Theory of Computation

Course note based on Computability, Complexity, and Languages:
Fundamentals of Theoretical Computer Science, 2nd edition,
authored by Martin Davis, Ron Sigal, and Elaine J. Weyuker.

course note prepared by

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Week 1, Spring 2008
About This Course Note

- It is prepared for the course *Theory of Computation* taught at the National Taiwan University in Spring 2008.
- It is available from Tyng-Ruey Chuang’s web site:
  
  http://www.iis.sinica.edu.tw/~trc/

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  http://creativecommons.org/licenses/by-sa/2.5/tw/
This course aims to cover . . .

- the development of computability theory using an extremely simple abstract programming language,
- the various different formulations of computability and their equivalence,
- the expressiveness and limitation of various kinds of automata and formal languages, and
- the basics of the theory of computational complexity.
By the end of this course, you should be able to . . .

▶ appreciate the existence of universal digital computers,
▶ understand there are well-defined functions that cannot be computed even by the universal computers,
▶ know that certain problems are truly harder than others,
▶ use various formalized computation models to solve your problems, and
▶ show that some problems are just too difficult for the models at hand.
Textbook


- Written for people who may know programming, but from a mathematical view of the subjects. Enjoyably readable but very rigorous.
- “It is our purpose . . . to provide an introduction to the various aspects of theoretical computer science for undergraduate and graduate students that is sufficiently comprehensive that . . . research papers will become accessible to our readers.” (the authors)
- We will cover just one half of the materials in the book.
Schedule (1/2)

02/20  Preliminaries; A Programming Language. (1.1–1.7; 2.1–2.2)
02/27  Computable Functions; Primitive Recursive Functions. (2.3–2.5; 3.1–3.4)
03/05  Coding Programs by Numbers. (3.5–3.8; 4.1)
03/12  The Halting Problem; Universality. (4.2–4.3)
03/19  Recursively Enumerable Sets. (4.4–4.5)
03/26  Diagonalization and Reducibility. (4.6–4.8)
04/02  A Computable Function That Is Not Primitive Recursive. (4.9)
04/09  Turing Machines. (6.1–6.4)
04/16  mid-term examination
Schedule (2/2)

04/23  Nondeterministic Turing Machines; Semi-Thue Processes. (6.5–6.5; 7.1–7.2)
04/30  Post’s Correspondence Problem. Grammars. (7.2–7.6)
05/07  Regular Languages, Part 1. (9.1–9.4)
05/14  Regular Languages, Part 2. (9.5–9.7)
05/21  Context-Free Languages, Part 1. (10.1–10.4)
05/28  Context-Free Languages, Part 2. (10.5–10.9)
06/04  Context-Sensitive Languages. (11.1–11.3)
06/11  Polynomial-Time Computability. (15.1–15.4)
06/18  final examination
Outline of Today’s Lecture

- Review some preliminary materials.
- Define an abstract programming language $S$ that is extremely simple.
- Write some programs in $S$. 
Cartesian Product

- If $S_1, S_2, \ldots, S_n$ are given sets, then we write $S_1 \times S_2, \times \cdots \times S_n$ for the set of all $n$-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_1 \in S_1, a_2 \in S_2, \ldots, a_n \in S_n$.
- $S_1 \times S_2, \times \cdots \times S_n$ is called the *Cartesian product* of $S_1, S_2, \ldots, S_n$.
- In case $S_1 = S_2 = \cdots = S_n = S$ we write $S^n$ for the Cartesian product $S_1 \times S_2, \times \cdots \times S_n$. 

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 Cartesian Product
Functions

- A function $f$ is a set whose members are ordered pairs (i.e., 2-tuples) and has the special property

  $$(a, b) \in f \text{ and } (a, c) \in f \implies b = c.$$  

  We write $f(a) = b$ to mean that $(a, b) \in f$. 

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  We write $f(a) = b$ to mean that $(a, b) \in f$.

- The set of all $a$ such that $(a, b) \in f$ for some $b$ is called the *domain* of $f$. The set of all $f(a)$ for $a$ in the domain of $f$ is called the *range* of $f$. 

  [Note: The partial function and its definition were not included in the text provided.]
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- A partial function on a set $S$ is a function whose domain is a subset of $S$. If a partial function on $S$ has the domain $S$, then it is called a total function.
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- A partial function on a set $S$ is a function whose domain is a subset of $S$. If a partial function on $S$ has the domain $S$, then it is called a total function.

