Supplymentary Proofs for the Paper Functional Pearl: Maximally Dense Segments

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This document records some proofs accompanying the paper Functional Pearl: Maximally Dense Segments. Theorems and lemmas are labelled with both their own numbers here and the numbers in the paper, if any. For example, **Lemma A.1** (3.1) is Lemma 3.1 in the paper.

A Preliminaries

To begin with, we recite the useful property about densities: for any lists $x, y ::$ $[\mathit{Elem}]$, and any operator $\oplus \in \{<_d, \leq_d, >_d, \geq_d\}$,

$$x \ \oplus \ (x +\!\!+ \, y) \quad \Leftrightarrow \quad x \oplus y \quad \Leftrightarrow \quad (x +\!\!+ \, y) \ \oplus \ y. \tag{1}$$

Recall that the function smsp is defined by:

$$\begin{array}{l} \mathit{smsp} :: [\mathit{Elem}] \rightarrow [\mathit{Elem}] \\ \mathit{smsp} = \mathit{min}_{\#} \circ \Lambda \mathit{max}_d \circ \Lambda((L \leq_b)? \circ \mathit{prefix}). \end{array}$$

When we write $\mathit{smsp}\ x$ we imply that $x \in \mathit{dom}\ \mathit{smsp}$. It is monotonic on the prefix ordering $\sqsubseteq$ in the sense below:

$$x \sqsubseteq y \ \Rightarrow \ \mathit{smsp}\ x \sqsubseteq \mathit{smsp}\ y, \tag{2}$$

$$x \sqsubseteq y \ \Rightarrow \ \mathit{smsp}\ x \leq_d \mathit{smsp}\ y. \tag{3}$$

Also, it is idempotent:

$$\mathit{smsp}\ (\mathit{smsp}\ x) = \mathit{smsp}\ x. \tag{4}$$

Let y be a prefix of x that is no shorter than $\mathit{smsp}\ x$, we reason:

$$\begin{array}{ll} & \mathit{smsp}\ x \sqsubseteq y \sqsubseteq x \\ \Rightarrow & \quad \{\ \text{by (2)}\ \} \\ & \mathit{smsp}\ (\mathit{smsp}\ x) \sqsubseteq \mathit{smsp}\ y \sqsubseteq \mathit{smsp}\ x \\ \equiv & \quad \{\ \text{by (4)}\ \} \\ & \mathit{smsp}\ x \sqsubseteq \mathit{smsp}\ y \sqsubseteq \mathit{smsp}\ x, \end{array}$$

thus $smsp\ y = smsp\ x$. We have just shown that

$$smsp\ x \sqsubseteq y \sqsubseteq x \ \Rightarrow\ smsp\ x = smsp\ y. \tag{5}$$

The function $trim$ is defined by:

$$trim\ x = \ max_{\#} \circ \Lambda((U \geq_b)? \circ prefix).$$

It is also monotonic on the prefix ordering:

$$x \sqsubseteq y \ \Rightarrow\ trim\ x \sqsubseteq trim\ y. \tag{6}$$

Apart from being idempotent, we also have:

$$trim\ (a : x) = trim\ (a : trim\ x). \tag{7}$$

Note that for any x, $trim\ x \geq_b L$ equivals that x has a prefix y such that $U \leq_b y \leq_b L$. Therefore,

$$trim\ x \geq_b L\ \wedge\ x' \sqsubseteq x\ \wedge\ x \geq_b L \ \ \Rightarrow\ \ trim\ x' \geq_b L. \tag{8}$$

If $u <_b U$ for some list u, then u is a prefix of $trim\ (u +\!\!+\ x)$. That is, there exists a function $trim_u$ such that:

$$trim\ (u +\!\!+\ x) = u +\!\!+\ trim_u\ x, \text{ if } u <_b U. \tag{9}$$

While $trim$ returns the longest prefix of a list no wider than U, $trim_u$ trims the list upto $U - breadth\ u$ units wide. If $u \geq_b U$, we let $trim_u\ x = [\,]$, thus in general $trim\ (u +\!\!+\ x) \sqsubseteq u +\!\!+\ trim_u\ x$. The function $trim_u$ satisfies a property similar to (5):

$$trim\ x \sqsubseteq y \sqsubseteq x \ \Rightarrow\ trim_u\ x = trim_u\ y. \tag{10}$$

The following is an important property relating $smsp$ and $trim_u$:

$$smsp\ (trim_u\ x) \sqsubseteq trim_u\ (smsp\ x). \tag{11}$$

The right-hand side computes the shortest, maximally dense prefix of x, before trimming it to some fixed width. The left-hand side computes $smsp$ only for trimmed x. In general the latter could be shorter. The x on the right-hand side may actually be replaced by any prefix of x no narrower than $trim_u\ x$. In particular:

$$smsp\ (trim_u\ x) \sqsubseteq trim_u\ (smsp\ (trim\ x)). \tag{12}$$

if $trim_u\ x$ and $trim\ x$ are in the domain of $smsp$.

