Proof Methods for Reasoning about Possibility and Necessity

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Key Words : Fuzzy set, Possibilistic logic, Modal logic, Kripke semantics, Tableau method, Natural deduction method.

Abstract

Possibilistic logic is a quantitative method for uncertainty reasoning which is closely related to Zadeh's fuzzy set theory. In this paper, we formulate it as a kind of multimodal logic and develop some proof methods for it, including tableau method and two styles of natural deduction methods. The completeness and soundness of these methods are proved. Finally, some potential applications and the possible research directions are pointed out.

1 Introduction

In the last thirty years, many uncertainty reasoning schemes have been proposed by logicians, scientists and engineers. Among them, possibilistic logic, which is proposed by Dubois and Prade[7] and is closely related to Zadeh's fuzzy set theory[29], is one of the most important approaches. The logic arises naturally in the following scenario[5].

Consider a state-descriptive information system(SDIS) $\Im = \langle U, PV\rangle$, where U is the set of all possible states of the world under consideration, and PV is a set of propositional variables which describe somethings about the world. For each $u \in U$, assume that u is complete with respect to PV in the sense that for each $p \in PV$, p is either true or false under the state u, we will write $u(p) = 1$ (resp. $=0$) if p is true (resp. false) under state u. Then, the *truth set* of the proposition p, written as $|p|$, is definied as the set of all states u such that $u(p) = 1$. Let us define a *query formula* f to the SDIS as a Boolean combination of the propositions in PV. The truth value of f under a state u is completely determined by the classical logic truth table and the truth values of the propositional variables under u. Thus, the definition of the truth set can be extended to any queries accordingly.

Now, if a piece of fuzzy information is stored in the SDIS, then the truth value of a formula f can be determined according to the stored information. A piece of fuzzy information μ is defined as a fuzzy subset (i. e. a possibility distribution) of U. Formally, $\mu : U \to [0, 1]$ is a piece of fuzzy information. The fuzzy information μ is called normalized iff $\sup_{u\in U} \mu(u) = 1$. In general, we require the fuzzy information is normalized. In this case, two measures Π and $N : 2^U \to [0, 1]$ are defined as follows,

$$\Pi(X) = \sup_{u\in X} \mu(u)$$

$$N(X) = 1 - \Pi(U\backslash X).$$

Let $\Pi(f) = \Pi(|f|)$ and $N(f) = N(|f|)$, then the following properties hold for all formulas f, f_1, and f_2:

(i) $\Pi(f)$ and $N(f) \in [0, 1]$

(ii) $N(f) = 1 - \Pi(\neg f)$,

(iii) $\Pi(\top) = 1$ and $\Pi(\bot) = 0$, and

(iv) $\Pi(f_1 \vee f_2) = \max(\Pi(f_1), \Pi(f_2))$.

By attaching two measures to classical logic formulas, possibilistic logic is obtained. More specifically, if f is a well-formed formula (wff) of propositional logic, then $(f(Nc))$ and $(f(\Pi c))$ where $c \in [0,1]$ are wffs of possibilistic logic. When f is a classical clause, the wffs are called possibilistic clauses. The intended meanings of the wffs are $N(f) \geq c$ and $\Pi(f) \geq c$ respectively. Then, the resolution rule of possibilistic logic is defined between two possibilistic clauses as follows:

$$\frac{(f\ w_1) \quad (g\ w_2)}{(R(f,g)\ w_1 * w_2)}$$

where $R(f,g)$ is the classical resolvent of f and g, and $*$ is defined by

$$\begin{aligned}
&(N\ c) * (N\ d) = (N\ min(c,d)) \\
&(N\ c) * (\Pi\ d) = \begin{cases} (\Pi\ d) & if\ c + d > 1 \\ (\Pi\ 0) & if\ c + d \leq 1 \end{cases} \\
&(\Pi\ c) * (\Pi\ d) = (\Pi\ 0)
\end{aligned}$$

By the axiom (iv) above, any wff of the form $(f(Nc))$ can be converted into an equivalent set of possibilistic clauses, so if only necessity measure N is concerned, the refutational completeness of this resolution rule can be proved[5]. However, if possibility-valued wffs are involved, then the completeness result does not hold in general. The following simple example will explain this point further.

Example 1.1 Consider the assumption set $B = \{(p \wedge q(\Pi 0.7)), (p \wedge q \supset r\ (N\ 0.6))\}$ and the wff $f = (r\ (\Pi\ 0.7))$ in possibilistic logic. Then obviously, f should be derivable from B.[1] However, since $(p \wedge q\ (\Pi\ 0.7))$ is not a possibilistic clause, the resolution rule can not be applied. A possible approach to overcome the difficulty may be to infer $(p\ (\Pi\ 0.7))$ and $(q\ (\Pi\ 0.7))$ firstly. This indeed improves on some situations if the second sentence is either $(p \supset r\ (N\ 0.6))$ or $(q \supset r\ (N\ 0.6))$, but it is useless for the present example

To lift the restriction of clausal form, we must consider more general deduction methods. Some non-clausal deduction methods for modal logics have been well studied by Fitting[13], so by formulating possibilistic reasoning as a kind of multimodal logic, we can generalize Fitting's methods to possibilistic logic. Recently, some modal formulations of possibilistic logic have been proposed[12, 21]. Here, we will review one of them, called quantitative modal logic(QML), and develop some nonclausal deduction methods for it. These include tableau methods and natural deduction methods.

In what follows, we will first review the syntax and semantics of QML. Then the advantages of QML in possibilistic reasoning is discussed. A basic theorem for the completeness of QML deduction methods, called model existence theorem, is proved in section 3. The above-mentioned proof methods are presented in section 4 and 5 sequentially. Finally, some related works are discussed and a brief conclusion is given.

[1] Replacing "$p \wedge q$" with a new propositional letter and using the given resolution rule will justify this.

2 Quantitative Modal Logic

2.1 Syntax

To define a new logical language, we must specify its syntax and semantics.

First, the alphabet of QML consists of logical constants, connectives and a (denumerable) set of propositional variables.

- Logical constants: $\top$ (*verum* or *truth constant*) and $\bot$ (*falsum* or *falsity constant*).
- Propositional variables: $p, q, r, \cdots$, etc.
- Classical connectives: $\neg$ (negation), $\vee$ (or), $\wedge$ (and), and $\supset$ (implication).
- Quantitative modal operators:$[c], \langle c\rangle, [c]^+$, and $\langle c\rangle^+$ for all $c \in [0,1]$[2].

. Second, the well-formed formulas (*wffs*) of QML are defined by the following recursive syntactic rules:

- all propositional variables and logical constants are wffs, also called atomic formulas,
- if f_1 and f_2 are wffs, so are $\neg f_1, f_1 \wedge f_2, f_1 \vee f_2$, and $f_1 \supset f_2$,
- if f is a wff, so are $[c]f, \langle c\rangle f, [c]^+ f$, and $\langle c\rangle^+ f$ for all $c \in [0,1]$,
- expressions except those determined by above are not wffs.

We usually use lower case letters (sometimes with indices) p, q, r to denote atomic formulas and f, g, h to denote wffs. In the present case, wffs are also called sentences. Parentheses are punctuation symbols used to avoid ambiguity in the formation of wffs. The formala "$f \equiv g$" is taken as an abbreviation of "$(f \supset g) \wedge (g \supset f)$".

Now, we introduce some notations which will be used later.

Definition 2.1 *Let S be a set of wffs. Then*

1. *Define the positive subformula set of S, denoted by S^+, as the smallest set containing S and being closed under the following conditions:*
   (a) *if $\neg f, \langle c\rangle f, \langle c\rangle^+ f, [c]f$, or $[c]^+ f \in S^+$, then $f \in S^+$.*
   (b) *if $f \wedge g, f \vee g, f \supset g \in S^+$, then $f, g \in S^+$.*
2. *Define the negation subformula set of S, denoted by S^-, as the following set:*
$$S^- = \{\neg f \mid f \in S^+\}.$$
3. *Define $Sub(S) = S^+ \cup S^-$.*

[2]Here, c is in fact a numeral designator representing a real number in [0,1], not a number itself. However, we will not distinguish a number from the symbol representing it rigorously. Furthermore, to keep the language countable, we may restrict c to be a rational number. However, the results we will present still hold without such restriction.

4. Define the parameter value set of $S, V(S)$ as follows:

$$V(S) = \{c, 1-c \mid \exists f \text{ such that } [c]f, \langle c\rangle f, [c]^+ f, or \langle c\rangle^+ f \in Sub(S)\} \cup \{0,1\}.$$

Also, $Sub(f) = Sub(\{f\})$ and $V(f) = V(\{f\})$ if f is a single wff. Note that if S is finite, then $Sub(S)$ and $V(S)$ are, too. Furthermore, an atomic wff or its negation is called a literal.

Following the ideas in [13], we may classify the non-literal wffs of QML into six categories according to the formula's main connective which is used to combine its direct subformulas. The classification will simplify the presentation of definitions and theorems in section 3. We list the six categories of wffs and their corresponding subformulas in table 1–4. In each table, the original wffs appear in the left of the vertical line and their corresponding subformula(s) in the right. For example, a wff of the form $\neg(f \vee g)$ is an α-formula, and its two components, called α_1 and α_2, are $\neg f$ and $\neg g$ respectively, while a wff $\neg[0.4]^+p$ is a $\pi(0.6)$-formula, and its component, called $\pi_0(0.6)$, is $\neg p$.

2.2 Semantics

Here we consider the semantics of QML. Define a *possibility frame* $F = \langle W, R\rangle$, where W is a set of possible words and $R : W^2 \rightarrow [0,1]$ is a fuzzy accessibility relation on W. Let PV and FA denote the set of all propositional variables and the set of all wffs respectively. Then a model of QML is a triple $M = \langle W, R, TA\rangle$, Where $\langle W, R\rangle$ is a possibility frame and $TA : W \times PV \rightarrow \{0,1\}$ is a truth value assignment for all worlds. A proposition p is said to be true in a world w iff $TA(w,p) = 1$. Given R, we can define a possibility distribution R_w for each $w \in W$ such that $R_w(s) = R(w,s)$ for all $s \in W$. Similarly, we can also define TA_w for each $w \in W$ such that $TA_w(p) = TA(w,p)$ for all $p \in PV$. Thus, mathematically, a model can be equivalently written as $\langle W, \langle R_w, TA_w\rangle_{w\in W}\rangle$, and intuitively, TA_w describes the *ontological state* of the world w, while R_w reflects the *epistemic state* of the world w.

