Theory of Computer Games: Selected Advanced Topics

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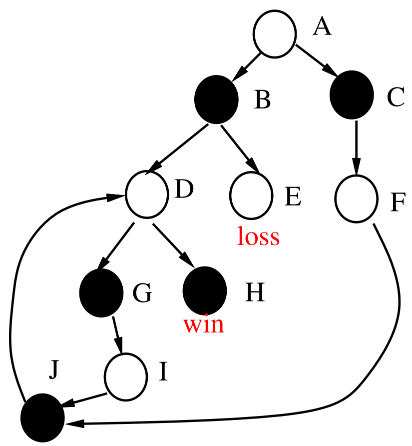
Abstract

- **Some advanced research issues.**
  - **The graph history interaction (GHI) problem.**
  - **Opponent models.**
  - **Searching chance nodes.**
  - **Proof-number search.**

Graph history interaction problem

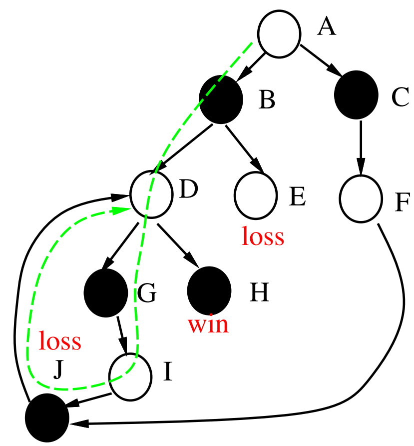
- **The graph history interaction (GHI) problem [Campbell 1985]:**
  - **In a game graph, a position can be visited by more than one paths from a starting position.**
  - **The value of the position depends on the path visiting it.**
    - ▷ *It can be win, loss or draw for Chinese chess.*
    - ▷ *It can only be draw for Western chess and Chinese dark chess.*
    - ▷ *It can only be loss for Go.*
- **In the transposition table, you record the value of a position, but not the path leading to it.**
  - **Values computed from rules on repetition cannot be used later on.**
  - **It takes a huge amount of storage to store all the paths visiting it.**
- **This is a very difficult problem to be solved in real time [Wu et al '05].**

GHI problem – example



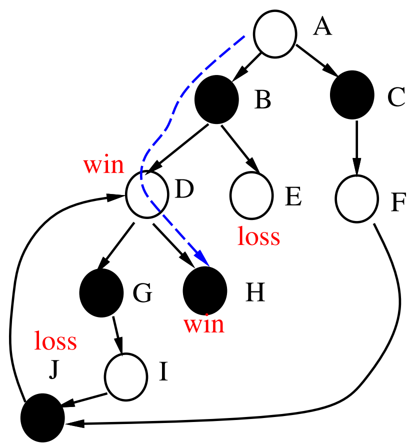
- **Assume the one causes loops loses the game.**

GHI problem – example



- **Assume the one causes loops loses the game.**
- $A \rightarrow B \rightarrow D \rightarrow G \rightarrow I \rightarrow J \rightarrow D$ **is loss because of rules of repetition.**
  - ▷ ***Memorized J as a loss position.***

GHI problem – example



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- $A \rightarrow B \rightarrow D \rightarrow G \rightarrow I \rightarrow J \rightarrow D$ **is loss because of rules of repetition.**
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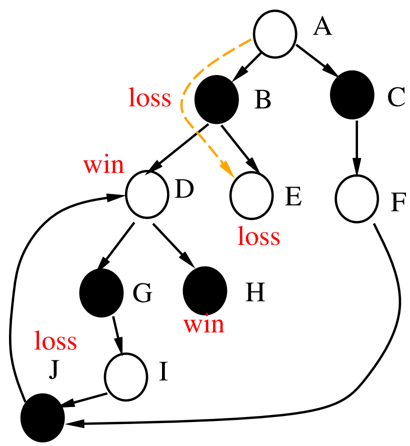
- $A \rightarrow B \rightarrow D \rightarrow H$ **is a win. Hence D is win.**

