Theory of Computation


course note prepared by

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About This Course Note

- It is prepared for the course *Theory of Computation* taught at the National Taiwan University in Spring 2008.
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Predicate

A *predicate*, or a *Boolean-valued function*, on a set $S$ is a *total* function $P$ on $S$ such that for each $a \in S$, either

$$P(a) = \text{TRUE} \quad \text{or} \quad P(a) = \text{FALSE}$$

We also identify the truth value TRUE with number 1 and the truth value FALSE with number 0.
Logic Connectives

The three *logic connectives*, or *propositional connectives*, $\sim$, $\lor$, $\land$ are defined by the two tables below.

$$
\begin{array}{c|c}
 p & \sim p \\
\hline
 0 & 1 \\
 1 & 0 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
 p & q & p \land q & p \lor q \\
\hline
 1 & 1 & 1 & 1 \\
 0 & 1 & 0 & 1 \\
 1 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 \\
\end{array}
$$
Characteristic Function

Given a predicate \( P \) on a set \( S \), there is a corresponding subset \( R \) of \( S \) consisting of all elements \( a \in S \) for which \( P(a) = 1 \). We write

\[
R = \{ a \in S \mid P(a) \}.
\]
Characteristic Function

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$$R = \{ a \in S \mid P(a) \}.$$ 

Conversely, given a subset $R$ of a given set $S$, the expression $x \in R$ defines a predicate $P$ on $S$:

$$P(x) = \begin{cases} 
1 & \text{if } x \in R \\
0 & \text{if } x \not\in R. 
\end{cases}$$

The predicate $P$ is called the *characteristic function* of the set $R$. 
Characteristic Function

Given a predicate $P$ on a set $S$, there is a corresponding subset $R$ of $S$ consisting of all elements $a \in S$ for which $P(a) = 1$. We write

$$R = \{a \in S \mid P(a)\}.$$  

Conversely, given a subset $R$ of a given set $S$, the expression $x \in R$ defines a predicate $P$ on $S$:

$$P(x) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{if } x \notin R. \end{cases}$$

The predicate $P$ is called the characteristic function of the set $R$. Note the easy translations between the two notations:

$$\{x \in S \mid P(x) \& Q(x)\} = \{x \in S \mid P(x)\} \cap \{x \in S \mid Q(x)\},$$

$$\{x \in S \mid P(x) \lor Q(x)\} = \{x \in S \mid P(x)\} \cup \{x \in S \mid Q(x)\},$$

$$\{x \in S \mid \sim P(x)\} = S - \{x \in S \mid P(x)\}.$$
Bounded Existential Quantifier

Let $P(t, x_1, \ldots, x_n)$ be a $(n + 1)$-ary predicate. Let predicate $Q(y, x_1, \ldots, x_n)$ be defined by

$$Q(y, x_1, \ldots, x_n) = P(0, x_1, \ldots, x_n) \lor P(1, x_1, \ldots, x_n) \lor \ldots \lor P(y, x_1, \ldots, x_n)$$

That is, $Q(y, x_1, \ldots, x_n)$ is true if there is a value $t \leq y$ such that $P(t, x_1, \ldots, x_n)$ is true. We write this predicate $Q$ as $(\exists t) \leq y P(t, x_1, \ldots, x_n)$.
Bounded Existential Quantifier

Let \( P(t, x_1, \ldots, x_n) \) be a \((n+1)\)-ary predicate. Let predicate \( Q(y, x_1, \ldots, x_n) \) be defined by

\[
Q(y, x_1, \ldots, x_n) = P(0, x_1, \ldots, x_n) \\
\lor P(1, x_1, \ldots, x_n) \\
\lor \ldots \\
\lor P(y, x_1, \ldots, x_n)
\]

That is, \( Q(y, x_1, \ldots, x_n) \) is true if there is a value \( t \leq y \) such that \( P(t, x_1, \ldots, x_n) \) is true. We write this predicate \( Q \) as

\[
(\exists t)_{\leq y} P(t, x_1, \ldots, x_n)
\]

“(\exists t)_{\leq y}” is called a bounded existential quantifier.
Bounded Universal Quantifier

Let $P(t, x_1, \ldots, x_n)$ be a $(n+1)$-ary predicate. Let predicate $Q(y, x_1, \ldots, x_n)$ be defined by

$$Q(y, x_1, \ldots, x_n) = P(0, x_1, \ldots, x_n)$$
$$& P(1, x_1, \ldots, x_n)$$
$$& \ldots$$
$$& P(y, x_1, \ldots, x_n)$$

That is, $Q(y, x_1, \ldots, x_n)$ is true if for all values $t \leq y$ such that $P(t, x_1, \ldots, x_n)$ is true. We write this predicate $Q$ as

$$\forall t \leq y P(t, x_1, \ldots, x_n)$$

"$\forall t \leq y$" is called a bounded universal quantifier.
Bounded Universal Quantifier