We recite Lemma 3.1 in the paper:

Lemma A.1 (3.1)**.** $u +\!\!+\ smsp\ x \sqsubset smsp\ (u +\!\!+\ x) \ \Rightarrow\ smsp\ x >_d smsp\ (u +\!\!+\ x).$

Proof. Denote $smsp\ x$ by x_1. The antecedent implies that there exists some non-empty x_2 such that $smsp\ (u +\!\!+\ x) = u +\!\!+\ x_1 +\!\!+\ x_2$. Since x_1 is a maximally density prefix of x, we have $x_1 \geq_d x_1 +\!\!+\ x_2$, which is equivalent to $x_1 \geq_d x_2$ by the density property (1). Since $u +\!\!+\ x_1 +\!\!+\ x_2$ is the shortest maximally dense prefix of $u +\!\!+\ x$, we have $u +\!\!+\ x_1 +\!\!+\ x_2 >_d u +\!\!+\ x_1$, which is equivalent to $x_2 >_d u +\!\!+\ x_1 +\!\!+\ x_2$ by (1). By transitivity we have $x_1 >_d u +\!\!+\ x_1 +\!\!+\ x_2$. □

For calculation, however, we will be using the contraposition of Lemma A.1:

Corollary A.2. $\mathit{smsp}\ (u \mathbin{+\!\!+} x) \geq_d \mathit{smsp}\ x\ \Rightarrow\ \mathit{smsp}\ (u \mathbin{+\!\!+} x) \sqsubseteq u \mathbin{+\!\!+} \mathit{smsp}\ x.$

B Catamorphism Required for Yoking

Recall the definition of *win*:

$$\begin{array}{ll} \mathit{win}\ [\,] & = [\,] \\ \mathit{win}\ (a : x) & = \mathit{wp}\ (\mathit{trim}\ (a : \mathit{win}\ x)), \end{array}$$

where *wp* is defined by:

$$\begin{array}{lll} \mathit{wp}\ :: & [\mathit{Elem}] \to [\mathit{Elem}] & \\ \mathit{wp}\ x \mid & L \leq_b x & = \mathit{smsp}\ x \\ \mid & \textbf{otherwise} & = x. \end{array}$$

Since $\mathit{trim} \subseteq \mathit{prefix}$ and $\mathit{smsp} \subseteq \mathit{prefix}$, the window $\mathit{win}\ x$ is always a prefix of $\mathit{wp}\ (\mathit{trim}\ x)$:

$$\mathit{win}\ x \sqsubseteq \mathit{wp}\ (\mathit{trim}\ x) \sqsubseteq x. \tag{13}$$

Now we prove Theorem 3.2 and Lemma 3.3:

Theorem B.1 (3.2)**.** $\mathit{mds}_{\mathsf{M}}\ (a : x) = \mathit{mds}_{\mathsf{M}}\ x \Uparrow_d \mathit{win}\ (a : x).$

Proof. By definition, $\mathit{mds}_{\mathsf{M}}\ (a : x) = \mathit{mds}_{\mathsf{M}}\ x \Uparrow_d \mathit{wp}\ (\mathit{trim}\ (a : x))$.

If $\mathit{trim}\ (a : x) <_b L$, by (13) and (6), $\mathit{trim}\ (a : \mathit{win}\ x)$ is narrower than L as well. Therefore both $\mathit{mds}_{\mathsf{M}}\ x \Uparrow_d \mathit{wp}\ (\mathit{trim}\ (a : x))$ and $\mathit{mds}_{\mathsf{M}}\ x \Uparrow_d \mathit{win}\ (a : x)$ reduce to $\mathit{mds}_{\mathsf{M}}\ x$.

If $\mathit{trim}\ (a : x) \geq_b L$, we first consider the case when x is in the domain of *mds* and $\mathit{smsp}\ (\mathit{trim}\ (a : x)) <_d \mathit{mds}\ x$. We reason:

$$\begin{array}{ll} & \mathit{mds}_{\mathsf{M}}\ x \Uparrow_d \mathit{wp}\ (\mathit{trim}\ (a : x)) \\ = & \quad \{\ \text{since } \mathit{trim}\ (a : x) \geq_b L\ \ \} \\ & \mathit{mds}_{\mathsf{M}}\ x \Uparrow_d \mathit{smsp}\ (\mathit{trim}\ (a : x)) \\ = & \quad \{\ \text{since } \mathit{mds}\ x >_d \mathit{smsp}\ (\mathit{trim}\ (a : x))\ \text{ and} \\ & \qquad \mathit{smsp}\ (\mathit{trim}\ (a : x)) \geq_d \mathit{win}\ (a : x), \text{ see below}\ \} \\ & \mathit{mds}_{\mathsf{M}}\ x \Uparrow_d \mathit{win}\ (a : x). \end{array}$$

The second step is valid because $\mathit{trim}\ (a : x) \geq_b L$ implies $\mathit{trim}\ (a : \mathit{win}\ x) \geq_b L$, to be shown as Corollary B.4. Therefore $\mathit{win}\ (a : x) = \mathit{smsp}\ (\mathit{trim}\ (a : \mathit{win}\ x))$, and we thus have $\mathit{smsp}\ (\mathit{trim}\ (a : x)) \geq_d \mathit{win}\ (a : x)$ by (2).

That leaves two other possibilities: $\mathit{mds}_{\mathsf{M}}\ x = \mathit{Nothing}$ or $\mathit{smsp}\ (\mathit{trim}\ (a : x)) \geq_d \mathit{mds}\ x$. For that case we claim that

$$\mathit{smsp}\ (\mathit{trim}\ (a : x)) = \mathit{smsp}\ (\mathit{trim}\ (a : \mathit{win}\ x))$$

holds, by Lemma B.2. □

Lemma B.2 (3.3)**.** For non-empty u such that $\mathit{trim}\ (u \mathbin{+\!\!+} x) \geq_b L$ we have:

$$\begin{array}{rl} & \mathit{mds}_{\mathsf{M}}\ x = \mathit{Nothing} \quad \vee \quad \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} x)) \geq_d \mathit{mds}\ x \\ \Rightarrow & \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} x)) = \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} \mathit{win}\ x)). \end{array}$$

Proof. If $u \geq_b U$, both sides of the consequent reduce to $\mathit{smsp}\ (\mathit{trim}\ u)$. Thus we consider the case when $u <_b U$. The proof is a strong induction on the length of x. The base case is $x = [\,]$ and both sides reduce to $\mathit{smsp}\ (\mathit{trim}\ u)$.