Given a model $M = \langle W, R, TA\rangle$, we can define the truth relation $\models_M \subseteq W \times FA$ as follows. First, for each world w and wff f, let $\Pi_w(f) = \sup_{s\in W}\{R_w(s) \mid s \models_M f\}$ and $N_w(f) = 1 - \Pi_w(\neg f)$

(1) $w \models_M p \Leftrightarrow TA(w,p) = 1, \forall p \in PV$,

(2) $w \models_M \top$ and $w \not\models_M \bot$,

(3) $w \models_M \neg f \Leftrightarrow w \not\models_M f$,

(4) $w \models_M f \wedge g \Leftrightarrow w \models_M f$ and $w \models_M g$,

(5) $w \models_M f \vee g \Leftrightarrow w \models_M f$ or $w \models_M g$,

(6) $w \models_M f \supset g \Leftrightarrow w \models_M \neg f$ or $w \models_M g$,

(7) $w \models_M \langle c\rangle f \Leftrightarrow \Pi_w(f) \geq c$,

Table 1: α wffs and their component formulas

α	α_1	α_2
$f \wedge g$	f	g
$\neg(f \vee g)$	$\neg f$	$\neg g$
$\neg(f \supset g)$	f	$\neg g$
$\neg\neg f$	f	f

Table 2: β wffs and their component formalas

β	β_1	β_2
$f \vee g$	f	g
$\neg(f \wedge g)$	$\neg f$	$\neg g$
$f \supset g$	$\neg f$	g

Table 3: ν and ν^+ wffs (with parameter c) and their component formulas

$\nu(c)$	$\nu_0(c)$	$\nu^+(c)$	$\nu_0^+(c)$
$.\ [c]f$	f	$[c]^+f$	f
$\neg\langle 1-c\rangle^+ f$	$\neg f$	$\neg\langle 1-c\rangle f$	$\neg f$

Table 4: π and π^+ wffs (with parameter c) and their component formulas

$\pi(c)$	$\pi_0(c)$	$\pi^+(c)$	$\pi_0^+(c)$
$\langle c\rangle f$	f	$\langle c\rangle^+ f$	f
$\neg[1-c]^+ f$	$\neg f$	$\neg[1-c]f$	$\neg f$

(8) $w \models_M \langle c\rangle^+ f \Leftrightarrow \Pi_w(f) > c$,

(9) $w \models_M [c]f \Leftrightarrow N_w(f) \geq c$,

(10) $w \models_M [c]^+ f \Leftrightarrow N_w(f) > c$.

Clauses (7)-(10) define the meaning of wffs with modal operators. Intuitively, $[c]f$ (resp. $[c]^+f$) means that $N(f) \geq c$ (resp. $N(f) > c$), and similarly, $\langle c\rangle f$ (resp. $\langle c\rangle^+ f$) denotes that $\Pi(f) \geq c$ (resp. $\Pi(f) > c$). Here, for convenience, we define $\sup \emptyset = 0$ and $\inf \emptyset = 1$.

A wff f is said to be valid in $M = \langle W, R, TA\rangle$, writen $\models_M f$ iff for all $w \in W, w \models_M f$. If S is a set of wffs, then $\models_M S$ means that for all $f \in S$, $\models_M f$. If $\mathbf{C}$ is a class of models and S is a set of of wffs, then we write $S \models_{\mathbf{C}} f$ to mean that for all $M \in \mathbf{C}$, $\models_M S$ implies $\models_M f$. A model with finite possible worlds is called finite model. A model is called serial iff for all w in W, $\sup_{s\in W} R(w,s) = 1$, and reflexive iff for all w in W, $R(w,w) = 1$. Throughout this paper, we use $\mathbf{K}$, $\mathbf{D}$, and $\mathbf{T}$ to denote the class of all models, serial models, and reflexive models. If $\mathbf{C}$ is a class of models, then $\mathbf{FC}$ denote the class of all finite models in $\mathbf{C}$. We also note that $\mathbf{T} \subseteq \mathbf{D} \subseteq \mathbf{K}$.

We can also see from the semantics that the main difference between QML and possibilistic logic is the former allow different possibility distributions for each world w, that is R_w, while the latter associate a constant possibility distribution to all worlds. Thus, we have enhanced the expressive power of standard possibilistic logic. For example, if from some information source, we get a rule " Smoking implies the possibility of cancer being at least 0.8", but the certainty (necessity) of the rule is at most 0.7 due to the reliability of the information source, then we can represent this fact as $\neg[0.7]^+(p \supset \langle 0.8\rangle q)$, where p and q denote "smoking" and "being cancered" respectively. Furthermore, the syntax make it easier to combine QML and other intensional logic. For instance, we can use $B\langle 0.6\rangle p$ and $\langle 0.6\rangle Bp$ to represent "the agent believes that the possibility of p is at least 0.6" and "the possibility of the agent believing p is at least 0.6" respectively. Recently, a complete axiomatic QML system for reasoning about higher order uncertainty have also been proposed[22]. This shows that QML has indeed some advantages on the representation of complex uncertainty wffs.

2.3 QML and Possibilistic Reasoning

To understand the relation between QML and possibilistic logic, we first show that all axioms for possibilistic logic can be realized in all serial models in the following sense.

Theorem 2.1 *(1)* $\models_{\mathbf{K}} [0]f \wedge \neg[1]^+ f \wedge \langle 0\rangle f \wedge \neg\langle 1\rangle^+ f$

(2) $\models_{\mathbf{K}} ([c]f \wedge \neg[c]^+ f) \equiv (\langle 1-c\rangle\neg f \wedge \neg\langle 1-c\rangle^+\neg f)$

(3) $\models_{\mathbf{D}} \langle 1\rangle\top$ *and* $\models_{\mathbf{K}} \neg\langle 0\rangle^+\bot$

(4) $\models_{\mathbf{K}} ([c](f\wedge g) \wedge \neg[c]^+(f \wedge g)) \equiv ([c]f \wedge \neg[c]^+ f \wedge [c]g) \vee ([c]g \wedge \neg[c]^+ g \wedge [c]f)$

Proof: The proof is nothing more a direct expansion according to the above-defined semantics. □

Since $\mathbf{D} \subseteq \mathbf{K}$, Theorem 2.1 is valid in al serial models. According to the possible-world semantics, Theorem 2.1 exactly corresponds to the four axioms for possibility and necessity measure in Section 1. Note that Theorem 2.1 (4) says that $N(f \wedge g) = c$ iff ($N(f) = c$ and $N(g) \geq c$) or ($N(g) = c$ and $N(f) \geq c$), i. e. $N(f \wedge g) = c$ iff $\min(N(f), N(g)) = c$, which is equivalent to say that $\Pi(f \vee g) = \max(\Pi(f), \Pi(g))$, since $N(f) = 1 - \Pi(\neg f)$. Thus, the consequence relation in serial models is exactly suitable for possibilistic reasoning. But how about $\mathbf{K}$ and $\mathbf{T}$? Recall that we can define a possibility distribution R_w for each world w in a possible world model according to its accessibility relation R, then the seriality of a model is equivalent to the normalization of all its corresponding possibility distribution[3]. Since the normalization requirement is what Dubois et al. impose on the semantics of possibilistic logic [9], the consequence relation in $\mathbf{K}$ is just the result of lifting this constraint. Thus, though the consequence relation for $\mathbf{K}$ is too weak for possibilistic reasoning, it can tell us which theorems of possibilistic logic may be derived without the normalization requirement. As for $\mathbf{T}$, since all reflexive models are also serial, we can do possibilistic reasoning by $\models_{\mathbf{T}}$. However, why the stronger consequence relation $\models_{\mathbf{T}}$ is useful? To understand this, let us generalize the notion of reflexivity. Given $e \in [0,1]$, we define a model $\langle W, R, TA \rangle$ as e-reflexive iff $\forall w \in W, R(w,w) \geq e$. We then have the following axiom AXT_e which is valid in all e-reflexive models.

$$\mathrm{AXT}_e : [1-e]^+ f \supset f$$

In general, AXT_{1-e} is called *e-threshold axiom* . It is named so because we accept a wff as true when its necessity measure is greater than e. Then AXT_1 , the strongest threshold axiom, i. e. 0-threshold axiom, is valid in all reflexive models. Although the results we will present for $\mathbf{T}$ may be generalized to any e-reflexive models , to keep simplicity and facilitate the comparison with ordinary modal logic, we will prove these results in $\mathbf{T}$ directly.

3 Model Existence Theorem

The model existence theorem provides a powerful tool to the proof of completeness for many formal systems, in particalar, for those without compactness, such as the infinitary logic $L_{\omega_1 \omega}$ [16]. Fitting first gives a systemic presentation of the model existence theorem for modal logic, and prove many important results using this theorem [13]. Interestingly, by modifying Fitting's argument, many results for ordinary modal logic may be transferred to QML without too much distortion.

The central idea of the model existence theorem is the so-called *consistency property.* To define it, we need the following functions. Throughout this section, let $F = Sub(G)$ for some finite set of wffs, G. Also recall the notation $V(F)$ from Section 2, and let $\alpha, \beta, \nu(c), \nu^+(c), \pi(c), \pi^+(c)$ denote a wff in the corresponding categories respectively.

Definition 3.1 *Define the world-alternative function K and strict world-alternative function K^+ on F as follows.*
$K, K^+ : 2^F \times V(F) \to 2^F$,

$$K(S,c) = \{\nu_0(c') \mid c' \geq c, \nu(c') \in S\} \cup \{\nu_0^+(c') \mid c' \geq c, \nu^+(c') \in S\}$$

[3] A possibility distribution $P : W \to [0,1]$ is normalized iff $\sup_{w \in W} P(w) = 1$.

and

$$K^{+}(S,c)=\{\nu_0(c') \mid c'>c, \nu(c')\in S\}\cup\{\nu_0^{+}(c') \mid c'\geq c, \nu^{+}(c')\in S\}.$$

Intuitively, $K(S,c)$ (resp. $K^{+}(S,c)$) contains all wffs f such that $N(f)\geq c$ (resp. $N(f)>c$) can be derived from S by using only duality property of N and Π (i.e. axiom (ii) of Section 1) and the transitivity of inequality relation.

Definition 3.2 *Let $\Theta\subseteq 2^F$ be a collection of subsets of F. Then*

(1) Θ is called a classical consistency property on F iff for all $S\in\Theta, S$ satisfies the following three conditions:

(a) $\alpha\in S\Rightarrow S\cup\{\alpha_1,\alpha_2\}\in\Theta$,

(b) $\beta\in S\Rightarrow S\cup\{\beta_1\}\in\Theta$, or $S\cup\{\beta_2\}\in\Theta$, and

(c) for any atomic wff p, S does not contain p and $\neg\, p$ simultaneously, and none of the following wffs are in $S:\nu^{+}(1),\pi^{+}(1),\neg\top,\bot$.

(2) A classical consistercy property Θ is a K-consistency property iff for all $S\in\Theta, S$ satisfies, in addition to (a)-(c), the following two conditions:

(k1) $c>0$ and $\pi(c)\in S\Rightarrow K^{+}(S,1-c)\cup\{\pi_0(c)\}\in\Theta$, and

(k2) $\pi^{+}(c)\in S\Rightarrow K(S,1-c)\cup\{\pi_0^{+}(c)\}\in\Theta$.

(3) A T-consistency property is a K-consistency property with the following condition:

(t) $\forall S\in\Theta, f\in K^{+}(S,0)\Rightarrow S\cup\{f\}\in\Theta$.

(4) A D-consistency property is a K-consistency property with the following condition:

(d) $\forall S\in\Theta, K^{+}(S,0)\in\Theta$.

Let $B\subseteq F$ and $L=K,T$, or D, then an L-consistency property Θ is called *B-compatible* iff for all $S\in\Theta$, and $f\in B$, $S\cup\{f\}\in\Theta$.