Let $P(t, x_1, \ldots, x_n)$ be a $(n + 1)$-ary predicate. Let predicate $Q(y, x_1, \ldots, x_n)$ be defined by

$$
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\& \ldots
\& P(y, x_1, \ldots, x_n)
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That is, $Q(y, x_1, \ldots, x_n)$ is true if for all value $t \leq y$ such that $P(t, x_1, \ldots, x_n)$ is true. We write this predicate $Q$ as

$$(\forall t)_{\leq y} P(t, x_1, \ldots, x_n)$$

“$(\forall t)_{\leq y}$” is called a bounded universal quantifier.
Proof by Contradiction

In a *proof by contradiction*, we begin by assuming the assertion we wish to prove is *false*. We then derive a contradiction based on this (faulty) assumption along with (faultless) logical reasoning. We then conclude that the original assertion must be true.
Proof by Contradiction: Example

Prove that the equation $2 = (m/n)^2$ has no solution $m, n \in N$. 

Proof. Assume $2 = (m/n)^2$ has a solution $m, n \in N$. Then it must also have a solution where not both $m$ and $n$ are even. This is so because we can repeatedly “cancel” 2 from $m$ and $n$ until at least one of them becomes odd, and still have the two “reduced” numbers as a solution. However, the equation $2 = (m/n)^2$ can be rewritten as $m^2 = 2n^2$ which shows that $m$ must be even. Let $m = 2k$, then $m^2 = (2k)^2 = 4k^2$. But this implies $n^2 = 2k^2$. Thus $n$ is even.

Now both $m$ and $n$ are even, which is a contradiction. We conclude that $2 = (m/n)^2$ has no solution $m, n \in N$. □
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Proof by Contradiction: Example

Prove that the equation $2 = (m/n)^2$ has no solution $m, n \in N$.

*Proof.* Assume $2 = (m/n)^2$ *has* a solution $m, n \in N$. Then it must also have a solution where not both $m$ and $n$ are even. This is so because we can repeatedly “cancel” 2 from $m$ and $n$ until at least one of them becomes *odd*, and still have the two “reduced” numbers as a solution.

However, the equation $2 = (m/n)^2$ can be rewritten as $m^2 = 2n^2$ which shows that $m$ must be even. Let $m = 2k$, then $m^2 = (2k)^2 = 4k^2$. But this implies $n^2 = 2k^2$. Thus $n$ is even. Now both $m$ and $n$ are even, which is a contradiction.

We conclude that $2 = (m/n)^2$ has no solution $m, n \in N$. □
Mathematical Induction

Given a predicate $P(x)$, and the assertion "$P(n)$ is true for all $n \in \mathbb{N}$", we can use mathematical induction to try to establish this assertion. One proceeds by proving a pair of auxiliary statements about $P(x)$, namely,

$$P(0)$$

and

For all $n \in \mathbb{N}$, $P(n)$ implies $P(n + 1)$
Mathematical Induction

Given a predicate $P(x)$, and the assertion “$P(n)$ is true for all $n \in \mathbb{N}$”, we can use mathematical induction to try to establish this assertion. One proceeds by proving a pair of auxiliary statements about $P(x)$, namely,

$$P(0)$$

and

$$\text{For all } n \in \mathbb{N}, \ P(n) \implies P(n+1)$$

In the second statement above, $P(n)$ is called an induction hypothesis. If both statements above are proved to be true, one then concludes that

$$\text{For all } n \in \mathbb{N}, \ P(n)$$
Mathematical Induction: Example

Prove that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n}(2i + 1) = (n + 1)^2$. 

Proof. For $n = 0$, then $\sum_{i=0}^{0}(2i + 1) = 1 = (0 + 1)^2$, which is true.

It remains to show that for all $n \in \mathbb{N}$, if $\sum_{i=0}^{n}(2i + 1) = (n + 1)^2$ is true, then $\sum_{i=0}^{n+1}(2i + 1) = (n + 2)^2$ is also true.

We expand $\sum_{i=0}^{n+1}(2i + 1)$ by its definition,

$\sum_{i=0}^{n+1}(2i + 1) = \sum_{i=0}^{n}(2i + 1) + 2(n + 1) + 1 = (n + 1)^2 + 2(n + 1) + 1$ (by induction hypothesis)

$= (n + 2)^2$.

We conclude that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n}(2i + 1) = (n + 1)^2$. □
Mathematical Induction: Example

Prove that for all \( n \in \mathbb{N} \), \( \sum_{i=0}^{n}(2i + 1) = (n + 1)^2 \).