For the inductive case, we assume that Lemma B.2 holds for x and all lists shorter than x, and attempt to prove Lemma B.2 for $b : x$, where $\mathit{trim}\ (u \mathbin{+\!\!+} b : x) \geq_b L$. The key step is the use of Corollary A.2, and the first few steps basically try to transform the conclusion to the form $\mathit{smsp}\ (u \mathbin{+\!\!+} \ldots) \sqsubseteq u \mathbin{+\!\!+} \mathit{smsp}\ (\ldots)$ such that Corollary A.2 is applicable. In the proof below we abbreviate $\mathit{mds}_{\mathsf{M}}\ x = \mathit{Nothing} \vee \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : x)) \geq_d \mathit{mds}\ x$ to P:

$$\begin{array}{rl} & \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : x)) = \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} \mathit{win}\ (b : x))) \\ \Leftarrow & \{ \text{ by (13), (6), and (5) } \} \\ & \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : x)) \sqsubseteq \mathit{trim}\ (u \mathbin{+\!\!+} \mathit{win}\ (b : x)) \\ \equiv & \{ \text{ since } u <_b U, \text{ introduce } \mathit{trim}_u \text{ by (9) } \} \\ & \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : x)) \sqsubseteq u \mathbin{+\!\!+} \mathit{trim}_u\ (\mathit{win}\ (b : x)) \\ \equiv & \{ \text{ definition of } \mathit{win} \ \} \\ & \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : x)) \sqsubseteq u \mathbin{+\!\!+} \mathit{trim}_u\ (\mathit{wp}\ (\mathit{trim}\ (b : \mathit{win}\ x))) \end{array}$$

We pause here to do a case analysis. Consider the case when $\mathit{trim}\ (b : \mathit{win}\ x) <_b L$. Since $\mathit{trim}\ (u \mathbin{+\!\!+} b : x) \geq_b L$, by Lemma B.5 we have $\mathit{trim}\ (u \mathbin{+\!\!+} b : \mathit{win}\ x) \geq_b L$. Thus by Lemma B.6, the left-hand side is a prefix of the right-hand side.

If $\mathit{trim}\ (b : \mathit{win}\ x) \geq_b L$, $\mathit{wp}\ (\mathit{trim}\ (b : \mathit{win}\ x))$ simplifies to $\mathit{smsp}\ (\mathit{trim}\ (b : \mathit{win}\ x))$. Similarly by Lemma B.6, if $\mathit{trim}_u\ (b : \mathit{win}\ x) <_b L$, the prefix relation holds. Thus we can safely apply (12):

$$\begin{array}{rl} & \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : x)) \sqsubseteq u \mathbin{+\!\!+} \mathit{trim}_u\ (\mathit{smsp}\ (\mathit{trim}\ (b : \mathit{win}\ x))) \\ \Leftarrow & \{ \text{ by (12) } \} \\ & \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : x)) \sqsubseteq u \mathbin{+\!\!+} \mathit{smsp}\ (\mathit{trim}_u\ (b : \mathit{win}\ x)) \\ \Leftarrow & \{ \text{ by induction: } \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : x)) \ = \\ & \qquad \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : \mathit{win}\ x)) \ \} \\ & \mathit{smsp}\ (\mathit{trim}\ (u \mathbin{+\!\!+} b : \mathit{win}\ x)) \sqsubseteq u \mathbin{+\!\!+} \mathit{smsp}\ (\mathit{trim}_u\ (b : \mathit{win}\ x)) \wedge P \\ \equiv & \{ \text{ since } u <_b U, \text{ introduce } \mathit{trim}_u \text{ by (9) } \} \\ & \mathit{smsp}\ (u \mathbin{+\!\!+} \mathit{trim}_u\ (b : \mathit{win}\ x)) \sqsubseteq u \mathbin{+\!\!+} \mathit{smsp}\ (\mathit{trim}_u\ (b : \mathit{win}\ x)) \wedge P \\ \Leftarrow & \{ \text{ Corollary A.2 } \} \\ & \mathit{smsp}\ (u \mathbin{+\!\!+} \mathit{trim}_u\ (b : \mathit{win}\ x)) \geq_d \mathit{smsp}\ (\mathit{trim}_u\ (b : \mathit{win}\ x)) \wedge P \\ \equiv & \{ \text{ since } u <_b U, \text{ dismiss } \mathit{trim}_u \text{ by (9) } \} \\ & \mathit{smsp}\ (u \mathbin{+\!\!+} \mathit{trim}_u\ (b : \mathit{win}\ x)) \geq_d \mathit{smsp}\ (\mathit{trim}_u\ (b : \mathit{win}\ x)) \wedge P \end{array}$$

$$
\begin{array}{cl}
\equiv & \{ \text{ induction } \} \\
& smsp\ (trim\ (u +\!\!+ b : x)) \geq_d smsp\ (trim_u\ (b : win\ x)) \land P \\
\Leftarrow & \{ \text{ since } trim_u\ (b : win\ x) \sqsubseteq trim\ (b : x), \text{ by } (3) \ \} \\
& smsp\ (trim\ (u +\!\!+ b : x)) \geq_d smsp\ (trim\ (b : x)) \land P \\
\equiv & \{ \text{ definition of } mds_{\mathsf{M}}, (b : x) \geq_b L \ \} \\
& mds_{\mathsf{M}}\ (b : x) = Nothing \quad \lor \quad smsp\ (trim\ (u +\!\!+ b : x)) \geq_d mds\ (b : x).
\end{array}
$$

□

B.1 Some Lemmas about Lengths and Widths

To support the proofs above, we need some more lemmas about lengths and widths of lists. We first prove the lemma below:

Lemma B.3. For $x :: [Elem]$ we have (a) $trim\ x \leq_b L \Rightarrow win\ x = trim\ x$, and (b) $trim\ x \geq_b L \Rightarrow win\ x \geq_b L$.