It can be checked that for any wff f, if $f\in F$, then the corresponding component formulas f_1 and f_2 (in the case of f being α or β wff) or f_0 (in the other cases) are all in F by the requirement of $F=Sub(G)$. This simple fact is critical to the well-definedness of the consistency property since we require that Θ is a subset of 2^F.

Obviously, the consistency property is closely related to the satisfiability of wffs. The intuition is formally stated as the following theorem. However, before the presentation of the main theorem, let us illustrate these definitions with a simple example.

Example 3.1 If Θ is a K-consistency property and $S=\{\langle 0.7\rangle p_1,[0.4]p_2,[0.3]^{+}p_3\}\in\Theta$, then from the consistency of S, we can infer that there is a model $M=\langle W,R,TA\rangle$ and a world $w\in W$ such that $w\models S$. Since $w\models\langle 0.7\rangle p_1$, under the assumption that W is finite (an assumption which can be made without loss of generality by Corollary 3.1), there exists another world u such that $R(w,u)\geq 0.7$ and $u\models p_1$. Now, because $w\models[0.4]p_2\wedge[0.3]^{+}p_3$, we have $N_w(p_2)\geq 0.4$ and $N_w(p_3)>0.3$, i.e., $\Pi_w(\neg p_2)\leq 0.6$ and $\Pi_w(\neg p_3)<0.7$, so for $u\in W$ such that

$R(w,u) \geq 0.7$, we have $u \models p_2 \wedge p_3$. Thus, $u \models K^+(S, 0.3) \cup \{p_1\}$ since $K^+(S, 0.3) = \{p_2, p_3\}$. That is, $K^+(S, 0.3) \cup \{p_1\}$ should be also consistent. This explains the $(k1)$ condition of definition 3.2(2). Other conditions of consistency properties can be explained essentially in the same way.

Theorem 3.1 *(***Model Existence theorem***) Let L=K,T,or D. If F is finite, B⊆F, and Θ is a B-compatible L-consistency property on F, then there exists a finite model $M = \langle W, R, TA \rangle$ in* **L** *such that $\models_M B$ and for all $f \in S \in \Theta$, there exists a world $w \in W$ such that $w \models_M f$.*

proof: To prove this theorem, we use a constructive argument, First, noting that Θ is ordered by $\subseteq$ and all elements of Θ is finite sets, we can construct the model $M_\Theta = \langle W, R, TA \rangle$ as follows:

W: the set of maximal members in Θ,

$TA(S,p) = 1 \Leftrightarrow p \in S, \forall S \in W$ and $p \in PV$, and

R must satisfy that for all $S, S' \in W$, and $c \in V(F)$:

(r1) if $c \neq 0$, then $K(S,c) \subseteq S' \Leftrightarrow R(S,S') > 1 - c$

(r2) $K^+(S,c) \subseteq S' \Leftrightarrow R(S,S') \geq 1 - c$.

Then, we use a series of lemmas to show that M_Θ is well-defined and satisfies the requirement of the theorem.

Lemma 3.1 *There exists R satisfying (r1) and (r2).*

proof: According to Definition 3.1, we have the following results for all $S \in 2^F$ and $c, c' \in V(F)$:

(i) $K^+(S,c) \subseteq K(S,c)$,

(ii) if $c \geq c'$, then $K(S,c) \subseteq K(S,c')$,

(iii) if $c \geq c'$, then $K^+(S,c) \subseteq K^+(S,c')$, and

(iv) if $c > c'$, then $K(S,c) \subseteq K^+(S,c')$

Let $V(F) = \{c_1, c_2, c_3, \cdots, c_n,\}$ such that $1 = c_1 > c_2 > c_3 > \cdots > c_n = 0$. Then, we have the following inclusion chain for any $S \in 2^F$:

$$\emptyset = K^+(S,c_1) \subseteq K(S,c_1) \subseteq \cdots \subseteq K^+(S,c_i) \subseteq K(S,c_i) \subseteq K^+(S,c_{i+1}) \subseteq \cdots \subseteq K^+(S,c_n)$$

Thus, given S and S', there are three possible cases:

Case 1: for some i such that $1 \leq i \leq n-1$, $K^+(S,c_i) \subseteq S'$ and $K(S,c_i) \not\subseteq S'$. In this case, the only value of $R(S,S')$ which satisfies the conditions (r1) and (r2) is $1 - c_i$.

Case 2: for some i such that $1 \leq i \leq n-1$, $K(S,c_i) \subseteq S'$ and $K^+(S,c_{i+1}) \not\subseteq S'$. In this case, if $1 - c_i < R(S,S') < 1 - c_{i+1}$) then (r1) and (r2) can be satisfied.

Case 3: if $K^+(S,c_n) \subseteq S'$, then $R(S,S') = 1$

□

Lemma 3.2 *If Θ is a T-consistency property, then M_Θ is a reflexive model.*

proof: Since any S in W is the maximal member of Θ, then by condition (t), we have $K^+(S,0) \subseteq S$, and this implies $R(S,S) = 1$ by (r2). $\Box$

Lemma 3.3 *If Θ is a D-consistency property, then M_Θ is a serial model.*

proof: Since for any S in W, $K^+(S,0) \in \Theta$ by condition (d), and each element of Θ is a finite set, we can find the maximal extension of $K^+(S,0)$, say S', in Θ. Then $S' \in W$, and $K^+(S,0) \subseteq S'$, so $R(S,S') = 1$ and $\sup_{S' \in W} R(S,S') = 1$. $\Box$

Lemma 3.4 *For any $S \in W$ and wff $f \in S$, $S \models_{M_\Theta} f$.*

proof: By induction on the structure of f. Note here S refer both to a set of wffs and a world in model M_Θ. For convenience, we drop the subscript M_Θ in the following proof, and write $S \models f$ only to mean that f is true in the world S. We consider the following exhaustive cases.
Case 1: f is an atomic wff. By the definition of $M_\Theta, TA(S,f) = 1$, since $f \in S$. Thus, $S \models f$ according to the possible world semantics.
Case 2: $f = \neg p$ for some atomic wff p. By condition (c) of Definition 3.2, $\neg p \in S$ implies $p \notin S$, and so $TA(S,p) = 0$, This in turn implies $S \models f$.
Case 3: f is an α wff. Then $S \cup \{\alpha_1, \alpha_2\} \in \Theta$, and since S is a maximal member of Θ, this means $\alpha_1, \alpha_2 \in S$. By induction hypothesis, $S \models \alpha_1$ and $S \models \alpha_2$, so $S \models f$.
Case 4: f is a β wff. Then $S \cup \{\beta_1\} \in \Theta$ or $S \cup \{\beta_2\} \in \Theta$, so $S \models \beta_1$ or $S \models \beta_2$ by the maximality of S. Thus, $S \models f$.
Case 5:f is a $\pi(c)$ wff with $c > 0$. (note that if $c = 0, S \models \pi(c)$ naturally). Then $K^+(S, 1-c) \cup \{\pi_0(c)\} \in \Theta$, so let S' be the maximal extension of $K^+(S,1-c) \cup \{\pi_0(c)\}$ in Θ, we have $R(S,S') \geq c$ since $K^+(S,1-c) \subseteq S'$. Furthermore, $\pi_0(c) \in S'$, so $S' \models \pi_0(c)$ by induction hypothesis. Thus , $S \models f$, by the definition of $\pi_0(c)$ wffs and their semantics.
Case 6:f is a $\pi^+(c)$ wff with $c < 1$. (Note that if $c = 1$, then $\pi^+(1) \notin S$). This case is similar to case 5.
Case 7: f is a $\nu(c)$ wff. Let S' be a element of W. If $S' \models \neg\nu_0(c)$, then $\nu_0(c) \notin S'$ by induction hypothesis. This implies $K(S,c) \not\subseteq S'$ since $\nu_0(c) \in K(S,c)$ by definition, and in turn $1 - R(S,S') \geq c$ by the definition of M_Θ. Thus we have $\inf\{1 - R(S,S') \mid S' \models \neg\nu_0(c), S' \in W\} \geq c$, i. e. $S \models f$.
Case 8:f is a $\nu^+(c)$ wff. The proof is similar to that of Case 7, except that the property of W being finite is used.$\Box$

Note that our induction basis are literals (i. e. Case 1 and 2), not just atomic formulas, so we can infer the results in all other cases.

Now, we can complete the proof of the model existence theorem. First, since Θ is B-compatible, we have $B \subseteq S$ for any $S \in W$, by the maximality of S. Thus for all $S \in W, S \models f$ if $f \in B$ by Lemma 3.4. That is, $\models_{M_\Theta} B$. Moveover, since for all $f \in S \in \Theta$, there exists a maximal extension of S, say S', in W, so $S' \models f$ by Lemma 3.4. Finally, W is finite and M_Θ is a model in **L** when Θ is an L-consistency property by Lemma 3.2 and 3.3. $\Box$

The following corollary is proved in [21] by the soundness and completeness of the axiomatic systems.

Corollary 3.1 $B \vdash_L f \Leftrightarrow B \models_{\mathbf{L}} f \Leftrightarrow B \models_{\mathbf{FL}} f$

This corollary is useful in the correctness of proof methods developed later and in the proof of the decidability theorem for QML. We will only consider the decidability theorem in this section, and devote the following sections to the development of the proof methods. As above, we assume $L = K, T$, or D, and let B be a finite set of wffs, f be a wff, and $F = Sub(B \cup \{f\})$.

Lemma 3.5 *$B \not\models_{\mathbf{L}} f$ iff there exists a B-compatible L-consistency property Θ on F such that $\{\neg f\} \in \Theta$.*

Proof: ($\Leftarrow$ part) This is just a restatement of Theorem 3.1.
($\Rightarrow$ part) $B \not\models_{\mathbf{L}} f$ implies $B \not\models_{\mathbf{FL}} f$ by Corollary 3.1. This in turn implies the existence of a finite L-model $M = \langle W, R, TA \rangle$ such that $\models_M S$ and $w \models_M \neg f$ for some $w \in W$. We can define Θ as follows:

$$\Theta = \{S \subseteq F \mid w \models_M S, w \in W\}$$

We can show that Θ is a B-compatible L-consistency property. Then, by the definition of Θ, we have $\{\neg f\} \in \Theta$. As for the process of verifying that Θ satisfies the appropriate conditions in Definition 3.2, it is routine. We only point out that the finiteness of M is used in the verification of condition (k1) and (d), and leave the detail to the reader.□

Theorem 3.2 *The problem " $B \models_{\mathbf{L}} f$? " is decidable.*

Proof: Since F is finite, we can test each subset Θ of 2^F to see whether it is a B-compatible L-consistency property (by checking the appropriate conditions in Definition 3.2) and whether $\{\neg f\} \in \Theta$ (by membership test). Since each test step is decidable and the process is finitely terminating, the problem is decidable.□

4 Tableau Method for Analytic Systems

Let $L = K, D$, or T. Define the analytic system L as the set of all wffs f such that $\models_{\mathbf{L}} f$. The remainder of the paper is devoted to the development of different proof methods for analytic systems. In this section, we first consider the tableau method. This is essentially a proof-tree construction approach. In this method, an attempted proof of a wff is to place the negated form of the wff as the root of the proof tree, and tableau rules are applied to extend the branches of the tree until some closed condition is encountered, or no more rules are applicable. Then, if all branches of the tree is closed, then the wff is proved as a theorem of the system. Otherwise, an open branch of the tree will construct a countermodel of the wff. In fact, in the presentation of Fitting's tableau method for ordinary modal logic [13], the model existence theorem is derived intuitively from the method by considering the consistency property as the collection of possible open branches. The main feature of the analytic systems is that the tableau rules are completely dependent on the component formulas of the parent node. Thus, if we label nodes of the proof tree by the wffs derived according to the rules, every node of the proof tree is labelled by a wff in $Sub(f)$, where f is the label of the root. Consequently, the proof tree is finite, and the proof process will terminate. These systems are named as analytic just because we can derive the proof of a theorem by *analyzing* the structure (i. e. the subformulas) of the theorem to be proved.

