so it substantially extends the application scope of possibilistic logic. Though the Hilbert-style axiomatic system for these logics have been presented, the proof methods of them are still urgently needed for the automated theorem proving in such logics.

Among the well-know proof methods for classical logic, the tableau method has been extensively developed in [13] and successfully extended to modal logic in [7]. Tableau method is essentially a proof-tree construction approach. In this method, an attempted proof of a well-formed formula(wff) is to place the negated form of the wff as the root of the proof tree, and tableau rules are applied to extend the branches of the tree until some closed condition is encountered, or no more rules are applicable. If all branches of the tree is closed, then the wff is proved as a theorem of the system. Otherwise, an open branch of the tree will construct a counter-model of the wff.

For the tableau methods of modal logic, an important problem is how to deal with multiple worlds due to application of modal rules. We start with an attempted proof of a wff, and the classical tableau rules is applied , but when a modal wff is derived in the course, we may also switch to an alternate world, and the proof is continued in the alternate world. The difficulty of this method is that we need to keep a lots of alternate tableaus in the same time, and the proof will get rather complex even a simple wff is given at the beginning of the proof.

To avoid the creation of multiple tableaus, we need a device to distinguish the wffs in different worlds. This is achieved by using prefixed tableau in [7], which can be viewed as a precursor of the more general methodology of labelled deduction[8]. While the labelled deductive system is a very general paradigm for many different modes of reasoning, the manipulation of labels need some meta-linguistic algebra. In [1, 2], it is shown that the meta-level construct can be internalized to the object language in hybrid logic by allowing the reference to the states explicitly as nominals. Thus, the proof methods for hybrid logic can be developed without resort to meta-linguistic apparatus.

In this paper, we show that the hybrid logic methodology can also be used in the modal logic for possibilistic and similarity-based reasoning. To achieve the purpose, we present two hybrid logics and their tableau systems. One is the extension of quantitative modal logic introduced in [12], which is a kind of two-valued graded modal logic, so the tableau system for multi-modal logic in [2] can be easily modified to fit our use except the interaction of modalities. The other is an extension of fuzzy modal logic. Though its syntax is completely same as the basic hybrid language in [1], it has a many-valued semantics, so we present its tableau system by using annotated formulas.

In the rest of the section, we will review the basic notions of possibility theory and possibilistic logic. In the next two sections, a hybrid graded modal logic and fuzzy modal logic and their tableau systems are respectively presented. Finally, we summarize the results and discuss some further research directions in the conclusion section.

1.1 Possibility theory and possibilistic logic

Possibility theory is developed by Zadeh from fuzzy set theory[15]. Given a universe W, a *possibility distribution* on W is a function $\pi : W \to [0,1]$. In general, the normalized condition is required, i.e., $\sup_{w \in W} \pi(w) = 1$ must hold. The possibility and necessity measures, denoted by Π and N respectively, are defined as $\Pi, N : 2^W \to [0,1]$ such that $\Pi(S) = \sup_{w \in S} \pi(w)$ and $N(S) = 1 - \Pi(\overline{S})$, where $\overline{S}$ is the complement of S with respect to W.

Based on possibility theory, Dubois and Prade propose the possibilistic logic(PL)[3, 4, 5]. The wffs of PL are one of the forms $(\varphi\ Nc)$ or $(\varphi\ \Pi c)$, where φ is a classical logic formulas and $c \in (0,1]$. A model for PL is a possibility distribution $\pi : \Omega \to [0,1]$, where Ω is the set of all classical logic interpretations. For any classical wff φ, $|\varphi| = \{\omega \mid \omega \models \varphi\}$ is the truth set of φ. Then, the possibility and necessity measures can be defined for the classical logic wffs, i.e., $N(\varphi) = N(|\varphi|)$, $\Pi(\varphi) = \Pi(|\varphi|)$. For a PL model π, define $\pi \models (\varphi\ Nc)$ iff $N(\varphi) \geq c$ and $\pi \models (\varphi\ \Pi c)$ iff $\Pi(\varphi) \geq c$.