- We write $f(a) \downarrow$ and say that $f(a)$ is defined if $a$ is in the domain of $f$; if $a$ is not in the domain of $f$, we write $f(a) \uparrow$ and say that $f(a)$ is undefined.
Examples of Functions

Let $f$ be the set of ordered pairs $(n, n^2)$ for $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, $f(n) = n^2$. The domain of $f$ is $\mathbb{N}$. The range of $f$ is the set of perfect squares. $f$ is a total function.
Examples of Functions

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- Assuming \( \mathbb{N} \) is our universe, an example of a partial function on \( \mathbb{N} \) is given by \( g(n) = \sqrt{n} \). The domain of \( g \) is the set of perfect squares. The range of \( g \) is \( \mathbb{N} \). \( g \) is not a total function.
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- Assuming $\mathbb{N}$ is our universe, an example of a partial function on $\mathbb{N}$ is given by $g(n) = \sqrt{n}$. The domain of $g$ is the set of perfect squares. The range of $g$ is $\mathbb{N}$. $g$ is not a total function.

- For a partial function $f$ on a Cartesian product $S_1 \times S_2, \times \cdots \times S_n$, we write $f(a_1, \ldots, a_n)$ rather than $f((a_1, \ldots, a_n))$. 
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- For a partial function $f$ on a Cartesian product $S_1 \times S_2, \times \cdots \times S_n$, we write $f(a_1, \ldots, a_n)$ rather than $f((a_1, \ldots, a_n))$.

- A partial function $f$ on a set $S^n$ is called an $n$-ary partial function on $S$, or a function of $n$ variables on $S$. We use unary and binary for 1-ary and 2-ary, respectively.
The Programming Language $\mathcal{I}$

- **Values**: natural numbers only, but of unlimited precision.
The Programming Language $S$

- **Values**: natural numbers only, but of unlimited precision.
- **Variables**:
  - Input variables $X_1, X_2, X_3, \ldots$
  - An output variable $Y$
  - Local variables $Z_1, Z_2, Z_3, \ldots$
The Programming Language $\mathcal{S}$

- **Values:** natural numbers only, but of unlimited precision.
- **Variables:**
  - Input variables $X_1, X_1, X_3, \ldots$
  - An output variable $Y$
  - Local variables $Z_1, Z_1, Z_3, \ldots$
- **Instructions:**
  - $V \leftarrow V + 1$ Increase by 1 the value of the variable $V$.
  - $V \leftarrow V - 1$ If the value of $V$ is 0, leave it unchanged; otherwise decrease by 1 the value of $V$.
  - IF $V \neq 0$ GOTO $L$ If the value of $V$ is nonzero, perform the instruction with label $L$ next; otherwise proceed to the next instruction in the list.
The Programming Language $\mathcal{L}$

- **Values**: natural numbers only, but of unlimited precision.
- **Variables**:
  - Input variables $X_1, X_1, X_3, \ldots$
  - An output variable $Y$
  - Local variables $Z_1, Z_1, Z_3, \ldots$
- **Instructions**:
  - $V \leftarrow V + 1$ Increase by 1 the value of the variable $V$.
  - $V \leftarrow V - 1$ If the value of $V$ is 0, leave it unchanged; otherwise decrease by 1 the value of $V$.
  - IF $V \neq 0$ GOTO $L$ If the value of $V$ is nonzero, perform the instruction with label $L$ next; otherwise proceed to the next instruction in the list.
- **Labels**: $A_1, B_1, C_1, D_1, E_1, A_2, B_2, C_2, D_2, E_2, A_3, \ldots$
- **Exit label**: $E$. 

▶ All variables and labels are in the global scope.
The Programming Language $\mathcal{S}$

- **Values:** natural numbers only, but of unlimited precision.
- **Variables:**
  - Input variables $X_1, X_1, X_3, \ldots$
  - An output variable $Y$
  - Local variables $Z_1, Z_1, Z_3, \ldots$
- **Instructions:**
  - $V \leftarrow V + 1$ Increase by 1 the value of the variable $V$.
  - $V \leftarrow V - 1$ If the value of $V$ is 0, leave it unchanged; otherwise decrease by 1 the value of $V$.
  - $\text{IF } V \neq 0 \text{ GOTO } L$ If the value of $V$ is nonzero, perform the instruction with label $L$ next; otherwise proceed to the next instruction in the list.
- **Labels:** $A_1, B_1, C_1, D_1, E_1, A_2, B_2, C_2, D_2, E_2, A_3, \ldots$
- **Exit label:** $E$.
- **All variables and labels are in the global scope.**
Programming in $\mathcal{L}$