Proof. For \( n = 0 \), then \( \sum_{i=0}^{0}(2i + 1) = 1 = (0 + 1)^2 \), which is true.
Mathematical Induction: Example

Prove that for all $n \in N$, $\sum_{i=0}^{n}(2i + 1) = (n + 1)^2$.

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\[
\sum_{i=0}^{n+1}(2i + 1) = \sum_{i=0}^{n}(2i + 1) + 2(n + 1) + 1
\]

\[
= (n + 1)^2 + 2(n + 1) + 1 \quad \text{(by induction hypothesis)}
\]

\[
= (n + 2)^2.
\]
Mathematical Induction: Example

Prove that for all \( n \in \mathbb{N} \), \( \sum_{i=0}^{n} (2i + 1) = (n + 1)^2 \).

Proof. For \( n = 0 \), then \( \sum_{i=0}^{0} (2i + 1) = 1 = (0 + 1)^2 \), which is true. It remains to show that for all \( n \in \mathbb{N} \), if \( \sum_{i=0}^{n} (2i + 1) = (n + 1)^2 \) is true, then \( \sum_{i=0}^{n+1} (2i + 1) = (n + 2)^2 \) is also true. We expand \( \sum_{i=0}^{n+1} (2i + 1) \) by its definition,

\[
\sum_{i=0}^{n+1} (2i + 1) = \sum_{i=0}^{n} (2i + 1) + 2(n + 1) + 1 = (n + 1)^2 + 2(n + 1) + 1 \quad \text{(by induction hypothesis)} = (n + 2)^2.
\]

We conclude that for all \( n \in \mathbb{N} \), \( \sum_{i=0}^{n} (2i + 1) = (n + 1)^2 \). \( \square \)
Initial Functions

The following functions are called *initial functions*:

\[
\begin{align*}
    s(x) & = x + 1, \\
    n(x) & = 0, \\
    u^n_i(x_1, \ldots, x_n) & = x_i, \quad 1 \leq i \leq n.
\end{align*}
\]

Note: Function \( u^n_i \) is called the *projection function*. For example, \( u^4_3(x_1, x_2, x_3, x_4) = x_3 \).
A class of total functions $\mathcal{C}$ is called a *PRC* class if

- the initial functions belong to $\mathcal{C}$,
- a function obtained from functions belonging to $\mathcal{C}$ by either composition or recursion also belongs to $\mathcal{C}$. 
Computable Functions are Primitive Recursively Closed

**Theorem 3.1.** The class of computable functions is a PRC class.

**Proof.** We have shown computable functions are closed under composition and recursion (Theorem 1.1 & 2.2). We need only verify the initial functions are computable. They are computed by the following programs.

\[
\begin{align*}
\text{s}(x) &= x + 1 & Y &\leftarrow X + 1; \\
n(x) &\quad \text{the empty program;} \\
u^n_i(x_1, \ldots, x_n) &\quad Y \leftarrow X_i.
\end{align*}
\]
A function is called *primitive recursive* if it can be obtained from the initial functions by a finite number of applications of composition and recursion.
A function is called *primitive recursive* if it can be obtained from the initial functions by a finite number of applications of composition and recursion.

Note that, by the above definition and the definition of Primitive Recursively Closed (PRC), it follows that:

**Corollary 3.2.** The class of primitive recursive function is a PRC class.
**Theorem 3.3.** A function is primitive recursive if and only if it belongs to every PRC class.

**Proof.**

\((\Leftarrow)\) If a function belongs to every PRC class, then by Corollary 3.2, it belongs to the class of primitive recursive functions.

\((\Rightarrow)\) If \(f\) is primitive recursive, then there is a list of functions \(f_1, f_2, \ldots, f_n\) such that \(f_n = f\) and for each \(f_i, 1 \leq i < n\), either

- \(f_i\) is an initial function, or
- \(f_i\) can be obtained from the preceding functions in the list by composition or recursion.

However, the initial functions belong to any PRC class \(C\). Furthermore, all functions obtained from functions in \(C\) by composition or recursion also belong to \(C\). It follows that each function \(f_1, f_2, \ldots, f_n = f\) in the above list is in \(C\).
Corollary 3.4. Every primitive recursive function is computable.

Proof. By Theorem 3.4, every primitive recursive function belongs to the PRC class of computable functions so is computable. □
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Note that,

► If a function $f$ is shown to be primitive recursive, by the above Corollary, $f$ can be expressed as a program in language $\mathcal{S}$.

► Not only we know there is program in $\mathcal{S}$ for $f$, by Theorem 3.1 (1.1 & 2.2), we also know how to write this program.