GHI problem – example



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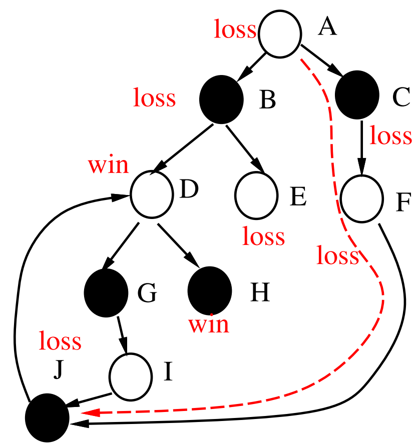
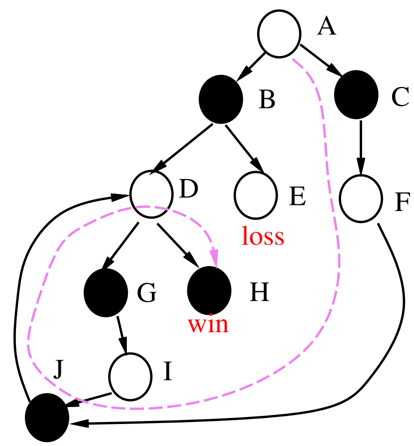
- $A \rightarrow B \rightarrow E$ **is a loss. Hence B is loss.**

GHI problem – example



- **Assume the one causes loops loses the game.**
- $A \to B \to D \to G \to I \to J \to D$ **is loss because of rules of repetition.**
  - *Memorized J as a loss position.*
- $A \to B \to D \to H$ **is a win. Hence** D **is win.**
- $A \to B \to E$ **is a loss. Hence** B **is loss.**
- $A \to C \to F \to J$ **is loss because** J **is recorded as loss.**
- A **is loss because both branches lead to loss.**

GHI problem – example



- **Assume the one causes loops loses the game.**
- $A \to B \to D \to G \to I \to J \to D$ **is loss because of rules of repetition.**
  - ▷ ***Memorized J as a loss position.***
- $A \to B \to D \to H$ **is a win. Hence D is win.**
- $A \to B \to E$ **is a loss. Hence B is loss.**
- $A \to C \to F \to J$ **is loss because J is recorded as loss.**
- A **is loss because both branches lead to loss.**
- **However,** $A \to C \to F \to J \to D \to H$ **is a win.**

Comments

- **Using DFS to search the above game graph from left first or from right first produces two different results.**
- **Position A is actually a win position.**
  - **Problem: memorize J is a loss is only valid when the path leading to it causes a loop.**
- **Storing the path leading to a position in a transposition table requires too much memory.**
- **It is still a research problem to use a more efficient data structure.**

Opponent models

- **In a normal alpha-beta search, it is assumed that you and the opponent use the same strategy.**
  - **What is good to you is bad to the opponent and vice versa!**
  - **Hence we can reduce a minimax search to a NegaMax search.**
  - **This is normally true when the game ends, but may not be true in the middle of the game.**
- **What will happen when there are two strategies or evaluating functions f_1 and f_2 so that**
  - **for some positions p, $f_1(p)$ is better than $f_2(p)$**
    - ▷ ***"better" means closer to the real value $f(p)$***
  - **for some positions q, $f_2(q)$ is better than $f_1(q)$**
- **If you are using f_1 and you know your opponent is using f_2, what can be done to take advantage of this information.**
  - **This is called OM (opponent model) search [Carmel and Markovitch 1996].**
    - ▷ ***In a MAX node, use f_1.***
    - ▷ ***In a MIN node, use f_2.***