- A program is a list (i.e., a finite sequence) of instructions.
- The output variable $Y$ and the local variables $Z_i$ initially have the value 0.
- A program halts when there is no more instruction to execute.
- A program also halts if an instruction labeled $L$ is to be executed, but there is no instruction with that label.
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- A program halts when there is no more instruction to execute.
- A program also halts if an instruction labeled $L$ is to be executed, but there is no instruction with that label.
- What does this program do?

\[
[A] \quad X \leftarrow X - 1 \\
Y \leftarrow Y + 1 \\
\text{IF } X \neq 0 \text{ GOTO } A
\]
A Bug?

What does this program do?

[A]  
\[ \begin{align*} 
X & \leftarrow X - 1 \\
Y & \leftarrow Y + 1 \\
\text{IF } X \neq 0 \text{ GOTO } A 
\end{align*} \]
A Bug?

What does this program do?

[A] \[ \begin{align*}
X & \leftarrow X - 1 \\
Y & \leftarrow Y + 1 \\
\text{IF } X \neq 0 \text{ GOTO } A
\end{align*} \]

The above program *computes* the function

\[ f(x) = \begin{cases} 
1 & \text{if } x = 0 \\
x & \text{otherwise.}
\end{cases} \]
A Program That Computes $f(x) = x$

[A] IF $X \neq 0$ GOTO $B$
    $Z \leftarrow Z + 1$
    IF $Z \neq 0$ GOTO $E$

[B] $X \leftarrow X - 1$
    $Y \leftarrow Y + 1$
    $Z \leftarrow Z + 1$
    IF $Z \neq 0$ GOTO $A$

▶ What does $Z$ actually do?
▶ What does the following do?
$Z \leftarrow Z + 1$
IF $Z \neq 0$ GOTO $L$
A Program That Computes $f(x) = x$

[A] IF $X \neq 0$ GOTO $B$
    $Z \leftarrow Z + 1$
    IF $Z \neq 0$ GOTO $E$

[B] $X \leftarrow X - 1$
    $Y \leftarrow Y + 1$
    $Z \leftarrow Z + 1$
    IF $Z \neq 0$ GOTO $A$

What does $Z$ actually do?
A Program That Computes $f(x) = x$

[A] IF $X \neq 0$ GOTO $B$
    $Z \leftarrow Z + 1$
    IF $Z \neq 0$ GOTO $E$

[B] $X \leftarrow X - 1$
    $Y \leftarrow Y + 1$
    $Z \leftarrow Z + 1$
    IF $Z \neq 0$ GOTO $A$

- What does $Z$ actually do?
- What does the following do?
  
  $Z \leftarrow Z + 1$
  IF $Z \neq 0$ GOTO $L$
A Macro for Unconditional GOTO

Before macro expansion:

[A] IF $X \neq 0$ GOTO $B$
GOTO $E$

[B] $X \leftarrow X - 1$
$Y \leftarrow Y + 1$
GOTO $A$

Fresh local variables are always used during macro expansions.
A Macro for Unconditional GOTO

Before macro expansion:

[A] IF \( X \neq 0 \) GOTO B

GOTO E

[B] \( X \leftarrow X - 1 \)

\( Y \leftarrow Y + 1 \)

GOTO A

After macro expansion:

[A] IF \( X \neq 0 \) GOTO B

\( Z_1 \leftarrow Z_1 + 1 \)

IF \( Z_1 \neq 0 \) GOTO E

[B] \( X \leftarrow X - 1 \)

\( Y \leftarrow Y + 1 \)

\( Z_2 \leftarrow Z_2 + 1 \)

IF \( Z_2 \neq 0 \) GOTO A
A Macro for Unconditional GOTO

Before macro expansion:

[A] IF \( X \neq 0 \) GOTO B
    GOTO E
[B] \( X \leftarrow X - 1 \)
    \( Y \leftarrow Y + 1 \)
    GOTO A

After macro expansion:

[A] IF \( X \neq 0 \) GOTO B
    \( Z_1 \leftarrow Z_1 + 1 \)
    IF \( Z_1 \neq 0 \) GOTO E
[B] \( X \leftarrow X - 1 \)
    \( Y \leftarrow Y + 1 \)
    \( Z_2 \leftarrow Z_2 + 1 \)
    IF \( Z_2 \neq 0 \) GOTO A

Fresh local variables are always used during macro expansions.
Copy The Value of Variable $X$ to Variable $Y$

- [A] \[ \text{IF } X \neq 0 \text{ GOTO } B \]
  \[ \text{GOTO } E \]
- [B] \[ X \leftarrow X - 1 \]
  \[ Y \leftarrow Y + 1 \]
  \[ \text{GOTO } A \]

Anything wrong?