► Furthermore, the program so written will always terminate.
Corollary 3.4. Every primitive recursive function is computable. 
Proof. By Theorem 3.4, every primitive recursive function belongs to the PRC class of computable functions so is computable. □

Note that,

- If a function \( f \) is shown to be primitive recursive, by the above Corollary, \( f \) can be expressed as a program in language \( \mathcal{S} \).
- Not only we know there is program in \( \mathcal{S} \) for \( f \), by Theorem 3.1 (1.1 & 2.2), we also know how to write this program.
- Furthermore, the program so written will always terminate.

However, if a function \( f \) is computable (that is, it is total and expressible in \( \mathcal{S} \)), it is not necessarily that \( f \) is primitive recursive. (A counter example will be shown later in this course.)
Function $f(x, y) = x + y$ is Primitive Recursive

Function $f$ can be defined by the recursion equations:

\[
\begin{align*}
    f(x, 0) &= x, \\
    f(x, y + 1) &= f(x, y) + 1.
\end{align*}
\]

The above can be rewritten as

\[
\begin{align*}
    f(x, 0) &= u_1^1(x), \\
    f(x, y + 1) &= g(y, f(x, y), x),
\end{align*}
\]

where

\[
g(x_1, x_2, x_3) = s(u_2^3(x_1, x_2, x_3)).
\]
Function $h(x, y) = x \cdot y$ Is Primitive Recursive

Function $h$ can be defined by the recursion equations:

$$
\begin{align*}
h(x, 0) & = 0, \\
h(x, y + 1) & = h(x, y) + x.
\end{align*}
$$

The above can be rewritten as

$$
\begin{align*}
h(x, 0) & = n(x), \\
h(x, y + 1) & = g(y, h(x, y), x),
\end{align*}
$$

where

$$
\begin{align*}
g(x_1, x_2, x_3) & = f(u_2^3(x_1, x_2, x_3), u_3^3(x_1, x_2, x_3)), \\
f(x, y) & = x + y.
\end{align*}
$$
Function $h(x) = x!$ is Primitive Recursive

Function $h(x)$ can be defined by

\[
\begin{align*}
    h(0) &= 1, \\
    h(t + 1) &= g(t, h(t)),
\end{align*}
\]

where

\[
g(x_1, x_2) = s(x_1) \cdot x_2.
\]

Note that $g$ is primitive recursive because

\[
g(x_1, x_2) = s(u_1^2(x_1, x_2)) \cdot u_2^2(x_1, x_2).
\]
Function $power(x, y) = x^y$ is Primitive Recursive

Function $power$ can be defined by

$$power(x, 0) = 1,$$
$$power(x, y + 1) = power(x, y) \cdot x.$$

Note that these equations assign the value 1 to the "indeterminate" $0^0$.

The above definition can be further rewritten into . . . .
The Predecessor Function Is Primitive Recursive

The predecessor function \( \text{pred}(x) \) is defined as follows:

\[
\text{pred}(x) = \begin{cases} 
  x - 1 & \text{if } x \neq 0 \\
  0 & \text{if } x = 0.
\end{cases}
\]

Note that function \( \text{pred} \) corresponds to the instruction \( X \leftarrow X - 1 \) in programming language \( S \).

The above definition can be further rewritten into . . . .
Function $x \cdot y$ Is Primitive Recursive

Function $x \cdot y$ is defined as follows:

$$x \cdot y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

Note that function $x \cdot y$ is different from function $x - y$, which is undefined if $x < y$. In particular, $x \cdot y$ is total while $x - y$ is not.
Function \( x \dot{-} y \) Is Primitive Recursive

Function \( x \dot{-} y \) is defined as follows:

\[
x \dot{-} y = \begin{cases} 
  x - y & \text{if } x \geq y \\
  0 & \text{if } x < y.
\end{cases}
\]

Note that function \( x \dot{-} y \) is different from function \( x - y \), which is undefined if \( x < y \). In particular, \( x \dot{-} y \) is total while \( x - y \) is not.

Function \( x \dot{-} y \) is primitive recursive because

\[
x \dot{-} 0 = x,
\]
\[
x \dot{-} (t + 1) = \text{pred}(x \dot{-} t).
\]

The above definition can be further rewritten into . . . .
Function $|x - y|$ Is Primitive Recursive

Function $|x - y|$ can be defined as follows:

$$|x - y| = (x - y) + (y - x)$$
Function $|x - y|$ is Primitive Recursive

Function $|x - y|$ can be defined as follows:

$$|x - y| = (x - y) + (y - x)$$

It is primitive recursive because the above definition can be further rewritten into . . . .
Is Function $\alpha(x)$ below Primitive Recursive?

Function $\alpha(x)$ is defined as:

$$\alpha(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$
Is Function $\alpha(x)$ below Primitive Recursive?