4.1 Tableau Method for Propositional Logic

In Smullyan [26], the tableau methods for propositional logic has been extensively developed. Then, Fitting [13] successfully extends the method to modal logic. Here, we first recall Smullyan's method. Then in the next subsection, we prove the soundness and completeness of the tableau method for QML's analytic systems.

Smullyan's method is consist of a set of tableau rules. When some branch of the proof tree contains the premises of some rule, the conclusions of the rule can be added to end of the branch. Essentially, there are two kinds of rules. The first ones, *the extension rules*, are to add the conclusions to the end of a branch direectly, and the added nodes are organized as a branch and attached to the end of the original branch. The second ones, *the forking rules*, are to add the conclusions as the sons of the end node of the original branch. In the propositional case, all forking rules have exactly two conclusions, so one is the left son and the other is the right son, and our proof tree is binary. The tableau rules for propositional logic[13] is presented in Figure 1.

1. Extension rules:

$$\wedge : \frac{f \wedge g}{\begin{array}{c} f \\ g \end{array}} \qquad \neg\vee : \frac{\neg(f \vee g)}{\begin{array}{c} \neg f \\ \neg g \end{array}} \qquad \neg\neg : \frac{\neg\neg f}{f} \qquad \neg \supset: \frac{\neg(f \supset g)}{\begin{array}{c} f \\ g \end{array}}$$

2. Forking rules:

$$\vee : \frac{f \vee g}{f \mid g} \qquad \neg\wedge : \frac{\neg(f \wedge g)}{\neg f \mid \neg g} \qquad \supset: \frac{f \supset g}{\neg f \mid g}$$

Figure 1: The Tableau Rules for Propositional Logic

Observing that all premises of the extension rules are α wffs and all premises of the forking rules are β wffs, we can summarize the tableau rules of propositional logic into two rules in accordance with the classification of propositional wffs. In other words, the tableau rules are the following α and β rules.

$$\alpha \text{ rule} : \frac{\alpha}{\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array}} \qquad \beta \text{ rule} : \frac{\beta}{\beta_1 \mid \beta_2}$$

Now, a *tableau* (or a *proof tree*) is a labelled binary tree whose nodes are labelled by wffs. For convenience, we identify the nodes of a tableau with their labels. A tableau rule is applicable to a tableau when some branch of the tableau contains the premise of the rule. After the application of an α rule, the new tableu is obtained by attaching the conclusions to the end of the branch to which the rule is applied, where one conclusion is the son of the old branch end, and the other conclusion is the grandson of it. After applying a β rule, the new tableau is obtained by attaching the two conclusions as the left and right son of the old branch end. Thus, a tableau rule transforms an old tableau into a new one. However, the transformation is nondeterministic, i.e., there may be many rules applicable to a tableau in the same time and one real application of them results in the new tableau. A *derivation* is thus a sequence of tableaus $T_1, T_2, \cdots, T_n$ such that T_1 is a single node (called the root) tree, and T_i is the resultant tableau of applying some rule which is applicable to T_{i-1} for each $1 < i \leq n$. A branch in a tableau is called *(classically) closed* if it satisfies one of the following three conditions:

- It contains p and $\neg p$ for some propositional variable p
- It contains $\bot$
- It contains $\neg\top$

A tableau is called (classically) closed if each branch of it is closed. A *tableau proof* for a wff f is a derivation $T_1, T_2, \cdots, T_n$ such that T_1 is a single node labelled as $\neg f$ and T_n is closed. We will use $\vdash_{PL^t} f$ to mean that there exists one tableau proof for f.

4.2 Tableau Methods for QML Analytic Systems

The tableau method for propositional logic is also called semantic tableau method because the tableau rules reflect the truth conditions of any wffs in a given world. For example, α rule means an α wff is true in a world iff its component wffs are both true in the world. However, when we turn to QML, we will need to consider the truth condition of wffs in different worlds. For example, by possible world semantics, if $\langle 0.5\rangle^+ f$ is true in a world w, then there exists one world w' such that $R(w, w') > 0.5$ and f is true in w'. Thus, we need to process the truth set in different possible worlds.

The most obvious approach is to allow the creation of *alternate tableaus* during the course of a derivation. This may be accomplished by allowing the nesting of tableaus, and the resultant proof procedure is just a combination of tableau method for propositional logic and the recursive call mechanism. In other words, we start with an attempted proof of a wff in QML, and the classical tableau rules is applied unproblematically, but when a modal wff is derived in the course, we may also switch to an alternate world, and the proof is continued in the alternate world. The difficulty of this method is that we need to keep a lots of alternate tableaus in the same time, and the proof will get rather complex even a simple wff is given at the beginning of the proof.

Fortunately, for the analytic systems, having jumped from one world to another, we do not need to go back again. This suggests that in the tableau methods for these systems, if we create an alternate tableau at some point in the course of a derivation, we can forget about the earlier tableau to which it is an alternate. Thus we only need to consider a single tableau at any given time.

Observing this, Fitting suggests a further simplificcation. Rather than actually creating alternate tableaus and discarding the old tableau, we could simply update the original one to reflect conditions in each current world as we pass through it. This save the time to duplicate the common parts of the old and the new tableaus. However, what might these updating rules be like. Consider the wff $\langle 0.5\rangle^{+}f$ occurring on a branch of a tableau. Intuitively, the wff is true in the current world w under consideration. Thus, there should be a world w' such that $R(w, w') > 0.5$ and f is true in w'. Suppose we want to update the branch to reflect a jump from the world w to w'. Then, obviously, we should add f to the end of the branch. But the question is, in general what additional information may we take with us in such a jump? The answers may be different for different logics. In the case of the system K, let S be the set of wffs occurring on the branch to be modified, then by the definition of possible world semantics, $K(S, 0.5)$ should be all true in the world w'. Thus, we should cross out all wffs in this branch and add $K(S, 0.5) \cup \{f\}$ to the end of it. In some cases the wffs to be added is a subset of the original branch. In those case, we just cross out the unnecessary wffs, and remain the needed wffs.

However, because of the way of the tableaus being written as a tree, it may happen that a branch is to be modified and part of it is common to several other branches which are not to be modified. In this case one can add to the ends of the branches that are not to be altered fresh occurrences of the wffs that will be crossed out, then the above-mentioned process of crossing-out and addition can be executed unproblematically.

Thus, in addition to the α and β rules, QML analytic systems contains tableau rules listed in Figure 2.

We can use the classification of wffs to simplify the presentation of the tableau rules as follows:

1. π rules

$$\pi(c) : (c > 0) \qquad \frac{S, \pi(c)}{K^{+}(S, 1-c), \pi_0(c)} \qquad\qquad \pi^{+}(c) : \qquad \frac{S, \pi^{+}(c)}{K(S, 1-c), \pi_0^{+}(c)}$$

2. ν rule for D:

$$\frac{S}{K^{+}(S, 0)}$$

3. ν rule for T:

$$\nu(c) : (c > 0) \qquad \frac{\nu(c)}{\nu_0(c)} \qquad\qquad \nu^{+}(c) : \qquad \frac{\nu^{+}(c)}{\nu_0^{+}(c)}$$

Now, we have three kind of rules for QML tableau method. The first ones are the extension rules which include α rules and the ν rules for the system T. The second ones are the forking rules which include β rules. The last ones are the modification rules which include π rules and the ν rule for the system D. The applicability condition of these rules are same. A rule is applicable to a tableau iff the *uncrossed-out* nodes in some branch of the tableau contains all premises of that rule. However, the effect of these rules are very different. The effect of extension and forking rules have been described in the preceding subsection. Let S and C denote the premise

1. Common Rules to K, D, and T:

   (a) If $c > 0$, then

$$\langle c \rangle : \quad \frac{S, \langle c \rangle f}{K^{+}(S, 1-c), f} \qquad\qquad \neg[1-c]^{+} : \quad \frac{S, \neg[1-c]^{+} f}{K^{+}(S, 1-c), \neg f}$$

   (b) If $c \geq 0$, then

$$\langle c \rangle^{+} : \quad \frac{S, \langle c \rangle^{+} f}{K(S, 1-c), f} \qquad\qquad \neg[1-c] : \quad \frac{S, \neg[1-c] f}{K(S, 1-c), \neg f}$$

2. Special Rules for D and T:

   (a) Rule for D

$$\frac{S}{K^{+}(S, 0)}$$

   (b) Rule for T

      i. If $c > 0$, then

$$[c] : \quad \frac{[c] f}{f} \qquad\qquad \neg\langle 1-c \rangle^{+} : \quad \frac{\neg\langle 1-c \rangle^{+} f}{\neg f}$$

      ii. If $c \geq 0$, then

$$[c]^{+} : \quad \frac{[c]^{+} f}{f} \qquad\qquad \neg\langle 1-c \rangle : \quad \frac{\neg\langle 1-c \rangle f}{\neg f}$$

Figure 2: Tableau Rules for QML Analytic Systems

and conclusion set of a modification rule respectively. If some branch of a tableau contains S, and let P be the set of all wffs on that branch, then the resultant tableau after applying the modification rule to that branch of the tableau is obtained as follows. First, for each wff in $P \backslash C$, add a new copy of the wff to the end of all other branches to which the wff belongs, and cross out all wffs in $P \backslash C$ on the branch to be modified. Then, the wffs in $C \backslash P$ is organized as a new branch and attached to the end of the modified branch.

The definition of a derivation is same as that for propositional logic, and a branch is called closed iff it is classically closed or it satisfies one of the following two conditions:

- It contains $[1]^+ f$, or $\langle 1\rangle^+ f$.
- It contains $\neg[0] f$, or $\neg\langle 0\rangle f$.

Let B be a set of wffs. Then the *global assumption rule* with B is: under any condition, the element of B may be added to the end of any branches of a tableau. Let $L = K, D$, or T. We say that f is provable with global assumption B in system L by tableau method, written as $B \vdash_{L^t} f$, iff there a derivation of a closed tableau from a single node $\neg f$ by tableau rules for L and the global assumption rule with B.