The value of $X$ is "destroyed" while copied to $Y$!
Copy The Value of Variable $X$ to Variable $Y$

- **[A]** IF $X \neq 0$ GOTO $B$
  GOTO $E$
- **[B]** $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  GOTO $A$

- Anything wrong?
Copy The Value of Variable $X$ to Variable $Y$

- **$A$**
  - IF $X \neq 0$ GOTO $B$
  - GOTO $E$

- **$B$**
  - $X \leftarrow X - 1$
  - $Y \leftarrow Y + 1$
  - GOTO $A$

- Anything wrong?
- The value of $X$ is “destroyed” while copied to $Y$!
Copy The Value of Variable $X$ to Variable $Y$, Continued

- **[A]** IF $X \neq 0$ GOTO $B$
  GOTO $C$

- **[B]** $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  $Z \leftarrow Z + 1$
  GOTO $A$

- **[C]** IF $Z \neq 0$ GOTO $D$
  GOTO $E$

- **[D]** $Z \leftarrow Z - 1$
  $X \leftarrow X + 1$
  GOTO $C$

Anything wrong?

This program is correct only when $Y$ and $Z$ are initialized to the value 0. It cannot be used as a macro.
Copy The Value of Variable $X$ to Variable $Y$, Continued

- **[A]** IF $X \neq 0$ GOTO $B$
  GOTO $C$
- **[B]** $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  $Z \leftarrow Z + 1$
  GOTO $A$
- **[C]** IF $Z \neq 0$ GOTO $D$
  GOTO $E$
- **[D]** $Z \leftarrow Z - 1$
  $X \leftarrow X + 1$
  GOTO $C$

- Anything wrong?
Copy The Value of Variable $X$ to Variable $Y$, Continued

- **[A]** IF $X \neq 0$ GOTO $B$
  GOTO $C$

- **[B]** $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  $Z \leftarrow Z + 1$
  GOTO $A$

- **[C]** IF $Z \neq 0$ GOTO $D$
  GOTO $E$

- **[D]** $Z \leftarrow Z - 1$
  $X \leftarrow X + 1$
  GOTO $C$

- Anything wrong?

- This program is correct only when $Y$ and $Z$ are initialized to the value 0. It cannot be used as a macro.
A Macro for $V \leftarrow V'$

- $V \leftarrow 0$
  - [A] IF $V' \neq 0$ GOTO $B$
  - GOTO $C$
- $V \leftarrow V' - 1$
  - $V \leftarrow V' + 1$
  - $Z \leftarrow Z + 1$
  - GOTO $A$
- [C] IF $Z \neq 0$ GOTO $D$
  - GOTO $E$
- [D] $Z \leftarrow Z - 1$
  - $V' \leftarrow V' + 1$
  - GOTO $C$
A Macro for $V \leftarrow V'$

- $V \leftarrow 0$
  
  - [A] IF $V' \neq 0$ GOTO $B$
  - GOTO $C$

- $B$
  
  - $V \leftarrow V' - 1$
  - $V \leftarrow V + 1$
  - $Z \leftarrow Z + 1$
  - GOTO $A$

- [C] IF $Z \neq 0$ GOTO $D$
  - GOTO $E$

- [D]
  
  - $Z \leftarrow Z - 1$
  - $V' \leftarrow V' + 1$
  - GOTO $C$

- Anything wrong?
A Macro for $V \leftarrow V'$

- $V \leftarrow 0$
- [A] IF $V' \neq 0$ GOTO $B$
  GOTO $C$
- [B] $V \leftarrow V' - 1$
  $V \leftarrow V + 1$
  $Z \leftarrow Z + 1$
  GOTO $A$
- [C] IF $Z \neq 0$ GOTO $D$
  GOTO $E$
- [D] $Z \leftarrow Z - 1$
  $V' \leftarrow V' + 1$
  GOTO $C$

Anything wrong?