Function $\alpha(x)$ is defined as:

$$\alpha(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

It is primitive recursive because . . . .
Is the function $d(x, y)$ below primitive recursive?

$$
d(x, y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y 
\end{cases}
$$
$x = y$ Is Primitive Recursive

Is the function $d(x, y)$ below primitive recursive?

$$d(x, y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y 
\end{cases}$$

It is because $d(x, y) = \alpha(|x - y|)$. 
Is $x \leq y$ Primitive Recursive?
Is $x \leq y$ Primitive Recursive?

It is primitive recursive because $x \leq y = \alpha(x - y)$. 
Logic Connectives Are Primitive Recursively Closed

**Theorem 5.1.** Let $C$ be a PRC class. If $P$, $Q$ are predicates that belong to $C$, then so are $\sim P$, $P \lor Q$, and $P \land Q$. 
Logic Connectives Are Primitive Recursively Closed

**Theorem 5.1.** Let $\mathcal{C}$ be a PRC class. If $P$, $Q$ are predicates that belong to $\mathcal{C}$, then so are $\sim P$, $P \lor Q$, and $P \land Q$.

**Proof.** We define $\sim P$, $P \lor Q$, and $P \land Q$ as follows:

\[
\begin{align*}
\sim P &= \alpha(P) \\
P \land Q &= P \cdot Q \\
P \lor Q &= \sim (\sim P \land \sim Q)
\end{align*}
\]
Logic Connectives Are Primitive Recursively Closed

**Theorem 5.1.** Let $\mathcal{C}$ be a PRC class. If $P$, $Q$ are predicates that belong to $\mathcal{C}$, then so are $\sim P$, $P \lor Q$, and $P \& Q$.

*Proof.* We define $\sim P$, $P \lor Q$, and $P \& Q$ as follows:

\[
\sim P = \alpha(P) \\
P \& Q = P \cdot Q \\
P \lor Q = \sim (\sim P \& \sim Q)
\]

We conclude that $\sim P$, $P \lor Q$, and $P \& Q$ all belong to $\mathcal{C}$.

\[\square\]
Logic Connectives Are Primitive Recursive and Computable

**Corollary 5.2.** If $P$, $Q$ are primitive recursive predicates, then so are $\sim P$, $P \lor Q$, and $P \land Q$. 
Logic Connectives Are Primitive Recursive and Computable

**Corollary 5.2.** If $P$, $Q$ are primitive recursive predicates, then so are $\sim P$, $P \lor Q$, and $P \land Q$.

**Corollary 5.3.** If $P$, $Q$ are computable predicates, then so are $\sim P$, $P \lor Q$, and $P \land Q$. 
Is $x < y$ Primitive Recursive?

It is primitive recursive because $x < y \Leftrightarrow \neg (y \leq x)$. 

Is $x < y$ Primitive Recursive?

It is primitive recursive because

$$x < y \iff \sim (y \leq x).$$
Definition by Cases

**Theorem 5.4.** Let $C$ be a PRC class. Let functions $g$, $h$ and predicate $P$ belong to $C$. Let function

$$f(x_1, \ldots, x_n) = \begin{cases} g(x_1, \ldots, x_n) & \text{if } P(x_1, \ldots, x_n) \\ h(x_1, \ldots, x_n) & \text{otherwise.} \end{cases}$$

Then $f$ belongs to $C$. 
Definition by Cases

**Theorem 5.4.** Let $C$ be a PRC class. Let functions $g$, $h$ and predicate $P$ belong to $C$. Let function

$$f(x_1, \ldots, x_n) = \begin{cases} 
  g(x_1, \ldots, x_n) & \text{if } P(x_1, \ldots, x_n) \\
  h(x_1, \ldots, x_n) & \text{otherwise.}
\end{cases}$$

Then $f$ belongs to $C$.

**Proof.** Function $f$ belongs to $C$ because

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \cdot P(x_1, \ldots, x_n)$$

$$+ h(x_1, \ldots, x_n) \cdot \alpha(P(x_1, \ldots, x_n)).$$
Definition by Cases, More

**Corollary 5.5.** Let $\mathcal{C}$ be a PRC class. Let $n$-ary functions $g_1, \ldots, g_m, h$ and predicates $P_1, \ldots, P_m$ belong to $\mathcal{C}$, and let

$$P_i(x_1, \ldots, x_n) \& P_j(x_1, \ldots, x_n) = 0$$

for all $1 \leq i \leq j \leq m$ and all $x_1, \ldots, x_n$. If

$$f(x_1, \ldots, x_n) = \begin{cases} 
  g_1(x_1, \ldots, x_n) & \text{if } P_1(x_1, \ldots, x_n) \\
  \vdots & \vdots \\
  g_m(x_1, \ldots, x_n) & \text{if } P_m(x_1, \ldots, x_n) \\
  h(x_1, \ldots, x_n) & \text{otherwise.} 
\end{cases}$$

then $f$ also belongs to $\mathcal{C}$. 