Next, we will construct the correctness and completeness of the tableau methods. But before continuing, let us consider a rule in some details to see why the tableau rules are correct intuitively. Consider the $\langle c\rangle^+$ rule. The premises of the rule are $S \cup \{\langle c\rangle^+ f\}$, i.e. the wffs in this set are true in some world w. This implies that there exists a world w' such that $R(w, w') > c$ and f is true in w' since by definition, $\sup_{u \in W}\{R(w, u) \mid u \models_M f\} > c$. However, any wff g in the set $K(S, 1-c)$ satisfies that $[1-c]g$ is true in w since by the premises S is true in w. This in turn implies that for any world u such that $u \models_M \neg g$, $R(w, u) \leq c$. Thus $\neg g$ must be false in w', i.e. g is true in w' for any wff g in $K(S, 1-c)$. This argument verify that there exists one world such that the conclusions of the $\langle c\rangle^+$ are true in it. However, the problem arise when we consider the $\langle c\rangle$ rule since $\sup_{u \in W}\{R(w, u) \mid u \models_M f\} \geq c$ does not implies the existence of a world w' such that $R(w, w') \geq c$ if the set W is infinite. Fortuntely, we have Corollary 3.1, which tells us that it is sufficient to consider only finite models.

To construct the soundness result, we need some definitions. Let $L = K, D, T$ and S be a set of wffs. Then S is called *finitely L-satisfiable under assumption B* iff there exists a finite model $M = \langle W, R, TA\rangle$ in $\mathbf{L}$ such that $\models_M B$ and there exists a world $w \in W$ such that $w \models_M S$. A branch of a tableau is finitely L-satisfiable under assumption B iff the set of uncrossed-out wffs on it is. A tableau is finitely L-satisfiable under assumption B iff there exists one branch of it which is so. We now have the following lemma.

Lemma 4.1 *Let L=K,D, or T. If T_1 and T_2 are two tableaus and T_2 is the result of applying some tableau rule for L to T_1, then T_2 is finitely L-satisfiable under assumption B if T_1 is, where B is the set with which the global assumption rule is applied.*

Proof: Suppose T_1 is finitely L-atisfiable under B, then there exists one branch of it finitely L-satisfiable under B. If T_2 is obtained from T_1 by applying some tableau rule to any branch except that branch, then obviously, that branch remain satisfiable in the original model, and the result follows. Now, if the rule is exactly applied to that branch, then there some cases to be considered. Let $M = \langle W, R, TA\rangle$ be a finite L-model such that $\models_M B$ and w be a world in W such that the uncrossed-out wffs on the branch are true in it. Then

α rules: the new branch extending the original one with α_1 and α_2 is still true in w since α is.

β rules: the new branch extending the original one with β_1 or that extending with β_2 are true in w since β is.

$\pi(c)$ rules: the set of uncrossed-out wffs on the original branch contains $S \cup \{\pi(c)\}$, so by assumption $w \models_M S \cup \{\pi(c)\}$. Since W is finite and $c > 0$, this means there exists one world w' such that $w' \models_M \pi_0(c)$ and $R(w, w') \geq c$. This in turn implies $w' \models_M K^+(S, 1-c) \cup \{\pi_0(c)\}$. Since the uncrossed-out wffs on the modified branch in tableau T_2 are exactly the wffs in this conclusion set, this means tableau T_2 is also finitely L-satisfiable under assumption B. The witness is the model M and the new world w'.

$\pi^+(c)$ rules: the proof is similar as above.

ν rule in D: in this case, M is assumed to be a finite serial model. The property of seriality and finiteness implies that there exists a world w' in W such that $R(w, w') = 1$. This results in $w' \models_M K^+(S, 0)$ since $w \models_M S$. Thus the modified branch in the new tableau T_2 wits the fact that T_2 is finitely D-satisfiable under asasumption B.

$\nu(c)$ rule in T: in this case, M is assumed to be a reflexive model. Since $w \models_M \nu(c)$ and $c > 0$, then by reflexivity of M, we have $w \models_M \nu_0(c)$. Thus the extended branch in the tableau T_2 is true in w.

$\nu^+(c)$ rule in T: the proof is similar as above.

global assumption rule with B: since the wffs in B are true in all worlds in W, the extended branch wits the finite L-satisfiability of T_2.

□

Theorem 4.1 *(Soundness theorem of tableau method for analytic systems) Let L=K,D, or T, and B be a finite set of wffs, then $B \vdash_{L^t} f$ implies $B \models_{\mathbf{L}} f$*

Proof: If $B \vdash_{L^t} f$ and $B \not\models_{\mathbf{L}} f$, then there exist a finite model $M = \langle W, R, TA \rangle$ in $\mathbf{L}$ and a world $w \in W$ such that $\models_M B$ and $w \models_M \neg f$ by corollary 3.1. This fact results in the initial tableau with a single node $\neg f$ is finitely L-satisfiable under assumption B, then by applying the above lemma repeatedly until the closed tableau is derived. However, it is impossible for a closed tableau to be finitely L-satisfiable under asumption B, so a contradiction follows. □

We can now turn to the completeness of the analytic systems. Suppose S is a finite set of wffs, then an L-tableau for S under assumption B is any tableau that begins with a single branch consist of all wffs in S and continues by applications of α, β, π ,ν rules for L, and global assumption rule with B. S is called *L-consistent* under B if no L-tableaus for S are closed.

Theorem 4.2 *(Completeness theorem of tableau method for analytic systems) Let L=K,D, or T, and B be a finite set of wffs. Then $B \models_{\mathbf{L}} f$ implies $B \vdash_{L^t} f$.*

Proof: Define

$$\Theta = \{S \mid S \subseteq Sub(B \cup \{f\}), S \text{ is } L\text{−consistent under } B\}.$$

We will show that Θ is an B-compatible L-consistency property by verifying that Θ satisfies the conditions in Definition 3.2. Let $S \in \Theta$, then there are no closed tableaus for S under B. Thus, we have

(a) if $\alpha \in S$ and $S \cup \{\alpha_1, \alpha_2\} \notin \Theta$, then there is a closed tableau for the second set. But we can apply α rule to S first and the resultant tableau is a single branch containing all wffs in $S \cup \{\alpha_1, \alpha_2\}$. Consequently, there is a closed tableau for S after all, and this is a contradiction.

(b) if $\beta \in S$, $S \cup \{\beta_1\} \notin \Theta$, and $S \cup \{\beta_2\} \notin \Theta$, then there are closed tableaus for $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ respectively. Then we can apply β rule to the initial tableau for S, and then derive the two closed tableaus for the two branches respectively. Since in our modification rules, the part to be crossed out which is common to other branches is copied to those branches, the derivation of the two branches do not interfere with each other. Thus the final tableau is a closed tableau for S.

(c) if S contains $\bot, \neg\top, \pi^+(1)$, or $\nu^+(1)$, then the initial tableau of S is just a closed tableau for it.

(k1) if $c > 0$ and $\pi(c) \in S$, then we can apply $\pi(c)$ rule to the initial tableau of S and the resultant tableau is just $K^+(S, 1-c) \cup \{\pi_0(c)\}$. Since S has no closed tableau, $K^+(S, 1-c) \cup \{\pi_0(c)\}$ has not, either. Thus, $K^+(S, 1-c) \cup \{\pi_0(c)\} \in \Theta$.

(k2) the condition is verified analogously.

(d) apply the ν rule for D to the initial tableau for S, the resultant tableau is $K^+(S, 0)$, and so $K^+(S, 0) \in \Theta$ since S has no closed D-tableau.

(t) if $f \in K^+(S, 0)$, then $f = \nu_0(c)(c > 0)$ or $f = \nu_0^+(c)$ for some $\nu(c)(c > 0)$ or $\nu^+(c) \in S$, so we can apply the $\nu(c)$ or $\nu^+(c)$ rule for T to attach f to the end of the initial tableau for S. Since S has no closed tableaus, $S \cup \{f\} \in \Theta$.

Finally, because of the global assumption rule with B, Θ is B-compatible.

Now, if $B \not\vdash_{L^t} f$, then $\{\neg f\} \in \Theta$, so by the model existence theorem, $B \not\models_{\mathbf{L}} f$. □

5 Natural Deduction Methods

Natural deduction methods are also important proof methods which have been extensively used in classical logic[25]. In general, the natural deduction method is a kind of forward reasoning method in which we deduce conclusions from some assumptions, and then these assumptions are discharged according to some deduction rules. The final form of the conclusion may be a conditional sentence. Thus, the main characteristic feature of natural deduction methods are that we can make assumptions at any point of the deduction process, deduce things from them, and later discharge the assumptions.

According to the description above, the natural deduction system will contain some assumption-discharging rules. Generally, the rules of a natural deduction system may be classified into two kinds. One are the elimination rules in which the conclusions are structurally simpler than the premises because of the elimination of some connectives. The other are the introduction rules in which the conclusions are structurally more complicated than the premises because of the introduction of connectives. The rules to discharge assumptions are in general a kind of introduction rules since discharging some assumptions will make the conclusion more complex to express the assumptions explicitly in the conclusion.

There are different formats to express a natural deduction proof. Here we will choose the one developed by Fitting[13] for modal logic and modify it to meet our need.

First, a deduction is a sequence of wffs in boxes. The first item in a box is an assumption and other wffs in a box are derived according to the rules of the natural deduction systems. The boxes may be nested. When a new assumption is introduced, a new box must be created, with that assumption as the first item of the created box. On the other hand, when an assumption is discharged by the assumption-discharge rules, a box created for the assumption should be closed off, and the conclusion is written just below the box. A *natural deduction proof* (abbreviated as proof throughout this section) is a deduction in which all assumptions have been discharged.

Although the boxes may be nested, they should not be overlapped. In other words, a box should be closed off only when all boxes inside it have been closed off. A wff f and a box $\mathbf{B}$ are said to be *at the same nest level* if $\mathbf{B}$ and f are inside the same box *directly*. Here, "directly inside a box $\mathbf{B}$" means " inside $\mathbf{B}$, but not inside any boxes inside $\mathbf{B}$".

The natural deduction rules for classical logic[13] is presented in Figure 3.

The last line of a proof is the wff that have been proved by the deduction. The global assumption rule with a set of wffs B is that any member of B may be used as a line at any point of the deduction. We write $B \vdash_{PL^n} f$ to denote that there is a proof of the wff f with the global assumption B in the classical natural deduction system. Some important properties of classical natural deduction system have been studied extensively. We restate some of them from [13] for the sake of reference.

Lemma 5.1 *1. If $B, f \vdash_{PL^n} g$ and $B, \neg f \vdash_{PL^n} g$, then $B \vdash_{PL^n} g$.*

2. (Reverse rule) The following assumption discharging rule holds.

$$\begin{array}{c}\boxed{\begin{array}{c}\neg f\\ \vdots\\ \bot\end{array}}\\ f\end{array}$$

Lemma 5.2 *1. if $B, \alpha_1, \alpha_2 \vdash_{PL^n} f$, then $B, \alpha \vdash_{PL^n} f$*

2. if $B, \beta_1 \vdash_{PL^n} f$ and $B, \beta_1 \vdash_{PL^n} f$, then $B, \beta \vdash_{PL^n} f$

Now, we can develop the natural deduction method for the QML analytic systems on the classical deduction basis. We call a deduction or derivation in a box as a subordinate deduction (or just subdeduction). To develop the natural deduction method for the QML systems, we

1. Iteration rule: in a deduction, if f and box $\mathbf{B}$ are at the same nest level and f occurs above $\mathbf{B}$, then f may be repeated directly inside the box $\mathbf{B}$.