Opponent models – comments

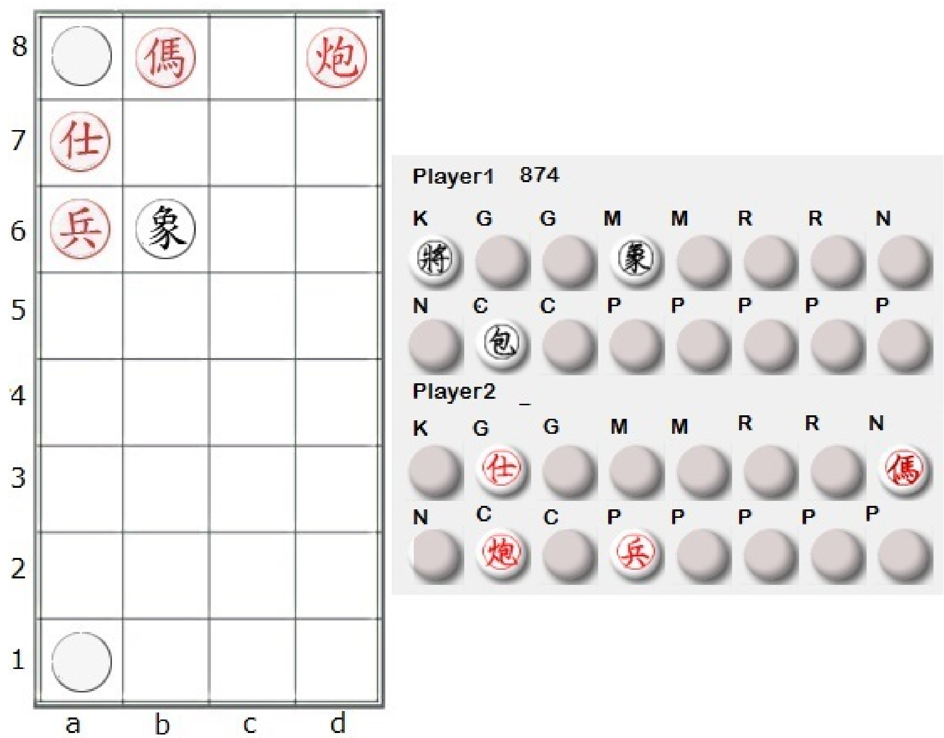
- **Comments:**
  - **Need to know your opponent's model precisely or to have some knowledge about your opponent.**
  - **How to learn the opponent model on-line or off-line?**
  - **When there are more than 2 possible opponent strategies, use a probability model (PrOM search) to form a strategy.**

Search with chance nodes

- **Chinese dark chess**
  - **Two player, zero sum, complete information**
  - **Perfect information**
  - **Stochastic**
  - **There is a chance node during searching [Ballard 1983].**
    - ▷ ***The value of a chance node is a distribution, not a fixed value.***
- **Previous work**
  - **Alpha-beta based [Ballard 1983]**
  - **Monte-Carlo based [Lancoto et al 2013]**