$V \leftarrow 0$ is not an instruction in $S$. 
A Macro for $V \leftarrow 0$
A Macro for $V \leftarrow 0$

\[
\text{[L]} \quad V \leftarrow V - 1 \\
\text{IF } V \neq 0 \text{ GOTO } L
\]
A Program That Computes $f(x_1, x_2) = x_1 + x_2$

$Y \leftarrow X_1$
$Z \leftarrow X_2$

[B] IF $Z \neq 0$ GOTO A
GOTO E

[A] $Z \leftarrow Z - 1$
$Y \leftarrow Y + 1$
GOTO B

Note that $Z$ is used to preserve the value of $X_2$ so that it will not be destroyed during the computation.
A Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$

\[
\begin{align*}
Z_2 & \leftarrow X_2 \\
[B] & \quad \text{IF } Z_2 \neq 0 \text{ GOTO } A \\
& \quad \text{GOTO } E \\
[A] & \quad Z_2 \leftarrow Z_2 - 1 \\
& \quad Z_1 \leftarrow X_1 + Y \\
& \quad Y \leftarrow Z_1 \\
& \quad \text{GOTO } B
\end{align*}
\]
A Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$

1. $Z_2 \leftarrow X_2$
2. [B] IF $Z_2 \neq 0$ GOTO A
3. GOTO E
4. [A] $Z_2 \leftarrow Z_2 - 1$
5. $Z_1 \leftarrow X_1 + Y$
6. $Y \leftarrow Z_1$
7. GOTO B

OK!
A Shorter Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$?

$Z_2 \leftarrow X_2$

$[B] \quad \text{IF } Z_2 \neq 0 \text{ GOTO } A$
GOTO $E$

$[A] \quad Z_2 \leftarrow Z_2 - 1$
$Y \leftarrow X_1 + Y$
GOTO $B$
A Shorter Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$?

- $Z_2 \leftarrow X_2$
- [B] IF $Z_2 \neq 0$ GOTO A
- GOTO E
- [A] $Z_2 \leftarrow Z_2 - 1$
- $Y \leftarrow X_1 + Y$
- GOTO B

- NO GOOD!
A Shorter Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$?

- $Z_2 \leftarrow X_2$
- \[B\] IF $Z_2 \neq 0$ GOTO $A$
  GOTO $E$
- \[A\] $Z_2 \leftarrow Z_2 - 1$
  $Y \leftarrow X_1 + Y$
  GOTO $B$

- NO GOOD!
- Why?
The macro for \( f(x_1, x_2) = x_1 + x_2 \)

\[
\begin{align*}
Y & \leftarrow X_1 \\
Z & \leftarrow X_2 \\
[B] & \quad \text{IF } Z \neq 0 \text{ GOTO } A \\
& \quad \text{GOTO } E \\
[A] & \quad Z \leftarrow Z - 1 \\
& \quad Y \leftarrow Y + 1 \\
& \quad \text{GOTO } B
\end{align*}
\]
The macro for \( f(x_1, x_2) = x_1 + x_2 \)

\[
\begin{align*}
Y & \leftarrow X_1 \\
Z & \leftarrow X_2 \\
[B] & \text{ IF } Z \neq 0 \text{ GOTO } A \\
& \text{ GOTO } E \\
[A] & \text{ Z } \leftarrow Z - 1 \\
& \text{ Y } \leftarrow Y + 1 \\
& \text{ GOTO } B
\end{align*}
\]

Macro expanding \( Y \leftarrow X_1 + Y \):

\[
\begin{align*}
Y & \leftarrow X_1 \\
Z & \leftarrow Y \\
[B] & \text{ IF } Z \neq 0 \text{ GOTO } A \\
& \text{ GOTO } E \\
[A] & \text{ Z } \leftarrow Z - 1 \\
& \text{ Y } \leftarrow Y + 1 \\
& \text{ GOTO } B
\end{align*}
\]
The macro for \( f(x_1, x_2) = x_1 + x_2 \)

\[
Y \leftarrow X_1 \\
Z \leftarrow X_2 \\
[B] \quad \text{IF } Z \neq 0 \quad \text{GOTO A} \\
\quad \text{GOTO E}
\]

\[
[A] \quad Z \leftarrow Z - 1 \\
\quad Y \leftarrow Y + 1 \\
\quad \text{GOTO B}
\]

Macro expanding \( Y \leftarrow X_1 + Y \):

\[
Y \leftarrow X_1 \\
Z \leftarrow Y \\
[B] \quad \text{IF } Z \neq 0 \quad \text{GOTO A} \\
\quad \text{GOTO E}
\]

\[
[A] \quad Z \leftarrow Z - 1 \\
\quad Y \leftarrow Y + 1 \\
\quad \text{GOTO B}
\]