2. Contradiction rules:

$$\frac{\begin{array}{c} f \\ \neg f \end{array}}{g} \qquad\qquad \frac{\bot}{g} \qquad\qquad \frac{\neg\top}{g}$$

3. Elimination rules:
   α elimination rules:

$$\wedge E: \ \frac{f \wedge g}{\begin{array}{c} f \\ g \end{array}} \qquad \neg\vee E: \ \frac{\neg(f \vee g)}{\begin{array}{c} \neg f \\ \neg g \end{array}} \qquad \neg\supset E: \ \frac{\neg(f \supset g)}{\begin{array}{c} f \\ \neg g \end{array}} \qquad \neg\neg E: \ \frac{\neg\neg f}{f}$$

   β elimination rules:

$$\vee E: \ \frac{\begin{array}{c} \neg f \\ f \vee g \end{array}}{g} \quad \frac{\begin{array}{c} \neg g \\ f \vee g \end{array}}{f} \qquad \neg\wedge E: \ \frac{\begin{array}{c} f \\ \neg(f \wedge g) \end{array}}{\neg g} \quad \frac{\begin{array}{c} g \\ \neg(f \wedge g) \end{array}}{\neg f} \qquad \supset E: \ \frac{\begin{array}{c} f \\ f \supset g \end{array}}{g} \quad \frac{\begin{array}{c} \neg g \\ f \supset g \end{array}}{\neg f}$$

4. Introduction rules:
   α introduction rules:

$$\wedge I: \ \frac{\begin{array}{c} f \\ g \end{array}}{f \wedge g} \qquad \neg\vee I: \ \frac{\begin{array}{c} \neg f \\ \neg g \end{array}}{\neg(f \vee g)} \qquad \neg\supset I: \ \frac{\begin{array}{c} f \\ \neg g \end{array}}{\neg(f \supset g)} \qquad \neg\neg I: \ \frac{f}{\neg\neg f}$$

   β introduction rules (assumption discharging rules):

$$\vee I: \ \begin{array}{c} \boxed{\begin{array}{c} \neg f \\ \vdots \\ g \end{array}} \\ f \vee g \end{array} \quad \begin{array}{c} \boxed{\begin{array}{c} \neg g \\ \vdots \\ f \end{array}} \\ f \vee g \end{array} \qquad \neg\wedge I: \ \begin{array}{c} \boxed{\begin{array}{c} f \\ \vdots \\ \neg g \end{array}} \\ \neg(f \wedge g) \end{array} \quad \begin{array}{c} \boxed{\begin{array}{c} g \\ \vdots \\ \neg f \end{array}} \\ \neg(f \wedge g) \end{array}$$

$$\supset I: \ \begin{array}{c} \boxed{\begin{array}{c} f \\ \vdots \\ g \end{array}} \\ f \supset g \end{array} \quad \begin{array}{c} \boxed{\begin{array}{c} \neg g \\ \vdots \\ \neg f \end{array}} \\ f \supset g \end{array} \qquad \neg I: \ \begin{array}{c} \boxed{\begin{array}{c} f \\ \vdots \\ \neg f \end{array}} \\ \neg f \end{array}$$

Figure 3: Natural Deduction System for Classical Logic

need another kind of boxes. The original kind of boxes represnts the subdeduction under some assumptions, while the new kind of boxes represent a subdeduction in a possible world. Since our possible worlds are connected by a fuzzy accessibility relation, we must be able to reflect the strength of the connection between two worlds. We say a world w' is *c-accessible* (resp. *c^+-accessible*) from the world w iff $R(w, w') \geq c$ (resp. $R(w, w') > c$).

Thus, we have two types of boxes in the QML natural deduction systems. The type 1 boxes are just those of the classical natural deduction system. The type 2 boxes are labeled by c or c^+ where $c \in [0, 1]$. Intuitively, the deduction process proceeding in a possible world, while entering a type 2 box labelled with c or c^+ means to start a deduction in a new possible world which is c-accessible or c^+-accessible from the original world. Type 1 boxes and type 2 boxes are distinguished by their labels. If a box has no any label, then it is a type 1 box, otherwise, it is a type 2 box.

When we are doing deductions in a possible world, we may want to jump to another world to do some deductions. There are al least two ways to do this. The first choice is to do deductions in a generic world which is c-accessible from the present world for some c. When returning from the new world, we get some general results about any worlds which are c-accessible from the present world, and the results are used in the present world to deduce other results. The other choice is to do the same thing in a particular world, and if we can derive some contradiction in the particular world, then the contradiction can be returned to the present world. Thus, two styles of QML natural deduction systems are induced. If the former approach is adopted, the obtained system is the so-called A-style natural deduction system. If we follows the latter approach, the resultant system is called I-style system. We will discuss the two styles of systems in the following subsections.

5.1 I-style Natural Deduction System

The I-style natural deduction systems for K, D, and T include all inference rules of the classical natural deduction system. However, any rules involving boxes (i.e. the iteration rule and the β introduction rules) refer only to the type 1 boxes. Thus, we must develop some rules for type 2 boxes. The following is a complete list of the common rules for I-style natural deduction K, D, and T systems:

1. Type 2 box creation rule: if a wff $\pi(c)$ (resp. $\pi^+(c)$) occurs in the course of a deduction, a type 2 box with label c (resp. c^+) may be created. The created box is at the same nest level as the occurrence of $\pi(c)$ (resp. $\pi^+(c)$), and $\pi_0(c)$ (resp. $\pi_0^+(c)$) is put at the first line of the box.

2. Iteration rules for type 2 boxes: if $\mathbf{B}$ is a type 2 box with label c (c^+), and S is the set of all wffs that occur above $\mathbf{B}$ and have the same nest level as $\mathbf{B}$. Then any menber of $K^+(S, 1-c)$ ($K(S, 1-c)$) may be repeated directly inside $\mathbf{B}$.

3. Contradiction return rule: if $\perp$ is derived in a type 2 box, the box my be closed off, and $\perp$ is returned outside.

4. Contradiction rules: four additional contradiction rules with respectively the premises $\neg[0]f, \neg\langle 0\rangle f, [1]^+f$, and $\langle 1\rangle^+f$ are included.

Furthermore, we have special rules for the system D and T. The special rule for T is directly presented in Figure 4. The special rule for D is as follows: one may creat a type 2 box labeled with 1 at any point of the deduction. For convenience, the first line of the so created box is $\top$. Schematically, these rules are presented in Figure 4

The definition of deduction, proof, and global assumption rule is almost same as that for classical logic. However, we require a proof is a deduction in which all boxes (type 1 or 2) have been closed. We use $B \vdash_{L^i} f$ to denote that f has a proof in the I-style system for L under the global assumption rule with set B. It is easy to construct the completeness of the I-style systems, so we consider it first.

Theorem 5.1 *Let B be a finite set of wffs and f be a wff. Then $B \models_{\mathbf{L}} f$ implies $B \vdash_{L^i} f$*

Proof: First, let $F = Sub(B \cup \{f\})$, and $\Theta = \{S \subseteq F \mid B \not\vdash_{L^i} \bigwedge S \supset \bot\}$. We will verify that Θ ia a B-compatible L-consistency property. Then, if $B \not\vdash_{L^i} f$, then $\{\neg f\} \in \Theta$. Otherwise, $B \vdash_{L^i} \neg f \supset \bot$, and by $\supset E$ and the reverse rule, we have $B \vdash_{L^i} f$ after all. Thus, by model existence theorem $B \not\models_{\mathbf{L}} f$.

To verify that Θ ia a B-compatible L-consistency property, we go through the following steps. First, the verification of conditions (a) and (b) follows from Lemma 5.2. Second, condition (c) can be verified by using $\wedge E$, contradiction, and $\supset I$ rules.

To verify condition (k1) and (k2), we present only one case of them, and the other is similar. assume $c > 0$ and $\pi(c) \in S \in \Theta$, but $K^+(S, 1-c) \cup \{\pi_0(c)\} \notin \Theta$, then the following is an I-style natural deduction proof of $B \vdash_{L^i} \bigwedge S \supset \bot$.

	1. $\bigwedge S$	; assumption
	2. $\pi(c)$	; 1., $\wedge E$, $\pi(c) \in S$
	3. $S \backslash \{\pi(c)\}$	; 1., $\wedge E$
c :	4. $\pi_0(c)$	; 2., type 2 box creation
	5. $K^+(S, 1-c)$	; 3., iteration
	6. $\bigwedge K^+(S, 1-c) \wedge \pi_0(c)$	; 4., 5., $\wedge I$
	7. $\bigwedge K^+(S, 1-c) \wedge \pi_0(c) \supset \bot$	; $K^+(S, 1-c) \cup \{\pi_0(c)\} \notin \Theta$
	8. $\bot$	; 6., 7., $\supset E$
	9. $\bot$	; $\bot$ return
	10. $\bigwedge S \supset \bot$	; 1., 9., $\supset I$

Note that line 3. of the deduction is a set of wffs, not just a wff. This is a convention to avoid listing all wffs line by line. The kind of convention will be used implicitly when writing a natural deduction proof.

1. Creation rules:

$$
\begin{array}{c}
\vdots \\
\pi(c)(c>0) \\
\vdots \\
c : \begin{array}{|c|}\hline \pi_0(c) \\ \vdots \end{array}
\end{array}
\qquad\qquad
\begin{array}{c}
\vdots \\
\pi^+(c) \\
\vdots \\
c^+ : \begin{array}{|c|}\hline \pi_0^+(c) \\ \vdots \end{array}
\end{array}
$$

2. Iteration rules:

$$
\begin{array}{c}
\vdots \\
f \in K^+(S, 1-c)(c>0) \\
\vdots \\
c : \begin{array}{|c|}\hline \vdots \\ f \\ \vdots \end{array}
\end{array}
\qquad\qquad
\begin{array}{c}
\vdots \\
f \in K(S, 1-c) \\
\vdots \\
c^+ : \begin{array}{|c|}\hline \vdots \\ f \\ \vdots \end{array}
\end{array}
$$

3. Return rule:

$$
\begin{array}{c}
\begin{array}{|c|} \vdots \\ \bot \\ \hline \end{array} \\
\bot
\end{array}
$$

4. Additional contradiction rules:

$$
\frac{\neg[0]f}{g} \qquad \frac{\neg\langle 0\rangle f}{g} \qquad \frac{[1]^+ f}{g} \qquad \frac{\langle 1\rangle^+ f}{g}
$$

5. Special rules for T:

$$
\frac{\nu(c)(c>0)}{\nu_0(c)} \qquad\qquad \frac{\nu^+(c)}{\nu_0^+(c)}
$$

6. Special rule for D:

$$
\begin{array}{c}
\vdots \\
1 : \begin{array}{|c|}\hline \top \\ \vdots \end{array}
\end{array}
$$

Figure 4: The I-style Natural Deduction Rules for Systems K, D, T

If $L = D$, to verify that condition (d) is satisfied. Suppose that $S \in \Theta$ but $K^+(S,0) \notin \Theta$, then we can find an I-style proof of $B \vdash_{L^i} \bigwedge S \supset \bot$ as follows:

$$\begin{array}{l}
\boxed{\begin{array}{ll}
1.\bigwedge S & ;\text{assumption}\\
2.S & ;1.,\wedge E\\
1: & \boxed{\begin{array}{ll} 3.\top & ;\text{creation rule for } D\\ 4.K^+(S,0) & ;2.,\text{iteration}\\ 5.\bigwedge K^+(S,0)\supset\bot & ;K^+(S,0)\notin\Theta\\ 6.\bot & ;4.,5.,\supset E\end{array}}\\
7.\bot & ;\bot\text{ return}
\end{array}}\\
8.\bigwedge S\supset\bot \quad ;1.,7.,\supset I
\end{array}$$

If $L = T$, condition (t) is verified as follows. Let $f \in K^+(S,0)$ and $S \cup \{f\} \notin \Theta$. We will show that $B \vdash_{L^i} \bigwedge S \supset \bot$. Because $f \in K^+(S,0)$ implies that $f = \nu_0^+(c)$ for some $c \geq 0$ and $\nu^+(c) \in S$ or $f = \nu_0(c)$ for some $c > 0$ and $\nu(c) \in S$. Thus, by special rule for T, f is derivable from S, and then $\bot$ is derivable from S. Finally, once again, the result follows by using $\supset I$ rule.