The possibility and necessity measures can be extended to the domain of fuzzy propositions. An interpretation of fuzzy logic is a [0,1]-valued truth assignment to the set of fuzzy propositions, so if ω is a fuzzy logic interpretation and φ is a fuzzy proposition, then $\omega(\varphi) \in [0,1]$. Let Ω now denote the set of all fuzzy logic interpretations and $\pi : \Omega \to [0,1]$ be a possibility distribution, then the possibility and necessity measures of a fuzzy proposition φ, according to [5], are defined by

$$\Pi(\varphi) = \sup_{\omega \in \Omega} \min(\pi(\omega), \omega(\varphi)),$$

$$N(\varphi) = \inf_{\omega \in \Omega} \max(1 - \pi(\omega), \omega(\varphi)).$$

2 Hybrid Graded Modal Logic

2.1 The language

In the hybrid language, two disjoint sets of atomic formulas are given. One is the set of ordinary propositional symbols $\mathcal{P} = \{p, q, r, \ldots\}$ and the other is the set of nominals $\mathcal{N} = \{i, j, k, l, \ldots\}$. The nominals will be used to name the possible worlds or states. The language for hybrid graded modal logic(HGML) is defined as follows:

$$\begin{array}{rcl} \Phi & ::= & i \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \supset \psi \mid \\ & & [c]\varphi \mid [c]^{+}\varphi \mid \langle c \rangle \varphi \mid \langle c \rangle^{+}\varphi \mid i : \varphi \end{array}$$

where c is a numeral in $[0,1]$. The symbol Φ denotes the set of well-formed formulas(wff) of HGML. The operator "i :" is called satisfaction operator in [1, 2][1] and the wff of the form $i : \varphi$ is called satisfaction statement. The statement asserts that φ is true in the world denoted by i. Except satisfaction statements, the HGML language is just that of quantitative modal logic (QML) in [12]

The HGML language is interpreted on Kripke models. Define an HGML model as a triple $M = \langle W, R, V \rangle$, where W is a set of possible worlds (or states), $R : W \times W \to [0,1]$ is a fuzzy binary relation on W, and $V : \mathcal{P} \cup \mathcal{N} \to 2^W$ is a truth value assignment satisfying the constraint that for $i \in \mathcal{N}$, $|V(i)| = 1$, i.e., $V(i)$ is a singleton. In what follows, we will identify the set $V(i)$ with its unique element. Given R, we can define a possibility distribution π_w for each $w \in W$ such that $\pi_w(s) = R(w,s)$ for all $s \in W$.

Given a model $M = \langle W, R, V \rangle$, we can define the truth relation $\models_M \subseteq W \times \Phi$ as follows, where Π_w and N_w are the possibility and necessity measures corresponding to the possibility distribution π_w. The clauses for classical connectives are as usual.

(1) $w \models_M \langle c \rangle \varphi \Leftrightarrow \Pi_w(\varphi) \geq c$,

(2) $w \models_M \langle c \rangle^{+} \varphi \Leftrightarrow \Pi_w(\varphi) > c$,

(3) $w \models_M [c]\varphi \Leftrightarrow N_w(\varphi) \geq c$,

(4) $w \models_M [c]^{+}\varphi \Leftrightarrow N_w(\varphi) > c$,

(5)$w \models_M i : \varphi \Leftrightarrow V(i) \models_M \varphi$.

Clauses (1)-(5) define the meaning of wffs with modal operators. Intuitively, $[c]\varphi$ (resp. $[c]^{+}\varphi$) means that $N(\varphi) \geq c$ (resp. $N(\varphi) > c$), and similarly, $\langle c \rangle \varphi$ (resp. $\langle c \rangle^{+}\varphi$) denotes that $\Pi(\varphi) \geq c$ (resp. $\Pi(\varphi) > c$). Here, for convenience, we define $\sup \emptyset = 0$ and $\inf \emptyset = 1$. The last clause define the meaning of satisfaction statement. It means that $i : \varphi$ is true in any world if φ is true in the world denoted (or named) by i.