Proof. The lemma trivially holds when $x = [\,]$. Consider the case $a : x$.

1. If $trim\ (a : x) \leq_b L$, we examine two possibilities:

   (a) if $trim\ x \leq_b L$, we have

$$
\begin{array}{cl}
& win\ (a : x) \\
= & wp\ (trim\ (a : win\ x)) \\
= & wp\ (trim\ (a : trim\ x)) \\
= & \{ \text{ by } (7) \ \} \\
& wp\ (trim\ (a : x)) \\
= & \{ \text{ since } trim\ (a : x) \leq_b L \ \} \\
& = trim\ (a : x).
\end{array}
$$

   (b) If $trim\ x \geq_b L$, by induction $win\ x \geq_b L$. Denote $trim\ (a : x)$ by $a : x_1$. Since it is the longest prefix no wider than U, it must be the case that x can be decomposed into $a : x_1 +\!\!+ [b] +\!\!+ x_2$, with $a : x_1 +\!\!+ [b] >_b U$. Since $win\ x \geq_b L$ and $x_1 <_b L$, $x_1 +\!\!+ [b]$ must be a prefix of $win\ x$. Therefore we have on the one hand

$$
\begin{array}{cl}
& win\ (a : x) \\
= & wp\ (trim\ (a : win\ x)) \\
\sqsupseteq & wp\ (trim\ (a : x_1 +\!\!+ [b]) \\
= & wp\ (a : x_1) \\
= & trim\ (a : x).
\end{array}
$$

   On the other hand, $win\ (a : x) \sqsubseteq trim\ (a : x)$. Thus $win\ (a : x) = trim\ (a : x)$.

2. If $trim\ (a : x) \geq_b L$, we also examine two possibilities:

   (a) if $trim\ x \leq_b L$, by induction $win\ x = trim\ x$. By a reasoning similar to 1.(a) above, $win\ (a : x) = wp\ (trim\ (a : x))$, which equals $smsp\ (trim\ (a : x))$ because $trim\ (a : x) \geq_b L$, and $smsp$ always returns a prefix at least L units wide.

   (b) If $trim\ x \geq_b L$, we reason:

$$
\begin{aligned}
& trim\ x \geq_b L \\
\Rightarrow\ & \quad \{\text{ induction }\} \\
& win\ x \geq_b L \\
\Rightarrow\ & a : win\ x \geq_b L \\
& \quad \{\text{ by (8) }\} \\
\Rightarrow\ & trim\ (a : win\ x) \geq_b L \\
\Rightarrow\ & win\ (a : x) = smsp\ (trim\ (a : win\ x)) \\
\Rightarrow\ & win\ (a : x) \geq_b L.
\end{aligned}
$$

□

Corollary B.4. $trim\ (a : x) \geq_b L \ \Rightarrow\ trim\ (a : win\ x) \geq_b L.$

Proof. Extracted from case 2 in the proof of Lemma B.3. □

Lemma B.5. $trim\ (u +\!\!+ x) \geq_b L \ \Rightarrow\ trim\ (u +\!\!+ win\ x) \geq_b L.$

Lemma B.6. Let x, y be lists of elements and y' a prefix of y such that $y' \geq_b L$. If $trim\ y' <_b L$ (or $trim_x\ y' <_b L$), then $smsp\ (x +\!\!+ trim\ y) \sqsubseteq x +\!\!+ trim\ y'$ (or $smsp\ (x +\!\!+ trim\ y) \sqsubseteq x +\!\!+ trim_x\ y'$).

Proof. From $trim\ y' <_b L$ and $y' \geq_b L$ we infer that y' can be decomposed into $trim\ y' +\!\!+ [a] +\!\!+ y''$ such that $trim\ y' +\!\!+ [a] >_b U$ (or $U - breadth\ x$). Therefore, we also have $x +\!\!+ trim\ y' +\!\!+ [a] >_b U$, and thus $smsp\ (x +\!\!+ trim\ y)$ cannot extend further than $x +\!\!+ trim\ y'$ (or $x +\!\!+ trim_x\ y'$). □

C Decreasing Right-Skew Partition

Lemma C.1 (5.2)**.** Let z, x_1, x_2 be non-empty lists of elements such that $x_1 \leq_d x_2$, then at least one of $z +\!\!+ x_1 \leq_d z$ or $z +\!\!+ x_1 <_d z +\!\!+ x_1 +\!\!+ x_2$ is true.

Proof. If $z \geq_d x_1$, we have $z \geq_d z +\!\!+ x_1$ by (1). If $z <_d x_1$, we reason:

$$
\begin{aligned}
& z <_d x_1 \leq_d x_2 \\
\Rightarrow\ & \quad \{\ (1)\ \} \\
& z <_d z +\!\!+ x_1 <_d x_1 \leq_d x_2 \\
\Rightarrow\ & \quad \{\ (1)\ \} \\
& z +\!\!+ x_1 <_d z +\!\!+ x_1 +\!\!+ x_2.
\end{aligned}
$$

□

Lemma C.2 (5.1)**.** $z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\Lambda \mathit{prefix}\ x) = z \uparrow_d (z \mathbin{+\!\!+} x)$ if x is right-skew.