Furthermore, Θ is B-compatible because of the global assumption rule. □

The soundness of I-style systems is intuitively obvious. However, to construct the formal result, we need some more definitions.

Definition 5.1 *Given an I-style deduction for the wff f under the global assumption B. We can define the set of assumptioms alive at line n, A(n), for each line of the deduction as follows:*

(i) $A(0) = \emptyset$.

(ii) if line n is a menber of B or comes from a earlier lines by one of the iteration, contradiction, $\alpha I, \alpha E$, *or* βE *rules, then* $A(n) = A(n-1)$.

(iii) If part of the proof looks like

$$\begin{array}{c}
\vdots\\
(n)f_1\\
\boxed{\begin{array}{c}(n+1)f_2\\ \vdots\\ (m)f_3\end{array}}\\
(m+1)f_4
\end{array}$$

where (n) *represents that the line number is n, then* $A(n+1) = A(n) \cup \{f_2\}$ *and* $A(m+1) = A(n)$.

(iv) if part of the proof looks like

$$\begin{array}{rc}
 & \vdots\\
 & (n)f_1\\
c : (c^+ :) & \boxed{\begin{array}{c}(n+1)f_2\\ \vdots\\ (m)f_3\end{array}}\\
 & (m+1)f_4
\end{array}$$

then $A(n+1) = \emptyset$ *and* $A(m+1) = A(n)$.

Note that in each box of a deduction, there are two kinds of assumptions alive. The first one is that induced by the creation of a type 1 box, and the second one is those imported from the outside. According to the definition above, when we enter a type 1 box, the two kinds of assumptions are both included in the set of alive assumptions. However, when we enter a type 2 box, all assumptions outside it are forgetten. Thus, the alive assumption set in fact depends on the world in which we are doing deduction. Furthermore, if line (n) and line (m) are at the same nest level, then $A(n) = A(m)$ according to the definition.

In what follows, let $L = K, D$, or T, and B be a finite set of wffs which is used as the global assumption set of our deduction process.

Lemma 5.3 *(conditional validity lemma) Let P be a proof in I-style natural deduction system with B as the global assumption set, and $M = \langle W, R, TA \rangle$ be a finite model in* $\mathbf{L}$ *such that* $\models_M B$*. Then for each $w \in W$, we have $w \models_M \bigwedge A(n) \supset f$ for any line $(n)f$ occurring in P but not inside any type 2 boxes.*

Proof: We prove the lemma line by line through P inductively.

Induction basis: assume the first line of P is $(1)f$. Then it must be the cases that $f \in B$ or f is an assumption. If $f \in B$, then $w \models_M f$ since $\models_M B$, so the result follows since $A(1) = \emptyset$. If f is an assumption, then it is the first line of the outermost type 1 box, and by definition $A(1) = \{f\}$, so the result holds.

Induction step: assume that for all lines whose number $k < n$ and not inside any type 2 boxes, we have $w \models_M \bigwedge A(k) \supset f$. To prove the result holds for line (n). Since line (n) is not in any type 2 box, there are four cases to be considered as follows:

1. if line $(n)f$ is derived by one of the α introduction rules, α elimination rules, β elimination rules, contradiction rules, type 1 iteration rule, or special rules for T from some premise lines $(m_1)f_1$ and $(m_2)f_2$ (if the rule is one-premise rule, then only (m_1) is needed), then by definition, $A(m_i) \subseteq A(n)(i = 1, 2)$ where the containment is proper in the case of iteration rule. Since (n) is not inside any type 2 boxes, and $(m_1), (m_2)$ are either at the same nest level as (n) or is outside the type 1 box containing (n) , then $(m_1), (m_2)$ are not inside any type 2 boxes, either. Thus, the induction hypothesis implies $w \models_M \bigwedge A(m_i) \supset f_i$ for $i = 1, 2$, and this in turn implies $w \models_M \bigwedge A(n) \supset f_i$ for $i = 1, 2$. Now, we can check all the above-mentioned rules to see that $w \models_M f_1 \wedge f_2 \supset f$, so the result follows.

2. if $(n)f$ is the first item of a type 1 box as follows:

$$\begin{array}{c} \vdots \\ (n-1)f' \\ \boxed{\begin{array}{c} (n)f \\ \vdots \end{array}} \end{array}$$

then $A(n) = A(n-1) \cup \{f\}$, so $w \models_M \bigwedge A(n) \supset f$.

3. if $(n)f$ is derived by the β introduction (i.e. assumption discharging) rules as follows:

$$\begin{array}{c} (k)f_1 \\ \boxed{\begin{array}{c} (k+1)f_2 \\ \vdots \\ (n-1)f_3 \end{array}} \\ (n)f \end{array}$$

   then by induction hypothesis, $w \models_M \bigwedge A(n) \wedge f_2 \supset f_3$ since $A(n-1) = A(k+1) = A(k) \cup \{f_2\} = A(n) \cup \{f_2\}$. Note that these equations are derived from the facts that (k) and (n) are at the same nest level, and $(k+1)$ is the first line of a type 1 box while $(n-1)$ is the last line of the same box. Thus, $w \models_M \bigwedge A(n) \supset (\neg f_2 \vee f_3)$. This implies the result since all assumption discharging rules have the form so that $f \equiv \neg f_2 \vee f_3$ is valid in classical logic.

4. if $(n)\perp$ is returned from a type 2 box as follows: (the c^+ case is similar)

$$\begin{array}{rc} & \vdots \\ & (i)\pi(c) \\ & \vdots \\ & (k)f_1 \\ c: & \boxed{\begin{array}{c} (k+1)\pi_0(c) \\ \vdots \\ (n-1)\perp \end{array}} \\ & (n)\perp \end{array}$$

   Note that the last line of a type 2 box must be $\perp$ since the only return rule for it is the contradiction return rule. We will show that $w \not\models_M \bigwedge A(k)$. This in turn implies $w \models_M \bigwedge A(n) \supset \perp$ since $A(k) = A(n)$. Assume $w \models_M \bigwedge A(k)$, and let the type 2 box under consideration be **B**. Then by induction hypothesis, we have $w \models_M \bigwedge A(k) \supset \pi(c)$ since (i) and (k) are at the same nest level. Thus $w \models_M \pi(c)$. Since M is a finite model, we can find a world w' such that $R(w, w') \geq c$ and $w' \models_M \pi_0(c)$. This also implies $w' \models_M \bigwedge A(k+1) \supset \pi_0(c)$. Now, we can consider the box **B** as an independent deduction process except that the type 2 iteration rules may import wffs into it. Thus, we can restart the proof of the lemma from the first line of **B** recursively to prove that $w' \models_M \bigwedge A(j) \supset f$ for each line $(j)f$ in **B** but not inside any other type 2 boxes except those lines imported by the type 2 iteration rules. However, if line $(j)f$ is imported from the outside of **B**, then $f \in K^+(S, 1-c)$, where S is the set of all wffs above **B** and at the same nest level as it. Since $w \models_M \bigwedge A(k)$ and all wffs in S are at the same nest level as line (k), we have $w \models_M S$ by induction hypothesis. This in turn implies $w' \models_M f$. Thus $w' \models_M \bigwedge A(j) \supset f$ for each line $(j)f$ in **B** but not inside any other type 2 boxes. In particular, $w' \models_M \bigwedge A(n-1) \supset \perp$. However, $A(n-1) = A(k+1) = \emptyset$ since all assumption introduced in **B** have been discharged before line $(n-1)$, so $w' \models_M \perp$[4]. This is a contradiction, thus the assumption that $w \models_M \bigwedge A(k)$ is false. That is, $w \not\models_M \bigwedge A(k)$.

[4] Note that the rules forbid that a box is closed before all boxes inside it are closed.

□

According to the lemma, we can construct the soundness result.

Theorem 5.2 *Let B be a set of wffs and f be a wff. Then $B \vdash_{L^i} f$ implies $B \models_{\mathbf{L}} f$.*

Proof: By the preceding lemma, in particular, consider the last line of the proof. Assume $(n)f$ is the last line of the proof $B \vdash_{L^i} f$, since in a natural deduction proof for the I-style system, all boxes have been closed, line (n) does not occur in any type 2 boxes, and since all assumptions have been discharged, $A(n) = \emptyset$, we have $B \models_{\mathbf{FL}} f$ by the preceding lemma. Then the desired result follows from Corollary 3.1. □

5.2 A-style Natural Deduction System

The A-style natural deduction system still contains the classical deduction rules. It also use two types of boxes. The classical rules involve only type 1 boxes, while the rules for type 2 boxes are different from those for the I-style system.

The differences between the A-style system and the I-style system are listed as follows:

1. The (type 2 box) creation rule: a type 2 box with label $c (0 < c \leq 1)$ or $c^+ (0 \leq c < 1)$ may be created at any point of the proof. For convenience, the first item of the created box is $\top$.

2. The (type 2 box) return rule can be drawn schematically as follows:

$$c: \begin{array}{c} \boxed{\begin{array}{c} \vdots \\ \vdots \\ \nu_0^+(1-c) \end{array}} \\ \nu^+(1-c) \end{array} \qquad\qquad c^+ \begin{array}{c} \boxed{\begin{array}{c} \vdots \\ \vdots \\ \nu_0(1-c) \end{array}} \\ \nu(1-c) \end{array}$$

3. the special rule for D is as follows:

$$\frac{[0]^+ f}{\langle 1 \rangle f}$$

The other rules, including the type 2 box iteration rules, contradiction rules, and the special rules for the system T, are completely same as those of I-style natural deduction system.

The global assumption rule with set B is defined as above. The definition of deduction and proof is also same as that for I-style system. We use $B \vdash_{L^a} f$ to denote that there is a proof of f in the A-style system under the global assumption set B. Then the completeness theorem is constructed as follows.

Theorem 5.3 *If $B \models_{\mathbf{L}} f$ then $B \vdash_{L^a} f$*

Proof: Let $F = Sub(B \cup \{f\})$. Define

$$\Theta = \{S \subseteq F \mid B \not\vdash_{L^a} \bigwedge S \supset \bot\}.$$

We will verify that Θ is a B-compatible L-consistency property. Once again, the verification work is routine.