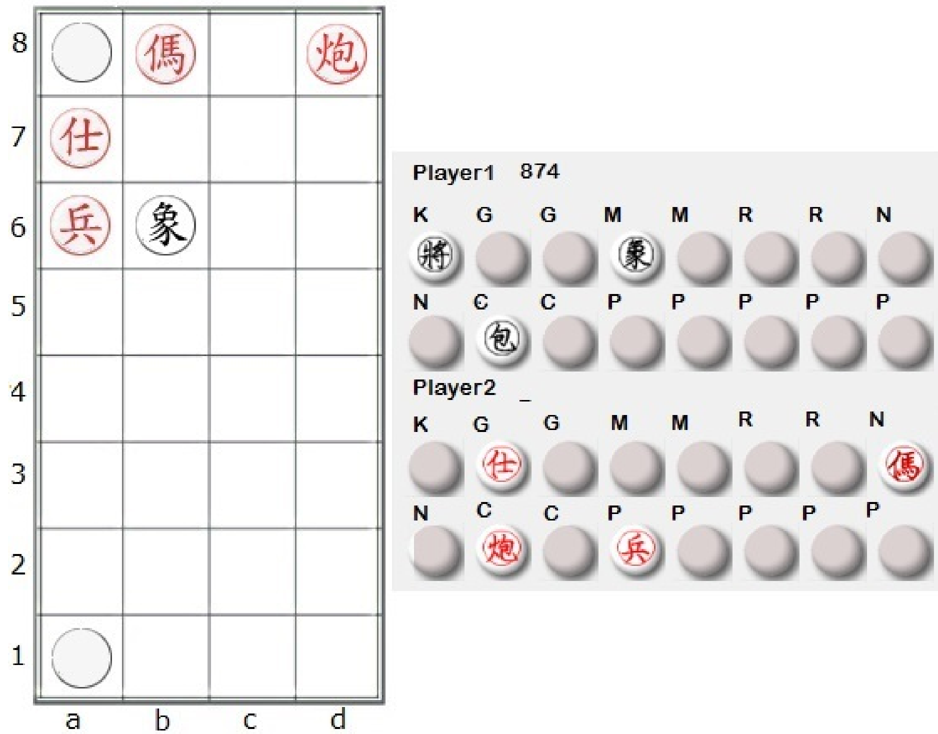
Example (1/3)

- It's black turn and black has 6 different possible legal moves including 4 of them being moving its elephant and two flipping moves at a1 or a8.
  - It is difficult for black to secure a win by moving its elephant in all 3 possible directions or capturing the red pawn at left.



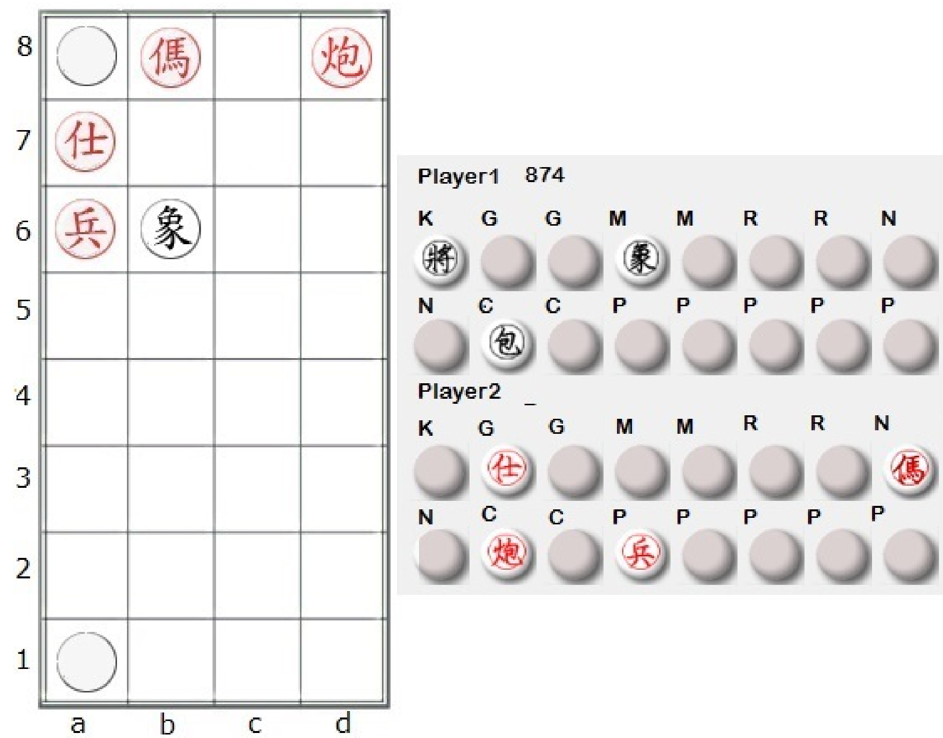
Example (2/3)

- If black flips a1, then it becomes one of the 2 following cases.
  - If a1 is black cannon, then it is difficult for red to win.
  - If a1 is black king, then it is difficult for black to lose.