Proof. A naïve induction like the following won't work:

$$
\begin{aligned}
& z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\Lambda \mathit{prefix}\ (a : x)) \\
= \;& z \uparrow_d z \mathbin{+\!\!+} [a] \mathbin{+\!\!+} \mathit{smwr}\ (z \mathbin{+\!\!+} [a])\ (\Lambda \mathit{prefix}\ x) \\
= \;& \quad \{ \text{ induction } \} \\
& \ldots
\end{aligned}
$$

because $a : x$ being right-screw doesn't imply that x is right-screw. We prove a generalised lemma: given two list x and y, if $y \mathbin{+\!\!+} x_1 \leq_d x_2$ for all x_1, x_2 such that $x_1 \mathbin{+\!\!+} x_2 = x$ and x_2 is non-empty, then we have

$$z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\mathsf{E}(y \mathbin{+\!\!+})\ (\Lambda \mathit{prefix}\ x)) = (z \mathbin{+\!\!+} y) \uparrow_d (z \mathbin{+\!\!+} y \mathbin{+\!\!+} x).$$

This can be proved using a simple structural induction on x. The base cases are $x = [\,]$ or x is a singleton list. For the inductive case, we assume that

$$y \mathbin{+\!\!+} x_1 \leq_d x_2 \text{ for all } x_1 \mathbin{+\!\!+} x_2 = a : x \text{ with } x_2 \text{ non-empty,} \tag{14}$$

and reason:

$$
\begin{aligned}
& z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\mathsf{E}\ (y \mathbin{+\!\!+}))\ (\Lambda \mathit{prefix}\ (a : x)) \\
= \;& z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\mathsf{E}\ (y \mathbin{+\!\!+}))\ (\{[\,]\} \cup (\mathsf{E}\ (a :)\ (\Lambda \mathit{prefix}\ x))) \\
= \;& (z \mathbin{+\!\!+} y) \uparrow_d (z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\mathsf{E}(y \mathbin{+\!\!+} [a] \mathbin{+\!\!+})\ (\Lambda \mathit{prefix}\ x))) \\
= \;& \quad \{ \text{ induction, since (14) implies} \\
& \qquad y \mathbin{+\!\!+} [a] \mathbin{+\!\!+} x'_1 \leq_d x'_2 \text{ for all } x'_1 \mathbin{+\!\!+} x'_2 = x \ \} \\
& (z \mathbin{+\!\!+} y) \uparrow_d (z \mathbin{+\!\!+} y \mathbin{+\!\!+} [a]) \uparrow_d (z \mathbin{+\!\!+} y \mathbin{+\!\!+} [a] \mathbin{+\!\!+} x) \\
= \;& \quad \{ \text{ by Lemma C.1, since (14) implies } y \mathbin{+\!\!+} [a] \leq_d x \ \} \\
& (z \mathbin{+\!\!+} y) \uparrow_d (z \mathbin{+\!\!+} y \mathbin{+\!\!+} [a] \mathbin{+\!\!+} x).
\end{aligned}
$$

□

Theorem C.3 (5.3)**.** $\mathit{smwr}\ z\ (\Lambda \mathit{prefix}\ (\mathit{concat}\ \mathit{xs})) = \mathit{smwr}\ z\ (\mathit{opts}\ \mathit{xs})$, if $\mathit{all}\ \mathit{rightskew}\ \mathit{xs}$ holds.

Proof. Note that *smwr* satisfies this property:

$$\mathit{smwr}\ z\ (\mathsf{E}(x \mathbin{+\!\!+}) \mathit{xs}) = x \mathbin{+\!\!+} \mathit{smwr}\ (z \mathbin{+\!\!+} x)\ \mathit{xs}. \tag{15}$$

We prove a generalisation: if $\mathit{all}\ \mathit{rightskew}\ \mathit{xs}$, then

$$z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\Lambda \mathit{prefix}\ (\mathit{concat}\ \mathit{xs})) = z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\mathit{opts}\ \mathit{xs}).$$

The proof is an induction on xs. The case for $\mathit{xs} = [\,]$ is trivial. For the case $x : \mathit{xs}$, we reason:

$$z \mathbin{+\!\!+} \mathit{smwr}\ z\ (\mathit{opts}\ (x : \mathit{xs}))$$

$$
\begin{aligned}
&= z \mathbin{+\!\!+} smwr\ z\ (\{[\,], x\} \cup \mathsf{E}(x\mathbin{+\!\!+})\ (opts\ xs))\\
&= z \uparrow_d (z \mathbin{+\!\!+} x) \uparrow_d z \mathbin{+\!\!+} smwr\ z\ (\mathsf{E}(x\mathbin{+\!\!+})\ (opts\ xs))\\
&= \quad \{\ \text{by (15)}\ \}\\
&\quad z \uparrow_d (z \mathbin{+\!\!+} x) \uparrow_d z \mathbin{+\!\!+} x \mathbin{+\!\!+} smwr\ (z \mathbin{+\!\!+} x)\ (opts\ xs)\\
&= \quad \{\ xs \text{ is a DRSP, induction}\ \}\\
&\quad z \uparrow_d (z \mathbin{+\!\!+} x) \uparrow_d z \mathbin{+\!\!+} x \mathbin{+\!\!+} smwr\ (z \mathbin{+\!\!+} x)\ (\Lambda prefix\ (concat\ xs))\\
&= \quad \{\ \text{since } x \text{ is right-skew, Lemma C.2}\ \}\\
&\quad z \mathbin{+\!\!+} smwr\ z\ (\Lambda prefix\ x) \uparrow_d z \mathbin{+\!\!+} x \mathbin{+\!\!+} smwr\ (z \mathbin{+\!\!+} x)\ (\Lambda prefix\ (concat\ xs))\\
&= \quad \{\ \text{by (15)}\ \}\\
&\quad z \mathbin{+\!\!+} smwr\ z\ (\Lambda prefix\ x) \uparrow_d z \mathbin{+\!\!+} smwr\ z\ (\mathsf{E}(x\mathbin{+\!\!+})\ (\Lambda prefix\ (concat\ xs)))\\
&= z \mathbin{+\!\!+} smwr\ z\ (\Lambda prefix\ x \cup \mathsf{E}\ (x\mathbin{+\!\!+})(\Lambda prefix(concat\ xs)))\\
&= z \mathbin{+\!\!+} smwr\ z\ (\Lambda prefix\ (x \mathbin{+\!\!+} concat\ xs))\\
&= z \mathbin{+\!\!+} smwr\ z\ (\Lambda prefix\ (concat\ (x : xs))).
\end{aligned}
$$