A wff φ is said to be valid in $M = \langle W, R, V \rangle$, written $\models_M \varphi$ iff for all $w \in W, w \models_M \varphi$. If **C** is a class of models, then $\models_{\mathbf{C}} \varphi$ means that for all $M \in$**C**, $\models_M \varphi$. A model is said to be finite-valued if the range of its

[1] In [2], it is denoted by $@_i$

accessibility relation, i.e., the set $R(W \times W)$ is finite. In particular, a model is finite-valued when W is finite. A wff is finitely valid iff it is valid in all *finite-valued* HGML models.

2.2 The tableau proof method

The rationale behind tableau proof method is to find a counter-model for the negation of a wff. If this is impossible, then the original wff is valid. Therefore, to test the validity of an HGML wff φ, let i be a nominal not occurring in φ, then a tableau proof tree will be started with the $\neg i : \varphi$ at its root. A tableau rule is applicable when some branch of the proof tree contains the premises of some rule, and the conclusions of the rule can be added to end of the branch after the application of the rule. The set of tableau rules is presented in figure 1. In the presentation, we follow the convention of [1] by using φ and ψ as metavariables over arbitrary wffs, s, t, and u, as metavariables over nominals, and a as a metavariable over new nominals(i.e., nominals that have not been used so far in the tableau construction).

These rules are divided into three groups. First, there are two kinds of classical rules. The *extension rules*, are to add the conclusions to the end of a branch directly, and the added nodes are organized as a branch and attached to the end of the original branch. The *forking rules*, are to add the conclusions as the sons of the end node of the original branch. In the propositional case, all forking rules have exactly two conclusions, so one is the left son and the other is the right son, and the proof tree is binary.

Second, the modal rules (a)-(d) can be seen as instances of those for multimodal logic in [2]. In particular, the soundness of the rules $\langle c\rangle$ and $\neg[1-c]^+$ relies on the finitary assumption of the models. For example, rule $\langle c\rangle$ says that if $\langle c\rangle\varphi$ is true in a state s, then there exists a state denoted by a such that $R(s,a) \geq c$ and φ is true in that state. However, according to our semantics, the truth of $\langle c\rangle\varphi$ in s means that $\sup\{R(s,w) \mid w \models_M \varphi\} \geq c$, so the existence of the above-mentioned state can be guaranteed only when the possible values of $R(s,w)$ is finite. The rule (e) is for the satisfaction operators. By the semantics, when a pair of nested satisfaction operators is given, the outermost does not have any effect on the evaluation of the formula.

Finally, the nominal rules reflect the fact that nominals are genuine labels. In [1], the set of nominal rules is a modal formulation of first-order logic axioms of equality. However, here, our bridge rules also include some interaction between modalities. For example, by using *Ref* and *Bridge* , we can infer $s : \langle d\rangle t$ from $s : \langle c\rangle t$ when $c > d$, where the conclusion is obviously weaker than the premises. Except the *Bridge* rules, the other nominal rules are the same as those in [1].

A branch in a tableau is closed if for some wff φ, one of the following conditions holds:

(i) $[1]^+\varphi$ or $\langle 1\rangle^+\varphi$ occurs in it,

(ii) $\neg[0]\varphi$ or $\neg\langle 0\rangle\varphi$ occurs in it, or

(iii) both φ and $\neg\varphi$ occur in it.