Example (3/3)

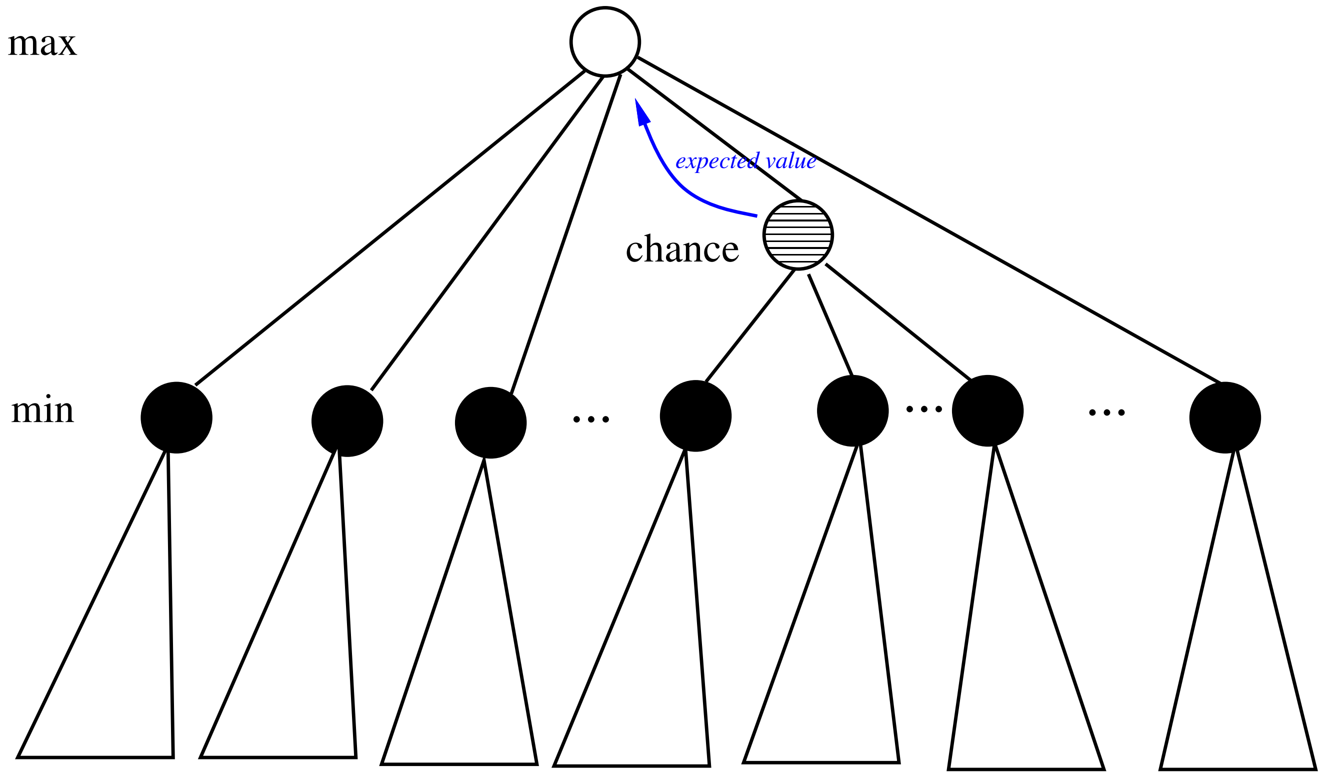
- **If black flips a8, then it becomes one of the 2 following cases.**
  - **If a8 is black cannon, then red cannon captures it immediately and results in a black lose.**
  - **If a8 is black king, then red cannon captures it immediately and results in a black lose.**