Proof. Proved by a mathematical induction on $m$. $\square$
Definition by Cases, More

**Corollary 5.5.** Let \( \mathcal{C} \) be a PRC class. Let \( n \)-ary functions \( g_1, \ldots, g_m, h \) and predicates \( P_1, \ldots, P_m \) belong to \( \mathcal{C} \), and let

\[
P_i(x_1, \ldots, x_n) \land P_j(x_1, \ldots, x_n) = 0
\]

for all \( 1 \leq i \leq j \leq m \) and all \( x_1, \ldots, x_n \). If

\[
f(x_1, \ldots, x_n) = \begin{cases} 
  g_1(x_1, \ldots, x_n) & \text{if } P_1(x_1, \ldots, x_n) \\
  \vdots & \vdots \\
  g_m(x_1, \ldots, x_n) & \text{if } P_m(x_1, \ldots, x_n) \\
  h(x_1, \ldots, x_n) & \text{otherwise.}
\end{cases}
\]

then \( f \) also belongs to \( \mathcal{C} \).

**Proof.** Proved by a mathematical induction on \( m \).
Iterated Operations

**Theorem 6.1.** Let $\mathcal{C}$ be a PRC class. If function $f(t, x_1, \ldots, x_n)$ belongs to $\mathcal{C}$, then so do the functions $g$ and $h$

$$g(y, x_1, \ldots, x_n) = \sum_{t=0}^{y} f(t, x_1, \ldots, x_n)$$

$$h(y, x_1, \ldots, x_n) = \prod_{t=0}^{y} f(t, x_1, \ldots, x_n)$$
Iterated Operations

**Theorem 6.1.** Let \( \mathcal{C} \) be a PRC class. If function \( f(t, x_1, \ldots, x_n) \) belongs to \( \mathcal{C} \), then so do the functions \( g \) and \( h \)

\[
\begin{align*}
g(y, x_1, \ldots, x_n) &= \sum_{t=0}^{y} f(t, x_1, \ldots, x_n) \\
h(y, x_1, \ldots, x_n) &= \prod_{t=0}^{y} f(t, x_1, \ldots, x_n)
\end{align*}
\]

*Proof.* Functions \( g \) and \( h \) each can be recursively defined as

\[
\begin{align*}
g(0, x_1, \ldots, x_n) &= f(0, x_1, \ldots, x_n), \\
g(t + 1, x_1, \ldots, x_n) &= g(t, x_1, \ldots, x_n) + f(t + 1, x_1, \ldots, x_n), \\
h(0, x_1, \ldots, x_n) &= f(0, x_1, \ldots, x_n), \\
h(t + 1, x_1, \ldots, x_n) &= h(t, x_1, \ldots, x_n) \cdot f(t + 1, x_1, \ldots, x_n).
\end{align*}
\]
Iterated Operations, More

**Corollary 6.2.** Let $\mathcal{C}$ be a PRC class. If function $f(t, x_1, \ldots, x_n)$ belongs to $\mathcal{C}$, then so do the functions

$$g(y, x_1, \ldots, x_n) = \sum_{t=1}^{y} f(t, x_1, \ldots, x_n)$$

and

$$h(y, x_1, \ldots, x_n) = \prod_{t=1}^{y} f(t, x_1, \ldots, x_n).$$

In the above, we assume that

$$g(0, x_1, \ldots, x_n) = 0,$$

$$h(0, x_1, \ldots, x_n) = 1.$$
Bounded Quantifiers

**Theorem 6.3.** If predicate $P(t, x_1, \ldots, x_n)$ belongs to some PRC class $\mathcal{C}$, then so do the predicates

$$(\forall t)_{\leq y} P(t, x_1, \ldots, x_n)$$

and

$$(\exists t)_{\leq y} P(t, x_1, \ldots, x_n)$$
Bounded Quantifiers

**Theorem 6.3.** If predicate $P(t, x_1, \ldots, x_n)$ belongs to some PRC class $C$, then so do the predicates

$$(\forall t) \leq y P(t, x_1, \ldots, x_n)$$

and

$$(\exists t) \leq y P(t, x_1, \ldots, x_n)$$

**Proof.** We need only observe that

$$(\forall t) \leq y P(t, x_1, \ldots, x_n) \iff \prod_{t=0}^{y} P(t, x_1, \ldots, x_n) = 1$$

and

$$(\exists t) \leq y P(t, x_1, \ldots, x_n) \iff \sum_{t=0}^{y} P(t, x_1, \ldots, x_n) \neq 0$$
Bounded Quantifiers, More