A tableau is closed if all its branches are closed. A wff φ is tableau provable if there is a closed tableau with $\neg i : \varphi$ as its root (where i is a nominal not occurring in φ). Then we have:

Theorem 1 *An HGML wff φ is tableau provable iff it is finitely valid.*

3 Hybrid Fuzzy Modal Logic

3.1 The language

The language for hybrid fuzzy modal logic(HFML) is the same as that of basic hybrid logic[1]:

$$\Phi ::= i \mid p \mid \neg\varphi \mid \varphi\wedge\psi \mid \varphi\vee\psi \mid \varphi \supset \psi \mid \Box\varphi \mid \Diamond\varphi \mid i : \varphi$$

However, unlike basic hybrid logic, the wffs of HFML are interpreted in fuzzy Kripke structure. An HFML model is a triple $M = \langle W, R, V\rangle$, where $\langle W, R\rangle$ is a possibility frame and $V : W \times (\mathcal{P} \cup \mathcal{N}) \to [0,1]$ is a [0,1]-valued truth assignment satisfying the constraint that for each $i \in \mathcal{N}$, there exists exactly one $w_0 \in W$ such that

$$V(w,i) = \begin{cases} 1, & \text{if } w = w_0, \\ 0, & \text{otherwise.} \end{cases}$$

In what follows, we will use $V(i)$ to denote the unique world w_0 satisfying $V(w_0, i) = 1$. From an HFML model M, we can define an evaluation function $e_M : W \times \Phi \to [0,1]$ inductively from V by

1. $e_M(w,a) = V(w,a)$ for all $a \in \mathcal{P} \cup \mathcal{N}$,
2. $e_M(w,\neg\varphi) = 1 - e_M(w,\varphi)$,
3. $e_M(w,\varphi \wedge \psi) = \min(e_M(w,\varphi), e_M(w,\psi))$,
4. $e_M(w,\varphi \vee \psi) = \max(e_M(w,\varphi), e_M(w,\psi))$,
5. $e_M(w,\varphi \supset \psi) = \max(1 - e_M(w,\varphi), e_M(w,\psi))$,
6. $e_M(w,\Diamond\varphi) = \sup_{u\in W} \min(R(w,u), e_M(u,\varphi))$,
7. $e_M(w,\Box\varphi) = 1 - e_M(w,\Diamond\neg\varphi)$,
8. $e_M(w,i : \varphi) = e_M(V(i),\varphi)$.

1. Boolean rules:

   (a) Extension rules:

$$\wedge: \frac{s:\varphi\wedge\psi}{\begin{array}{c}s:\varphi\\ s:\psi\end{array}} \qquad \neg\vee: \frac{\neg s:(\varphi\vee\psi)}{\begin{array}{c}\neg s:\varphi\\ \neg s:\psi\end{array}} \qquad \neg\supset: \frac{\neg s:(\varphi\supset\psi)}{\begin{array}{c}s:\varphi\\ \neg s:\psi\end{array}} \qquad \neg: \frac{s:\neg\varphi}{\neg s:\varphi} \qquad \neg\neg: \frac{\neg s:\neg\varphi}{s:\varphi}$$

   (b) Forking rules:

$$\vee: \frac{s:(\varphi\vee\psi)}{s:\varphi \mid s:\psi} \qquad \neg\wedge: \frac{\neg s:(\varphi\wedge\psi)}{\neg s:\varphi \mid \neg s:\psi} \qquad \supset: \frac{s:(\varphi\supset\psi)}{\neg s:\varphi \mid s:\psi}$$

2. Modal rules:

   (a) If $c > 0$, then

$$\langle c\rangle: \frac{s:\langle c\rangle\varphi}{\begin{array}{c}s:\langle c\rangle a\\ a:\varphi\end{array}} \qquad \neg[1-c]^+: \frac{\neg s:[1-c]^+\varphi}{\begin{array}{c}s:\langle c\rangle a\\ \neg a:\varphi\end{array}}$$

   (b)

$$\langle c\rangle^+: \frac{s:\langle c\rangle^+\varphi}{\begin{array}{c}s:\langle c\rangle^+ a\\ a:\varphi\end{array}} \qquad \neg[1-c]: \frac{\neg s:[1-c]\varphi}{\begin{array}{c}s:\langle c\rangle^+ a\\ \neg a:\varphi\end{array}}$$