Basic ideas for searching chance nodes

- **Assume a chance node x has a score probability distribution function $Pr(*)$ with the range of possible outcomes from 1 to N where N is a positive integer.**
  - **For each possible outcome i, we need to compute $score(i)$.**
  - **The expected value $E = \sum_{i=1}^{N} score(i) * Pr(x = i)$.**
  - **The minimum value is $m = \min_{i=1}^{N}\{score(i) \mid Pr(x = i) > 0\}$.**
  - **The maximum value is $M = \max_{i=1}^{N}\{score(i) \mid Pr(x = i) > 0\}$.**
- **Example: open game in Chinese dark chess.**
  - **For the first ply, $N = 14 * 32$.**
    - ▷ ***Using symmetry, we can reduce it to 7*8.***
  - **We now consider the chance node of flipping the piece at the cell a1.**
    - ▷ $N = 14$.
    - ▷ ***Assume $x = 1$ means a black King is revealed and $x = 8$ means a red King is revealed.***
    - ▷ ***Then $score(1) = score(8)$ since the first player owns the revealed king no matter its color is.***
    - ▷ $Pr(x = 1) = Pr(x = 8) = 1/14$.

Illustration



Bounds in a chance node

- **Assume the various possibilities of a chance node is evaluated one by one in the order that at the end of phase i, the ith choice is evaluated.**
  - **Assume $v_{min} \leq score(i) \leq v_{max}$.**
- **What are the lower and upper bounds, namely m_i and M_i, of the expected value of the chance node immediately after the end of phase i?**
  - $i = 0$.
    - ▷ $m_0 = v_{min}$
    - ▷ $M_0 = v_{max}$
  - $i = 1$, **we first compute $score(1)$, and then know**
    - ▷ $m_1 \geq score(1) * Pr(x = 1) + v_{min} * (1 - Pr(x = 1))$, *and*
    - ▷ $M_1 \leq score(1) * Pr(x = 1) + v_{max} * (1 - Pr(x = 1))$.
  - $\cdots$
  - $i = i^*$, **we have computed $score(1), \ldots, score(i^*)$, and then know**
    - ▷ $m_{i^*} \geq \sum_{i=1}^{i^*} score(i) * Pr(x = i) + v_{min} * (1 - \sum_{i=1}^{i^*} Pr(x = i))$, *and*
    - ▷ $M_{i^*} \leq \sum_{i=1}^{i^*} score(i) * Pr(x = i) + v_{max} * (1 - \sum_{i=1}^{i^*} Pr(x = i))$.

Changes of bounds: uniform case (1/2)

- **Assume the search window entering a chance node with $N = c$ choices is $[alpha, beta]$.**
  - **For simplicity, let's assume $Pr_i = \frac{1}{c}$, for all i, and the evaluated value of the ith choice is v_i.**
- **The value of a chance node after the first i choices are explored can be expressed as**
  - **an expected value $E_i = vsum_i/i$;**
    - $vsum_i = \sum_{j=1}^{i} v_j$
    - ***This value is returned only when all choices are explored.***
      $\Rightarrow$ The expected value of an un-explored child shouldn't be $\frac{v_{min}+v_{max}}{2}$.
  - **a range of possible values $[m_i, M_i]$.**
    - $m_i = (\sum_{j=1}^{i} v_j + v_{min} \cdot (c-i))/c$
    - $M_i = (\sum_{j=1}^{i} v_j + v_{max} \cdot (c-i))/c$
  - **Invariants:**
    - $E_i \in [m_i, M_i]$
    - $E_N = m_N = M_N$

Changes of bounds: uniform case (2/2)