□

D MDS with an Upper Bound

The window for the second algorithm is defined by:

$$
\begin{aligned}
&win2\ ::\ [Elem] \to ([Elem], [Elem], [Elem])\\
&win2\ [\,] \quad = ([\,], [\,], [\,])\\
&win2\ (a : x) = wp2\ (trim2\ (a \triangleright win2\ x))\\
&\quad \textbf{where}\ a \triangleright (w, y, v) = (a : w, y, v).
\end{aligned}
$$

The main theorem is:

Theorem D.1 (8.1)**.** *$mds_{\mathsf{M}}\ (a : x) = mds_{\mathsf{M}}\ x \upuparrows_d win2\ (a : x)$, where $x \upuparrows_d (w, y, z) = x \Uparrow_d (w \mathbin{+\!\!+} y \mathbin{+\!\!+} z)$.*

Proof. Similar to that of Theorem B.1. After some case anaysis we end up having to prove:

$$
\begin{aligned}
&mds_{\mathsf{M}}\ x = Nothing\ \vee\ smsp\ (trim\ (a : x)) \geq_d mds\\
\Rightarrow\ &smsp\ (trim\ (a : x)) = win2\ (a : x),
\end{aligned}
$$

A special case of Corollary D.3, to be shown later, states that the antecedent also implies $win2\ (a : x) = smsp\ (trim\ (a \triangleright win2\ x))$, that is, the outer-most step of $win2$ can be replaced by $smsp \circ trim \circ (a\triangleright)$. Thus we only have to prove, under the same antecedent, that

$$
smsp\ (trim\ (a : x)) = smsp\ (trim\ (a \triangleright win2\ x)),
$$

which is generalised and proved as Lemma D.2 below. □

Lemma D.2 (8.3)**.** If $smsp\ (trim\ (u +\!\!+\ x)))$ and either $mds_{\mathsf{M}}\ x = Nothing$ or $smsp\ (trim\ (u +\!\!+\ x)) \geq_d mds\ x$, we have

$$smsp\ (trim\ (u +\!\!+\ x)) = smsp\ (trim\ (u \mathrel{\triangleright\!\!\triangleright} win2\ x)), \tag{16}$$

where $u \mathrel{\triangleright\!\!\triangleright} (w, y, z) = u +\!\!+\ w +\!\!+\ y +\!\!+\ z$.

Proof. This is a proof similar to that of Theorem B.2. The case when $x = [\,]$ is easily dismissed. We look at the inductive case $b : x$. In the proof below we denote $mds_{\mathsf{M}}\ x = Nothing \lor smsp\ (trim\ (u +\!\!+\ b : x)) \geq_d mds\ x$ by P, and overload $trim_u$ to a function from a window to a list — $trim_u\ (w, x, y) = trim_u\ (w +\!\!+ x +\!\!+ y)$. The heuristic is to transform the expression so that Corollary A.2 is applicable:

$$\begin{array}{ll}
& smsp\ (trim\ (u +\!\!+\ b : x)) = smsp\ (trim\ (u \mathrel{\triangleright\!\!\triangleright} win2\ (b : x))) \\
\Leftarrow & \quad \{\ \text{by (5)}\ \} \\
& smsp\ (trim\ (u +\!\!+\ b : x)) \sqsubseteq trim\ (u \mathrel{\triangleright\!\!\triangleright} win2\ (b : x)) \\
\Leftarrow & \quad \{\ \text{induction}\ \} \\
& smsp\ (trim\ ((u +\!\!+\ [b]) \mathrel{\triangleright\!\!\triangleright} win2\ x)) \sqsubseteq trim\ (u \mathrel{\triangleright\!\!\triangleright} win2\ (b : x))\ \land\ P \\
\equiv & \quad \{\ \text{since } u <_b U, \text{ introduce } trim_u \text{ by (9)}\ \} \\
& smsp\ (u +\!\!+\ trim_u\ (b \triangleright win2\ x)) \sqsubseteq u +\!\!+\ trim_u\ (win2\ (b : x))\ \land\ P \\
\Leftarrow & \quad \{\ \text{Lemma D.3, see below}\ \} \\
& smsp\ (u +\!\!+\ trim_u\ (b \triangleright win2\ x)) \sqsubseteq u +\!\!+\ smsp\ (trim_u\ (b \triangleright win2\ x)\ \land\ P \\
\Leftarrow & \quad \{\ \text{by Corollary A.2}\ \} \\
& smsp\ (u +\!\!+\ trim_u\ (b \triangleright win2\ x)) \geq_d smsp\ (trim_u\ (b \triangleright win2\ x))\ \land\ P \\
\equiv & \quad \{\ \text{since } u <_b U, \text{ dismiss } trim_u \text{ by (9)}\ \} \\
& smsp\ (trim\ (u +\!\!+\ b \triangleright win2\ x)) \geq_d smsp\ (trim_u\ (b \triangleright win2\ x))\ \land\ P \\
\equiv & \quad \{\ \text{induction}\ \} \\
& smsp\ (trim\ (u +\!\!+\ b : x)) \geq_d smsp\ (trim_u\ (b \triangleright win2\ x))\ \land\ P \\
\Leftarrow & \quad \{\ \text{since } trim_u(b \triangleright win2\ x) \sqsubseteq trim\ (b : x)\ \} \\
& smsp\ (trim\ (u +\!\!+\ b : x)) \geq_d smsp\ (trim\ (b : x))\ \land\ P \\
\Leftarrow & mds_{\mathsf{M}}\ (b : x) = Nothing\ \lor\ smsp\ (trim\ (u +\!\!+\ b : x)) \geq_d mds(b : x).
\end{array}$$