The verification of condition (a), (b), (c), (t), and B-compatibility are same as the I-style system.

To verify that the conditions (k1) and (k2) are satisfied, we show one of them as a sample. Assume $\pi(c) \in S \in \Theta$ but $K^+(S, 1-c) \cup \{\pi_0(c)\} \notin \Theta$ for some $c > 0$. Then by definition of Θ, we have $B \vdash_{L^a} \bigwedge K^+(S, 1-c) \wedge \pi_0(c) \supset \perp$. We can generate a proof of $B \vdash_{L^a} \bigwedge S \supset \perp$ from this as follows:

	$1. \bigwedge S$	; assumption
	$2. S$	; 1., $\wedge E$
$c:$	$3. K^+(S, 1-c)$	; type 2 iteration rule
	$4. \pi_0(c)$	; assumption
	$\vdots$	; proof of $\bigwedge K^+(S, 1-c) \wedge \pi_0(c) \supset \perp$
	$5. \perp$	; 3., 4., type 1 iteration, $\supset E$
	$6. \neg\pi_0(c)$	; 4., 5., reverse
	$7. \neg\pi(c)$	; 6., $\nu^+(1-c)$ return rule
	$8. \perp$	; 2., 7., contradiction rule($\pi(c) \in S$)
	$9. \bigwedge S \supset \perp$	; 1., 8., $\supset I$

If $L = D$, to verify condition (d), suppose that $S \in \Theta$ but $K^+(S, 0) \notin \Theta$, then the proof of $B \vdash_{L^a} \bigwedge S \supset \perp$ is as follows:

	$1. \bigwedge S$	; assumption
	$2. S$	; 1., $\wedge E$
$1:$	$3. \top$	; creation rule for D
	$4. K^+(S, 0)$	; 2., iteration
	$5. \bigwedge K^+(S, 0) \supset \perp$	; $K^+(S, 0) \notin \Theta$
	$6. \perp$	; 4., 5., $\supset E$
	$7. f$	; 6., contradiction
	$8. [0]^+ f$	; $\nu^+(0)$ return
$1:$	$9. \perp$	; repeat step 3.–6.
	$10. \neg f$	; 9., contradiction
	$11. \neg\langle 1\rangle f$	; $\nu^+(0)$ return
	$12. \langle 1\rangle f$	; 8., D rule
	$13. \perp$	; 11., 12., contradiction
	$14. \bigwedge S \supset \perp$	; 1., 13., $\supset I$

□

Finally, the soundness theorem of the A-style system is constructed by following an analogous argument as that for I-style system.

Lemma 5.4 *(conditional validity lemma) Let P be a proof in A-style natural deduction system with B as the global assumption set, and $M = \langle W, R, TA\rangle$ be a finite model in* **L** *such that $\models_M B$. Then for each $w \in W$, we have $w \models_M \bigwedge A(n) \supset f$ for any line $(n)f$ occurring in P but not inside any type 2 boxes.*

Proof: The proof is almost same as that for the I-style system except case 4 in which now we have

$$\begin{array}{c} \vdots \\ \vdots \\ (k) f_1 \\ c : \boxed{\begin{array}{c} (k+1)\top \\ \vdots \\ (n-1)\nu_0^+(1-c) \end{array}} \\ (n)\nu^+(1-c) \end{array}$$

In terms of that proof, if $w \models_M \bigwedge A(k)$, then we can show that $w' \models_M f$ for each world w' such that $R(w, w') \geq c$ and each wff f imported into the type 2 box **B**. Thus, for each world w' such that $R(w, w') \geq c$, we have $w' \models_M \nu_0^+(1-c)$, and this implies $w \models_M \nu^+(1-c)$. Therefore, we have $w \models_M \bigwedge A(n) \supset \nu^+(1-c)$ since again $A(n) = A(k)$. $\Box$

Following the lemma, the soundness theorem is proved in the same way as I-style system, so we just state it without proof.

Theorem 5.4 *Let B be a set of wffs and f be a wff. Then $B \vdash_{L^a} f$ implies $B \models_{\mathbf{L}} f$.*

6 Summary and Future Directions

6.1 Summary

We summerize the main results of this paper in the following theorem.

Theorem 6.1 *Let B be a finite set of wffs and f be a wff. Then for L=K, D, or T, the following statements are equivalent:*

1. $B \models_{\mathbf{L}} f$ *(f is valid in all L-models of B)*
2. $B \models_{\mathbf{FL}} f$ *(f is valid in all* finite *L-models of B)*
3. $B \vdash_{L^t} f$ *(f is derivable from B by L-tableau methods)*
4. $B \vdash_{L^i} f$ *(f is derivable from B by I-style natural deduction methods for L)*
5. $B \vdash_{L^a} f$ *(f is derivable from B by A-style natural deduction methods for L)*

6.2 Resumption of the Example

We can now return to Example 1 to see how different proof methods are used to derive the expected results. First, we give a tableau proof for it.

$$
\begin{array}{c}
\neg\langle 0.7\rangle r \\ \hline
[0.6](p\wedge q\supset r) \\ \hline
\langle 0.7\rangle(p\wedge q) \\ \hline
\neg r \\
p\wedge q\supset r \\
p\wedge q \\ \hline
p \\
q \\ \hline
\begin{array}{c|c}
\begin{array}{c} \neg(p\supset q) \\ \hline \begin{array}{c|c} \neg p & \neg q \\ \times & \times \end{array} \end{array} & \begin{array}{c} r \\ \times \\ \end{array}
\end{array}
\end{array}
$$

Each horizontal line represents an application of some tableau rule. The applied rules from top to bottom are two global assumption rules, one $\pi(0.7)$ rule, one α rule and two β rules respectively.

Next, we show a I-style natural deduction proof for the same example.

$$
\begin{array}{|lll|}
\hline
1.\neg\langle 0.7\rangle r & & ;\text{assumption} \\
2.[0.6](p\wedge q\supset r) & & ;\text{global assumption} \\
3.\langle 0.7\rangle(p\wedge q) & & ;\text{global assumption} \\
\quad 0.7: & \begin{array}{|ll|}\hline
4.p\wedge q & ;3.,\text{type 2 box creation} \\
5.\neg r & ;1.,\text{iteration} \\
6.p\wedge q\supset r & ;1.,\text{iteration} \\
7.r & ;4.,6.,\supset E \\
8.\bot & ;5.,7.,\mathit{contradiction} \\ \hline
\end{array} & \\
9.\bot & & ;\bot\text{ return} \\ \hline
\end{array}
$$
$$10.\langle 0.7\rangle r \quad ;1.,9.,\text{Reverse (Lemma 5.1)}$$

Finally, A-style natural deduction proof is given by modifying the I-style proof slightly.

$$
\begin{array}{|lll|}
\hline
1.\neg\langle 0.7\rangle r & & ;\text{assumption} \\
2.[0.6](p\wedge q\supset r) & & ;\text{global assumption} \\
3.\langle 0.7\rangle(p\wedge q) & & ;\text{global assumption} \\
\quad 0.7: & \begin{array}{|ll|}\hline
4.\top & ;\text{type 2 box creation} \\
5.\neg r & ;1.,\text{iteration} \\
6.p\wedge q\supset r & ;1.,\text{iteration} \\
7.\neg(p\wedge q) & ;5.,6.,\supset E \\ \hline
\end{array} & \\
8.\neg\langle 0.7\rangle(p\wedge q) & & ;\nu^{+}(0.3)\text{return rule} \\
9.\bot & & ;3.,8.,\text{contradiction} \\ \hline
\end{array}
$$
$$10.\langle 0.7\rangle r \quad ;1.,9.,\text{Reverse}$$

6.3 Related Works

In this section, we consider some related work on modal approach to uncertainty reasoning.

First, Dubois et al. [9] give an intriguing semantic account of possibilistic logic based on incomplete states of knowledge, and they also suggest the graded modal operators $\Box_c$ and $\Diamond_c$, which are roughly equivalent to our $[1-c]^+$ and $\langle c\rangle$. However, their semantic unit is a complete or incomplete state of knowledge, so the accessibility relation betwee two states means the precisiation relation, while according to the discussion in section 2.2, our accessibility relation (i. e. the collection of all R_w) means the total epistemic states.

Second, Farinas del cerro and Herzig[12] also formulate possibility logic as a multimodal logic PL_P, where P is a finite subset of [0,1]. PL_P is essentially equivalent to our system D. However, there is no completeness result for PL_P being established. Furthermore, they define a conditional formula $f \geq g$ to denote $\Pi(f) \geq \Pi(g)$ and show that $f \geq g$ can be transalted into a wff of PL_P. However, the given translation is not truth-preserving. That is, it is possibile that $f \geq g$ is false but the resultant wff is true.

Third, Catach[4] considers some axiomatic schemata for general multimodal logics. Though his results can be applied to different intensional logics, such as epistemic, doxastic, temporal, dynamic logics and their combination, however, it seems that he only allows a finite set of atomic modal operators and then forms other modal operators from this finite set by composition and union. On the other hand, in QML, we have a set of modal operators whose cardinality is $\aleph_1$, so his results can not be applied to our case directly.

Finally, in probabilistic logic, there are some similar approaches, notably that of Fagin and Halpern[10]. However, due to the additivity of probability measure, it seems difficult to develop complete proof method for modal probability logic other than the axiomatic system.

6.4 Future Directions

Although the presentation of the paper have almost reached the end, this is by no means the termination of the QML research. There are still works remaining to be done. Among them, we list some possible research directions of QML.

First, the proof methods for analytic systems should be implemented to test the effeciency of the methods. We consider the natural deduction methods as the most easily implemented one since the box structure of natural deduction methods is very like the block structure of computer program language. However, the tableau method may be more effecient since the backward reasoning characteristic of it guarantee the convergence of an attempted proof. Of course, our inference rules are essentially non-deterministic, so some heuristics and control strategies for proof tree search should first be developed. Furthermore, other styles of proof methods may also be considered, such as the resolution method[1] and matrix proof method[28].

Second, we should consider the extension of QML to predicate logic. This will enhance the expressive power of our language. However, the proof methods and completeness will become far more complicated than the present case.

Third, QML is essentally a kind of intensional logics[14], so it is possible to combine QML with other intensional logics. In particular, we will be interested in the combination of QML with temporal logic [2] and with epistemic logic [15]. A logic for reasoning about knowledge and probability have been successful developed by Fagin et al.[10]. Therefore, we believe the combination of QML with epistemic logic and with temporal logic will produce more fruitful results due to the intensional feature of QML.

Finally, the logic program paradigm for QML may be considered. The extensions of logic programming to including modal logic [11] or possibilistic logic [6] have been explored by some researchers respectively. It seems that QML may provides a framework to integrate the two extensions.

Of course, other research directions are also possible, and we can not enumerate all of them here. We just emphasize once again that the analogy between possibilistic and modal reasonings will provide very fruitful results for uncertainty reasoning research. Since 1912, C.I. Lewis [20] opened the door to the modern modal logic, it has been about eighty years old, while possibilistic logic is at most in its teens. The links between them revealed by QML show that the trace of growth of the old discipline may provide much experience and insight into the younger one.

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