- **Let m_i and M_i be the current lower and upper bounds, respectively, of the expected value of this chance node immediately after the evaluation of the ith node.**
  - $m_i = (\sum_{j=1}^{i-1} v_j + v_i + v_{min} \cdot (c - i))/c$
  - $M_i = (\sum_{j=1}^{i-1} v_j + v_i + v_{max} \cdot (c - i))/c$
- **How to incrementally update m_i and M_i:**
  - $m_0 = v_{min}$
  - $M_0 = v_{max}$
  - $m_i = m_{i-1} + (v_i - v_{min})/c$
  - $M_i = M_{i-1} + (v_i - v_{max})/c$
- **The current search window is $[alpha, beta]$.**
  - **No more searching is needed when**
    - ▷ $m_i \geq beta$, ***chance node cut off I;***
      $\Rightarrow$ ***The lower bound found so far is good enough.***
      $\Rightarrow$ ***Similar to a beta cutoff.***
      $\Rightarrow$ ***The returned value is m_i.***
    - ▷ $M_i \leq alpha$, ***chance node cut off II.***
      $\Rightarrow$ ***The upper bound found so far is bad enough.***
      $\Rightarrow$ ***Similar to an alpha cutoff.***
      $\Rightarrow$ ***The returned value is M_i.***

Chance node cut off

- **When** $m_i \geq beta$, **chance node cut off I,**
  - **which means** $(\sum_{j=1}^{i-1} v_j + v_i + v_{min} \cdot (c-i))/c \geq beta$
  - $\Rightarrow v_i \geq B_{i-1} = c \cdot beta - (\sum_{j=1}^{i-1} v_j - v_{min} * (c-i))$
- **When** $M_i \leq alpha$, **chance node cut off II,**
  - **which means** $(\sum_{j=1}^{i-1} v_j + v_i + v_{max} \cdot (c-i))/c \leq alpha$
  - $\Rightarrow v_i \leq A_{i-1} = c \cdot alpha - (\sum_{j=1}^{i-1} v_j - v_{max} * (c-i))$
- **Hence set the window for searching the** i**th choice to be** $[A_{i-1}, B_{i-1}]$ **which means no further search is needed if the result is not within this window.**
- **How to incrementally update** A_i **and** B_i**?**
  - $A_0 = c \cdot (alpha - v_{max}) + v_{max}$
  - $B_0 = c \cdot (beta - v_{min}) + v_{min}$
  - $A_i = A_{i-1} + v_{max} - v_i$
  - $B_i = B_{i-1} + v_{min} - v_i$

Algorithm: Chance_Search

- **Algorithm** $F3.1'$**(position** p**, value** $alpha$**, value** $beta$**)**
  // max node
  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$**, then return** $f(p)$
    else begin
    - ▷ $m := -\infty$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ ***if*** p_i ***is to play a chance node*** n
      then $t := Star1\_F3.1'(p_i, n, max\{alpha, m\}, beta)$
    - ▷ ***else*** $t := G3.1'(p_i, max\{alpha, m\}, beta)$
    - ▷ ***if*** $t > m$ ***then*** $m := t$
    - ▷ ***if*** $m \geq beta$ ***then return***(m) ***// beta cut off***
    - ▷ ***end***
  - **end;**
  - **return** m

Algorithm: Chance_Search

- **Algorithm** $Star1\_F3.1'$**(position** p**, node** n**, value** $alpha$**, value** $beta$**)**
  - **// a chance node n with equal probability choices k_1, ..., k_c**
  - **determine the possible values of the chance node n to be $k_1, \ldots, k_c$**
  - $A_0 = c \cdot (alpha - v_{max}) + v_{max}$, $B_0 = c \cdot (beta - v_{min}) + v_{min}$;
  - $m_0 = v_{min}$, $M_0 = v_{max}$ **// current lower and upper bounds**
  - $vsum = 0$; **// current sum of expected values**
  - **for** $i = 1$ **to** c **do**
  - **begin**
    - ▷ ***let*** p_i ***be the position of assigning*** k_i ***to*** n ***in*** p;
    - ▷ $t := G3.1'(p_i, \max\{A_{i-1}, v_{min}\}, \min\{B_{i-1}, v_{max}\})$
    - ▷ $m_i = m_{i-1} + (t - v_{min})/c$, $M_i = M_{i-1} + (t - v_{max})/c$;
    - ▷ ***if*** $t \geq B_{i-1}$ ***then return*** m_i; ***// failed high, chance node cut off I***
    - ▷ ***if*** $t \leq A_{i-1}$ ***then return*** M_i; ***// failed low, chance node cut off II***
    - ▷ $vsum$ += t;
    - ▷ $A_i = A_{i-1} + v_{max} - t$, $B_i = B_{i-1} + v_{min} - t$;
  - **end**
- **return** $vsum/c$;