Note that

\[(\forall t)_<^y P(t, x_1, \ldots, x_n) \iff (\forall t)_{\leq y}[t = y \lor P(t, x_1, \ldots, x_n)],\]

and

\[(\exists t)_<^y P(t, x_1, \ldots, x_n) \iff (\exists t)_{\leq y}[t \neq y \land P(t, x_1, \ldots, x_n)].\]
Note that

\[(\forall t)_{<y} P(t, x_1, \ldots, x_n) \iff (\forall t)_{\leq y} [t = y \lor P(t, x_1, \ldots, x_n)],\]

and

\[(\exists t)_{<y} P(t, x_1, \ldots, x_n) \iff (\exists t)_{\leq y} [t \neq y \land P(t, x_1, \ldots, x_n)].\]

Therefore, both the quantifiers \((\forall t)_{<y}\) and \((\exists t)_{<y}\) are primitive recursively closed.

Bounded Quantifiers, More
$y \mid x$ is primitive recursive

The “$y$ is a divisor of $x$” predicate $y \mid x$ is primitive recursive because
The “$y$ is a divisor of $x$” predicate $y|x$ is primitive recursive because

$$y|x \iff (\exists t \leq x) (y \cdot t = x).$$
Prime($x$) Is Primitive Recursive

The “$x$ is a prime” predicate $\text{Prime}(x)$ is primitive recursive because
Prime(x) Is Primitive Recursive

The “x is a prime” predicate $\text{Prime}(x)$ is primitive recursive because

$$\text{Prime}(x) \iff x > 1 \quad \& \quad (\forall t)_{\leq x} [t = 1 \quad \lor \quad t = x \quad \lor \quad \sim (t|\!x)].$$
Bounded Minimalization

What does the following function $g$ do?

$$g(y, x_1, \ldots, x_n) = \sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n))$$
Bounded Minimalization

What does the following function $g$ do?

$$g(y, x_1, \ldots, x_n) = \sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n))$$

It computes the least value $t \leq y$ for which $P(t, x_1, \ldots, x_n)$ is true!
Bounded Minimalization

What does the following function \( g \) do?

\[
g(y, x_1, \ldots, x_n) = \sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n))
\]

It computes the least value \( t \leq y \) for which \( P(t, x_1, \ldots, x_n) \) is true!

To see why, let \( t_0 \leq y \) such that

\[
P(t, x_1, \ldots, x_n) = 0 \quad \text{for all } t < t_0,
\]

but

\[
P(t_0, x_1, \ldots, x_n) = 1
\]
Bounded Minimalization

What does the following function $g$ do?

$$g(y, x_1, \ldots, x_n) = \sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n))$$

It computes the least value $t \leq y$ for which $P(t, x_1, \ldots, x_n)$ is true!

To see why, let $t_0 \leq y$ such that

$$P(t_0, x_1, \ldots, x_n) = 1$$

but

$$P(t, x_1, \ldots, x_n) = 0 \quad \text{for all } t < t_0,$$

Then

$$\prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n)) = \begin{cases} 
1 & \text{if } u < t_0, \\
0 & \text{if } u \geq t_0.
\end{cases}$$
Bounded Minimalization

What does the following function $g$ do?

$$g(y, x_1, \ldots, x_n) = \sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n))$$

It computes the least value $t \leq y$ for which $P(t, x_1, \ldots, x_n)$ is true!

To see why, let $t_0 \leq y$ such that $P(t_0, x_1, \ldots, x_n) = 1$ but $P(t, x_1, \ldots, x_n) = 0$ for all $t < t_0$.

Then

$$\prod_{t=0}^{u} \alpha(P(t, x_1, \ldots, x_n)) = \begin{cases} 1 & \text{if } u < t_0, \\ 0 & \text{if } u \geq t_0. \end{cases}$$

Hence $g(y, x_1, \ldots, x_n) = \sum_{u<t_0} 1 = t_0$. 
Bounded Minimalization, Continued

Define

\[
\min_{t \leq y} P(t, x_1, \ldots, x_n) = \begin{cases} 
  g(y, x_1, \ldots, x_n) & \text{if } (\exists t)_{\leq y} P(t, x_1, \ldots, x_n), \\
  0 & \text{otherwise.}
\end{cases}
\]
Bounded Minimalization, Continued

Define

\[
\min_{t \leq y} P(t, x_1, \ldots, x_n) = \begin{cases} 
  g(y, x_1, \ldots, x_n) & \text{if } (\exists t)_{\leq y} P(t, x_1, \ldots, x_n), \\
  0 & \text{otherwise.}
\end{cases}
\]

Thus, \( \min_{t \leq y} P(t, x_1, \ldots, x_n) \), is the least value \( t \leq y \) for which \( P(t, x_1, \ldots, x_n) \) is true, if such exists; otherwise it assumes the (default) value 0.
Bounded Minimalization, Continued

Define

\[
\min_{t \leq y} P(t, x_1, \ldots, x_n) = \begin{cases} 
  g(y, x_1, \ldots, x_n) & \text{if } (\exists t) \leq y P(t, x_1, \ldots, x_n), \\
  0 & \text{otherwise.}
\end{cases}
\]

Thus, \( \min_{t \leq y} P(t, x_1, \ldots, x_n) \), is the least value \( t \leq y \) for which \( P(t, x_1, \ldots, x_n) \) is true, if such exists; otherwise it assumes the (default) value 0.