The above actually computes \( f(x_1, x_2) = 2 \cdot x_1 \)
A Program That Computes \( f(x_1, x_2) = x_1 \cdot x_2 \), Revisited

- Need to macro expand \( Z_1 \leftarrow X_1 + Y \).
- After macro expansion:

\[
\begin{align*}
Z_2 & \leftarrow X_2 \\
[B] \quad & \text{IF } Z_2 \neq 0 \text{ GOTO } A \\
& \text{GOTO } E \\
[A] \quad & Z_2 \leftarrow Z_2 - 1 \\
& Z_1 \leftarrow X_1 \\
& Z_3 \leftarrow Y \\
[B_2] \quad & \text{IF } Z_3 \neq 0 \text{ GOTO } A_2 \\
& \text{GOTO } E_2 \\
[A_2] \quad & Z_3 \leftarrow Z_3 - 1 \\
& Z_1 \leftarrow Z_1 + 1 \\
& \text{GOTO } B_2 \\
[E_2] \quad & Y \leftarrow Z_1 \\
& \text{GOTO } B
\end{align*}
\]
Note on The Macro Expansion

- The output variable $Y$ in the macro $f(x_1, x_2) = x_1 + x_2$ is now a fresh variable $Z_1$ in the expanded form.
- The local variable $Z$ in the macro $f(x_1, x_2) = x_1 + x_2$ is now a fresh variable $Z_3$ in the expanded form (as variables $Z_1$ and $Z_2$ are already used).
- Fresh labels $A_2, B_2, \text{ and } E_2$ are used in the expanded form (as the original labels $A, B, \text{ and } E$ are already used).
- The instruction GOTO $E_2$ only terminates the addition. The computation must continue to place following the addition. Hence, the instruction immediately following the addition is labeled $E_2$.
- *Unlimited supply of fresh local variables and local labels!*
- More about macro expansion next week.
A Final Example

What does this program compute?

\[
\begin{align*}
Y & \leftarrow X_1 \\
Z & \leftarrow X_2 \\
\text{[C]} & \text{ IF } Z \neq 0 \text{ GOTO A} \\
& \text{ GOTO E} \\
\text{[A]} & \text{ IF } Y \neq 0 \text{ GOTO B} \\
& \text{ GOTO A} \\
\text{[B]} & \text{ Y } \leftarrow Y - 1 \\
& \text{ Z } \leftarrow Z - 1 \\
& \text{ GOTO C}
\end{align*}
\]
A Final Example

What does this program compute?

\[
Y \leftarrow X_1 \\
Z \leftarrow X_2 \\
[C] \quad \text{IF } Z \neq 0 \text{ GOTO } A \\
\quad \text{GOTO } E \\
[A] \quad \text{IF } Y \neq 0 \text{ GOTO } B \\
\quad \text{GOTO } A \\
[B] \quad Y \leftarrow Y - 1 \\
\quad Z \leftarrow Z - 1 \\
\quad \text{GOTO } C \\
\]

If we begin with \( X_1 = 5 \) and \( X_2 = 2 \), . . .
A Final Example

- What does this program compute?

```
Y ← X_1
Z ← X_2
[C] IF Z ≠ 0 GOTO A
    GOTO E
[A] IF Y ≠ 0 GOTO B
    GOTO A
[B] Y ← Y − 1
    Z ← Z − 1
    GOTO C
```

- If we begin with $X_1 = 5$ and $X_2 = 2$, . . .
- If we begin with $X_1 = 2$ and $X_2 = 5$, . . .
A Final Example

- What does this program compute?
  
  ```
  Y ← X₁
  Z ← X₂
  [C] IF Z ≠ 0 GOTO A
  GOTO E
  [A] IF Y ≠ 0 GOTO B
  GOTO A
  [B] Y ← Y − 1
  Z ← Z − 1
  GOTO C
  ```

- If we begin with $X₁ = 5$ and $X₂ = 2$, . . .
- If we begin with $X₁ = 2$ and $X₂ = 5$, . . .
- This program computes the following partial function

  $$g(x₁, x₂) = \begin{cases} 
  x₁ - x₂ & \text{if } x₁ \geq x₂ \\
  ↑ & \text{if } x₁ < x₂
  \end{cases}$$