   (c)

$$[c]: \frac{s:[c]\varphi,\ \ s:\langle 1-c\rangle^+ t}{t:\varphi} \qquad \neg\langle c\rangle^+: \frac{\neg s:\langle c\rangle^+\varphi,\ \ s:\langle c\rangle^+ t}{\neg t:\varphi}$$

   (d)

$$[c]^+: \frac{s:[c]^+\varphi,\ \ s:\langle 1-c\rangle t}{t:\varphi} \qquad \neg\langle c\rangle: \frac{\neg s:\langle c\rangle\varphi,\ \ s:\langle c\rangle t}{\neg t:\varphi}$$

   (e)

$$Sat: \frac{s:t:\varphi}{t:\varphi} \qquad \neg Sat: \frac{\neg s:t:\varphi}{\neg t:\varphi}$$

3. Nominal rules

$$Ref: \frac{[s \text{ occurs on the branch}]}{s:s} \qquad Sym: \frac{s:t}{t:s} \qquad Nom: \frac{s:t,\ \ t:\varphi}{s:\varphi}$$

$$Bridge: \frac{s:\langle c\rangle t,\ \ t:u}{s:\langle d\rangle u\ (c\geq d)} \qquad \frac{s:\langle c\rangle t,\ \ t:u}{s:\langle d\rangle^+ u\ (c>d)} \qquad \frac{s:\langle c\rangle^+ t,\ \ t:u}{s:\langle c\rangle^+ u} \qquad \frac{s:\langle c\rangle^+ t}{s:\langle c\rangle t}$$

Figure 1. The tableau rules for HGML

Note that $e_M(w, \Box\varphi)$ and $e_M(w, \Diamond\varphi)$ correspond respectively to $N_w(\varphi)$ and $\Pi_w(\varphi)$ for the fuzzy proposition φ. Also, an HFML model is said to be finite-valued if both the ranges of R and V are finite, i.e., the set $R(W \times W) \cup V(W \times (\mathcal{P} \cup \mathcal{N}))$ is finite.

There have been quite a few variants of the semantics for fuzzy modal logic[9, 11, 10, 14]. The difference mainly lies on the interpretations of the implication and conjunction in the definition of necessity and possibility. Our semantics corresponds to the Kleene-Dienes implication and min conjunction given in [14]. Most of the semantic variants for fuzzy modal logic concerns about the so-called c-valid wffs (in particular, $c = 1$). A wff φ is c-valid if for all models M and worlds w, $e_M(w, \varphi) \geq c$. However, without regarding the nominals, it can be shown that there will be no any c-valid wffs according to our semantics when $c > \frac{1}{2}$. Though the addition of nominals to the language induces some 1-valid sentences, like $i : i$, they are too trivial to be useful in possibilistic or similarity-based reasoning. To circumvent the problem, we thus introduce the annotated wffs as follows.

Definition 1 *1. If φ is an HFML wff, then $\varphi \geq c$ and $\varphi > c$ is called annotated atoms for any $c \in [0, 1]$.*

2. The set of annotated wffs is the least set containing all annotated atoms and closed under classical connectives $\neg$, $\vee$, and $\wedge$.

3. An annotate literal is an annotated atom or its negation.

Definition 2 *Let $M = \langle W, R, V\rangle$ be an HFML model, and $w \in W$, then for any annotated wffs, the satisfaction relation is defined by*

(i) $w \models_M \varphi \geq c$ iff $e_M(w, \varphi) \geq c$,

(ii) $w \models_M \varphi > c$ iff $e_M(w, \varphi) > c$,

and the usual rules for classical connectives.

As usual, an annotated wff is true in a model if it is satisfied by all worlds of the model and it is valid in a class of models if it is true in all models belonging to the class. In particular, an annotated wff is finitely valid if it is true in all finite-valued models.