**Theorem 7.1.** \( \min_{t \leq y} P(t, x_1, \ldots, x_n) \) is in PRC class \( \mathcal{C} \) if \( P(t, x_1, \ldots, x_n) \) is in \( \mathcal{C} \).

**Proof.** By Theorems 5.4 and 6.3. \( \square \)
\( \lfloor \frac{x}{y} \rfloor \) Is Primitive Recursive

\( \lfloor \frac{x}{y} \rfloor \) is the “integer part” of the quotient \( \frac{x}{y} \).
\[ \lfloor \frac{x}{y} \rfloor \text{ is Primitive Recursive} \]

\[ \lfloor \frac{x}{y} \rfloor \text{ is the “integer part” of the quotient } \frac{x}{y}. \]

The equation
\[ \lfloor \frac{x}{y} \rfloor = \min_{t \leq x} [(t + 1) \cdot y > x] \]
shows that \( \lfloor \frac{x}{y} \rfloor \) is primitive recursive. Note that according to this definition, \( \lfloor x/0 \rfloor = 0 \).
$R(x, y)$, The Remainder Function, Is Primitive Recursive

$R(x, y)$ is the remainder when $x$ is divided by $y$. As we can write

$$R(x, y) = x \dot{-} (y \cdot \lfloor x/y \rfloor),$$

so that $R(x, y)$ is primitive recursive. Note that $R(x, 0) = x$. 

$p_n$, The $n$th Prime Number, Is Primitive Recursive

Note that $p_0 = 0$, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.
$p_n$, The $n$th Prime Number, Is Primitive Recursive

Note that $p_0 = 0$, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.

$p_n$ is defined by the following recursive equations

\[
\begin{align*}
p_0 &= 0, \\
p_{n+1} &= \min_{t \leq p_n! + 1} [\text{Prime}(t) \, \& \, t > p_n]
\end{align*}
\]

so it is primitive recursive.

Note that $p_n! + 1$ is not divisible by any of the primes $p_1, p_2, \ldots, p_n$. So, either $p_n! + 1$ is itself a prime or it is divisible by a prime greater than $p_n$. In either case, there is a prime $q$ such that $p_n < q \leq p_n! + 1$. 
To be precise, we shall first define a primitive recursive function

\[ h(y, z) = \min_{t \leq z} \text{Prime}(t) \& t > y. \]

Then we define another primitive function

\[ k(x) = h(x, x! + 1) \]

Finally, \( p_n \) is defined as

\[
P_0 = 0, \\
P_{n+1} = k(p_n),
\]

and it is concluded that \( p_n \) is primitive recursive.
Minimalization, With No Bound

We write

$$\min_y P(x_1, \ldots, x_n, y)$$

for the least value of $y$ for which the predicate $P$ is true if there is one. If there is no value of $y$ for which $P(x_1, \ldots, x_n, y)$ is true, then $\min_y P(x_1, \ldots, x_n, y)$ is undefined.
Minimalization, With No Bound

We write

$$\min_y P(x_1, \ldots, x_n, y)$$

for the least value of $y$ for which the predicate $P$ is true \textit{if there is one}. \textit{If there is no value of $y$ for which $P(x_1, \ldots, x_n, y)$ is true, then $\min_y P(x_1, \ldots, x_n, y)$ is undefined.}

Note that unbounded minimalization of a predicate can easily produce function which is not total. For example,

$$x - y = \min_z [y + z = x]$$

is undefined for $x < y$. 
Unbounded Minimalization is Partially Computable

**Theorem 7.2.** If \( P(x_1, \ldots, x_n, y) \) is a computable predicate and if

\[
g(x_1, \ldots, x_n) = \min_y P(x_1, \ldots, x_n, y)
\]

then \( g \) is a partially computable function.
Unbounded Minimalization is Partially Computable

**Theorem 7.2.** If $P(x_1, \ldots, x_n, y)$ is a computable predicate and if

$$g(x_1, \ldots, x_n) = \min_y P(x_1, \ldots, x_n, y)$$

then $g$ is a partially computable function.

**Proof.** The following program computes $g$:

$$[A] \quad \text{IF } P(X_1, \ldots, X_n, Y) \text{ GOTO E}$$
$$\quad Y \leftarrow Y + 1$$
$$\quad \text{GOTO A}$$