□

Lemma D.3 allows us to convert one application of $smsp \circ trim \circ (a\triangleright)$ to one step of $win2$. The difference is that the latter, when given a window (w, y, z), ignores y.

Lemma D.3. If $mds_{\mathsf{M}}\ x = Nothing$ or $smsp\ (trim\ (u +\!\!+ b \triangleright win2\ x)) \geq_d mds\ x$, then we have that for all u such that $u <_b U$:

$$smsp\ (trim_u\ (b \triangleright win2\ x)) \sqsubseteq trim_u\ (win2\ (b : x)).$$

Before we go on, it helps to recall that the function *wp2* calls *smsp2* if the window is at least L units wide:

$$\begin{aligned} \mathit{wp2}\ (w,y,v) \mid L \leq_b (w \mathbin{+\!\!+} y \mathbin{+\!\!+} v) &= \mathit{smsp2}\ (w,y,v) \\ \mid \mathbf{otherwise} \qquad\qquad &= (w,y,v). \end{aligned}$$

Given a window (w, y, v), *smsp2* respectively applies *smsp* to w and applies *smwr* $(w \mathbin{+\!\!+} y)$ to prefixes of v:

$$\begin{aligned} \mathit{smsp2}\ (w,y,v) \ = \ & \mathbf{let}\ w' = \mathit{smsp}\ w \\ & \qquad v' = \mathit{smwr}\ (w \mathbin{+\!\!+} y)\ (\Lambda \mathit{prefix}\ v) \\ & \mathbf{in}\ (w',[\,],[\,]) \uparrow_d (w',(w-w') \mathbin{+\!\!+} y, v'). \end{aligned}$$

The function $\mathit{trim2} :: ([\mathit{Elem}],[\mathit{Elem}],[\mathit{Elem}]) \rightarrow ([\mathit{Elem}],[\mathit{Elem}],[\mathit{Elem}])$ satisfies the following constraints.

1. If $(w, y, v) \leq_b L$, then $\mathit{trim2}\ (w, y, v) = (w \mathbin{+\!\!+} y \mathbin{+\!\!+} v, [\,], [\,])$.
2. If $w \mathbin{+\!\!+} y \leq_b U$, $\mathit{trim2}\ (w, y, v)$ leaves w and y unchanged, but may perform some trimming in v if necessary;
3. otherwise y and v are entirely dropped, and *trim2* returns some $(w', [\,], v')$ with $w' \mathbin{+\!\!+} v' = \mathit{trim}\ w$, to mimic in advance the behaviour that some *PTree*s might be spun off from the *DForest*. The list w' must be at least L units broad if that is the case for $w' \mathbin{+\!\!+} v'$.

Proof of Lemma D.3. Let $(w, y, z) = \mathit{win2}\ x$. We first find out what $\mathit{win2}\ (b : x) = \mathit{wp2}\ (\mathit{trim2}\ (b \triangleright (w, y, z)))$ could be:

1. **[Case** $(b : w, y, z) \leq_b L$]: For this case, $\mathit{wp2}\ (\mathit{trim2}\ (b : w, y, z)) = (b : w \mathbin{+\!\!+} y \mathbin{+\!\!+} z, [\,], [\,])$.
2. **[Case** $(b : w, y, z) >_b L \ \wedge \ (b : w \mathbin{+\!\!+} y) >_b U$]: For this case , $\mathit{trim2}\ (b : w, y, z) = (b : w_1, [\,], w_2)$ where $b : w_1 \mathbin{+\!\!+} w_2 = \mathit{trim}\ (b : w)$. The function *smsp2* then returns either
   (a) some $(\mathit{smsp}\ (b : w_1), y', w_2')$ such that $\mathit{smsp}\ (b : w_1) \mathbin{+\!\!+} y' = b : w_1$ and w_2' is the shortest optimal prefix of w_2, if $b : w_1 \mathbin{+\!\!+} w_2'$ is denser than $(\mathit{smsp}\ (b : w_1)$, or
   (b) $(\mathit{smsp}\ (b : w_1), [\,], [\,])$, otherwise.

   Either way, the returned window flattens to: $\mathit{smsp}\ (\mathit{trim}\ (b : w))$.
3. **[Case** $(b : w, y, z) >_b L \ \wedge \ (b : w \mathbin{+\!\!+} y) \leq_b U$]: For this case, $\mathit{trim2}\ (b : w, y, z) = (b : w, y, z_1)$ where z_1 is a prefix of z. The function *smsp2* then returns either some
   (a) $(\mathit{smsp}\ (b : w), y', z_1')$ such that $\mathit{smsp}\ (b : w) \mathbin{+\!\!+} y' = b : w \mathbin{+\!\!+} y$ and z_1' is the shortest optimal prefix of z_1, if $b : w \mathbin{+\!\!+} y \mathbin{+\!\!+} z_1'$ is denser than $\mathit{smsp}\ (b : w)$, or

(b) $(smsp\ (b : w), [\,], [\,])$, otherwise.