3.2 The tableau proof method

To test the validity of an annotated wff, we can first transform it into an equivalent conjunctive normal form, i.e., an annotated wff of the form $C_1 \wedge \cdots \wedge C_n$, where each C_i is a clause of the form $L_{i1} \vee \cdots \vee L_{im_i}$, with each L_{ij} being an annotated literal. Thus, without loss of generality, we can consider only the validity of an annotated clause. Now, given an clause $L_1 \vee \cdots \vee L_m$, the tableau proof is started with a single branch with exactly m nodes corresponding to $L_1, \ldots, L_m$. Let i be a nominal not occurring in the clause, then for $L = L_1, \ldots, L_m$, the node corresponding to L is

(i)$i : \varphi \geq c$(resp. $i : \varphi > c$) if $L = \neg(\varphi \geq c)$ (resp. $\neg(\varphi > c)$),

(ii)$\neg i : \varphi \geq 1 - c$ if $L = \varphi > c$,

(ii)$\neg i : \varphi > 1 - c$ if $L = \varphi \geq c$.

Thus each node of a tableau proof tree for annotated wff will be an annotated atom and the tableau rules will be applied to them to extend or split a branch until some closed condition is encountered or no more rules are applicable. A branch is closed if for some HFML wff φ, $\varphi > 1$ occurs in it or both $\varphi \rhd_1 c$ and $\neg\varphi \rhd_2 d$ where $\rhd_i$(i=1,2) denotes $\geq$ or $>$ occur in it and

(i) at least one of $\rhd_i$ is $>$ and $c + d \geq 1$, or

(ii) $\rhd_1 = \rhd_2 = \geq$ and $c + d > 1$.

The tableau rules for annotated wffs in HFML is presented in figure 2. To simplify the presentation, we use some uniform notations in [13]. Let α and β denote respectively the premises of extension and forking rules in figure 1, then α_i and β_i ($i = 1, 2$, in the case of $\neg$ and $\neg\neg$ rules, $i = 1$) are the conclusions of the corresponding rules. Furthermore, the symbol $\rhd$ denotes $\geq$ or $>$. When $\rhd = \geq$(resp. $>$), then $\bar{\rhd} = >$(resp. $\geq$). Note also that the soundness of the rules $\Diamond$ and $\neg\Box$ when $\rhd = \geq$ relies on the finitary assumption of the models. Thus, we have

Theorem 2 *An annotated clause is tableau provable iff it is finitely valid.*

4 Conclusion

We have presented two hybrid logics for possibilistic and similarity-based reasoning. The first one is an extension of graded modal logic based on possibility and necessity measures of crisp proposition. Though is is a kind of multi-modal logic, the semantics is two-valued, so a tableau system for the logic is analogous to that for basic hybrid logic except the interaction of modalities. The second one is an extension of fuzzy modal logic. Though its syntax is same as the basic hybrid logic, it has a many-valued semantics, so we have to develop its tableau proof method by using annotated formulas. The labelled deduction discipline is blended into the hybrid logic and has produced fruitful results for classical modal logic. Since possibilistic and similarity-based reasoning has very natural modal formulation, it is expected that they can also benefit from the hybrid paradigm.

1. Boolean rules:

$$\frac{\alpha \rhd c}{\begin{array}{c}\alpha_1 \rhd c\\ \alpha_2 \rhd c\end{array}} \qquad \frac{\beta \rhd c}{\beta_1 \rhd c \mid \beta_1 \rhd c}$$

2. Modal rules:

$$\Diamond : \quad \frac{(s : \Diamond\varphi) \rhd c}{\begin{array}{c}(s : \Diamond a) \rhd c\\ (a : \varphi) \rhd c\end{array}} \qquad \neg\Box : \quad \frac{(\neg s : \Box\varphi) \rhd c}{\begin{array}{c}(s : \Diamond a) \rhd c\\ (\neg a : \varphi) \rhd c\end{array}}$$

$$\Box : \quad \frac{(s : \Box\varphi) \rhd c, \ \ (s : \Diamond t) \unrhd c}{t : \varphi \rhd c} \qquad \neg\Diamond : \quad \frac{(\neg s : \Diamond\varphi) \rhd c, \ \ (s : \Diamond t) \unrhd c}{(\neg t : \varphi) \rhd c}$$

$$Sat : \quad \frac{(s : t : \varphi) \rhd c}{t : \varphi \rhd c} \qquad \neg Sat : \quad \frac{(\neg s : t : \varphi) \rhd c}{(\neg t : \varphi) \rhd c}$$

3. Nominal rules: Ref and Sym are the same as above if we identify φ with $\varphi \geq 1$.

$$Nom : \quad \frac{s : t \geq 1, \ \ t : \varphi \rhd c}{s : \varphi \rhd c} \qquad Bridge : \quad \frac{(s : \Diamond t) \rhd c, \ \ t : u \geq 1}{(s : \Diamond u) \rhd c}$$

4. Bookkeeping rules:

$$\frac{\varphi > c}{\varphi \geq c} \qquad c > d : \quad \frac{\varphi \geq c}{\varphi > d}$$

Figure 2. The tableau rules for HFML

References

[1] P. Blackburn. “Discipline as logic: treating labels as first class citizens”. In D. Basin, M. D’Agostino, D. Gabbay, S. Matthews, and L. Viganó, editors, *Labelled Deduction*, pages 81–105. Kluwer Academic Publishers, 2000.

[2] P. Blackburn. “Representation, reasoning, and relational structures: a hybrid logic manifesto”. *Logic Journal of IGPL*, 8(3):339–365, 2000.

[3] D. Dubois, J. Lang, and H. Prade. “Fuzzy sets in approximate reasoning, Part 2: logical approaches”. *Fuzzy Sets and Systems*, 40:203–244, 1991.

[4] D. Dubois, J. Lang, and H. Prade. “Possibilistic logic”. In D. Gabbay, C. Hogger, and J. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming, Vol 3 : Nonmonotonic Reasoning and Uncertain Reasoning*, pages 439–513. Clarendon Press - Oxford, 1994.

[5] D. Dubois and H. Prade. “An introduction to possibilistic and fuzzy logics”. In P. Smets, A. Mamdani, D. Dubois, and H. Prade, editors, *Non-Standard Logics for Automated Reasoning*, pages 253–286. Academic Press, 1988.

[6] F. Esteva, P. Garcia, L. Godo, and R. Rodriguez. “A modal account of similarity-based reasoning”. *International Journal of Approximate Reasoning*, pages 235–260, 1997.

[7] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. D. Reidel Publishing Company, 1983.

[8] D. Gabbay. *Labelled Deductive Systems*. Clarendon Press, 1996.

[9] L. Godo and R. Rodriguez. “A fuzzy modal logic for similarity reasoning”. In G. Chen, M. Ying, and K. Cai, editors, *Fuzzy Logic and Soft Computing*, pages 33–48. Kluwer Academic Publishers, 1999.

[10] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publisher, 1998.

[11] P. Hájek and D. Harmancová. “A many-valued modal logic”. In *Proceedings of the International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems(IPMU)*, pages 1021–1024, 1996.

[12] C. Liau and I. Lin. “Quantitative modal logic and possibilistic reasoning”. In B. Neumann, editor, *Proceedings of the 10th ECAI*, pages 43–47. John Wiley & Sons. Ltd, 1992.

[13] R. Smullyan. *First-Order Logic*. Springer-Verlag, 1968.

[14] H. Thiele. “On the definition of modal operators in fuzzy logic”. In *Proceedings of the 23rd International Symposium on Multiple-Valued Logic*, pages 62–67, 1993.

[15] L. Zadeh. “Fuzzy sets as a basis for a theory of possibility”. *Fuzzy Sets and Systems*, 1(1):3–28, 1978.