Case 1 contradicts our implicit assumption that $smsp\ (trim_u\ (b \triangleright win2\ x))$ is defined. If it is the case of 3(a), $b : w \mathbin{+\!\!+} y \mathbin{+\!\!+} z_1'$ is the optimal prefix of $b : w \mathbin{+\!\!+} y \mathbin{+\!\!+} z$, and $smsp\ (trim_u\ (b : w \mathbin{+\!\!+} y \mathbin{+\!\!+} z))$ should return the same result too.

For cases 2 and 3(b), it is sufficient to prove:

$$smsp\ (trim_u\ (b : w \mathbin{+\!\!+} y)) \sqsubseteq smsp\ (trim\ (b : w))$$

under the given assumption. Since $trim_u\ (b : w \mathbin{+\!\!+} y) \sqsubseteq trim\ (b : w \mathbin{+\!\!+} y))$, it is implied by Lemma D.4 below. □

Lemma D.4 (8.2)**.** Let $(w, y, z) = win2\ x$, we have:

$$smsp\ (trim\ (u \mathbin{+\!\!+} w \mathbin{+\!\!+} y)) = smsp\ (trim\ (u \mathbin{+\!\!+} w)),$$

if $mds_{\mathsf{M}}\ x = Nothing$ or $smsp\ (trim\ (u \mathbin{+\!\!+} w \mathbin{+\!\!+} y)) \geq_d mds\ x$.

Proof. The proof is an induction on x and is similar to that of Theorem B.2. When $x = [\,]$, both sides reduce to $smsp\ (trim\ u)$.

For the case $b : x$, let $(w, y, z) = win2\ x$, thus $b \triangleright win2\ x = (b : w, y, z)$, and $win2\ (b : x) = wp2\ (trim2\ (b : w, y, z))$. We do a case analysis similar to that in the previous lemma. It will turn out that we only need to prove:

$$smsp\ (trim\ (u \mathbin{+\!\!+} b : w \mathbin{+\!\!+} y)) = smsp\ (trim\ (u \mathbin{+\!\!+} b : w))$$

under the assumpton that $mds_{\mathsf{M}}\ (b : x) = Nothing$ or $smsp\ (trim\ (u \mathbin{+\!\!+} b : w \mathbin{+\!\!+} y)) \geq_d mds\ (b : x)$.

Both sides reduce to $smsp\ (trim\ u)$ if $u \geq_b U$, so we assume $u <_b U$. We abbreviate $mds_{\mathsf{M}}\ x = Nothing \vee smsp\ (trim\ (u \mathbin{+\!\!+} b : w \mathbin{+\!\!+} y)) >_d mds\ x$ to P:

$$\begin{array}{ll}
 & smsp\ (trim\ (u \mathbin{+\!\!+} b : w \mathbin{+\!\!+} y)) = smsp\ (trim\ (u \mathbin{+\!\!+} smsp\ (b : w))) \\
\Leftarrow & \quad \{\ \text{by (5)}\ \} \\
 & smsp\ (trim\ (u \mathbin{+\!\!+} b : w \mathbin{+\!\!+} y)) \sqsubseteq trim\ (u \mathbin{+\!\!+} smsp\ (b : w)) \\
\Leftarrow & \quad \{\ \text{induction}\ \} \\
 & smsp\ (trim\ (u \mathbin{+\!\!+} b : w)) \sqsubseteq trim\ (u \mathbin{+\!\!+} smsp\ (b : w))\ \wedge\ P \\
\equiv & \quad \{\ \text{since } u <_b U\ \} \\
 & smsp\ (u \mathbin{+\!\!+} trim_u\ (b : w)) \sqsubseteq u \mathbin{+\!\!+} trim_u\ (smsp\ (b : w))\ \wedge\ P \\
\Leftarrow & \quad \{\ \text{by (11)}\ \} \\
 & smsp\ (u \mathbin{+\!\!+} trim_u\ (b : w)) \sqsubseteq u \mathbin{+\!\!+} smsp\ (trim_u\ (b : w)))\ \wedge\ P \\
\Leftarrow & \quad \{\ \text{by Corollary A.2}\ \} \\
 & smsp\ (u \mathbin{+\!\!+} trim_u\ (b : w)) \geq_d smsp\ (trim_u\ (b : w))\ \wedge\ P \\
\equiv & \quad \{\ \text{since } u <_b U\ \} \\
 & smsp\ (trim\ (u \mathbin{+\!\!+} b : w)) \geq_d smsp\ (trim_u\ (b : w))\ \wedge\ P \\
\equiv & \quad \{\ \text{induction}\ \}
\end{array}$$

$$
\begin{array}{ll}
& smsp\ (trim\ (u \mathbin{+\!\!+} b : w \mathbin{+\!\!+} y)) \geq_d smsp\ (trim_u\ (b : w))\ \wedge\ P \\
\equiv & \quad \{\ \text{since } trim_u\ (b : w) \sqsubseteq trim\ x\ \ \} \\
& smsp\ (trim\ (u \mathbin{+\!\!+} b : w \mathbin{+\!\!+} y)) \geq_d smsp\ (trim\ x)\ \wedge\ P \\
\Leftarrow & mds_{\mathsf{M}}\ (b : x) = Nothing\ \vee\ smsp\ (trim\ (u \mathbin{+\!\!+} b : x \mathbin{+\!\!+} y)) \geq_d mds\ (b : x).
\end{array}
$$

$\square$