Example: Chinese dark chess

- **Assumption:**
  - **The range of the scores of Chinese dark chess is $[-10, 10]$ inclusive, $alpha = -10$ and $beta = 10$.**
  - $N = 7$.
  - $Pr(x = i) = 1/N = 1/7$.
- **Calculation:**
  - $i = 0$,
    - ▷ $m_0 = -10$.
    - ▷ $M_0 = 10$.
  - $i = 1$ **and if** $score(1) = -2$, **then**
    - ▷ $m_1 = -2 * 1/7 + -10 * 6/7 = -62/7 \simeq -8.86$.
    - ▷ $M_1 = -2 * 1/7 + 10 * 6/7 = 58/7 \simeq 8.26$.
  - $i = 1$ **and if** $score(1) = 3$, **then**
    - ▷ $m_1 = 3 * 1/7 + -10 * 6/7 = -57/7 \simeq -8.14$.
    - ▷ $M_1 = 3 * 1/7 + 10 * 6/7 = 63/7 = 9$.

General case

- **Assume the ith choice happens with a chance w_i/c where $c = \sum_{i=1}^{N} w_i$ and N is the total number of choices.**
  - $m_0 = v_{min}$
  - $M_0 = v_{max}$
  - $m_i = (\sum_{j=1}^{i-1} w_j \cdot v_j + w_i \cdot v_i + v_{min} \cdot (c - \sum_{j=1}^{i} w_j))/c$
    - ▷ $m_i = m_{i-1} + (w_i/c) \cdot (v_i - v_{min})$
  - $M_i = (\sum_{j=1}^{i-1} w_j \cdot v_j + w_i \cdot v_i + v_{max} \cdot (c - \sum_{j=1}^{i} w_j))/c$
    - ▷ $M_i = M_{i-1} + (w_i/c) \cdot (v_i - v_{max})$
  - $A_0 = (c/w_1) \cdot (alpha - v_{max}) + v_{max}$
  - $B_0 = (c/w_1) \cdot (beta - v_{min}) + v_{min}$
  - $A_{i-1} = (c \cdot alpha - (\sum_{j=1}^{i-1} w_j \cdot v_j - v_{max} \cdot (c - \sum_{j=1}^{i} w_j)))/w_i$
    - ▷ $A_i = (w_i/w_{i+1}) \cdot (A_{i-1} - v_i) + v_{max}$
  - $B_{i-1} = (c \cdot beta - (\sum_{j=1}^{i-1} w_j \cdot v_j - v_{min} \cdot (c - \sum_{j=1}^{i} w_j)))/w_i$
    - ▷ $B_i = (w_i/w_{i+1}) \cdot (B_{i-1} - v_i) + v_{min}$

Comments

- We illustrate the ideas using a fail soft version of the alpha-beta algorithm.
  - Original and dail hard version have a simpler logic in maintaining the search interval.
  - The semantic of comparing an exact returning value with an expected returning value is something that needs careful thinking.
  - May want to pick a chance node with a lower expected value but having a hope of winning, not one with a slightly higher expected value but having no hope of winning when you are in disadvantageous.
  - May want to pick a chance node with a lower expected value but having no chance of losing, not one with a slightly higher expected value but having a chance of losing when you are in advantage.
- Need to revise algorithms carefully when dealing with the original, fail hard or NegaScout version.
  - What does it mean to combine bounds from a fail hard version?
- Exist other improvements by considering better move orderings involving chance nodes.

How to use these bounds

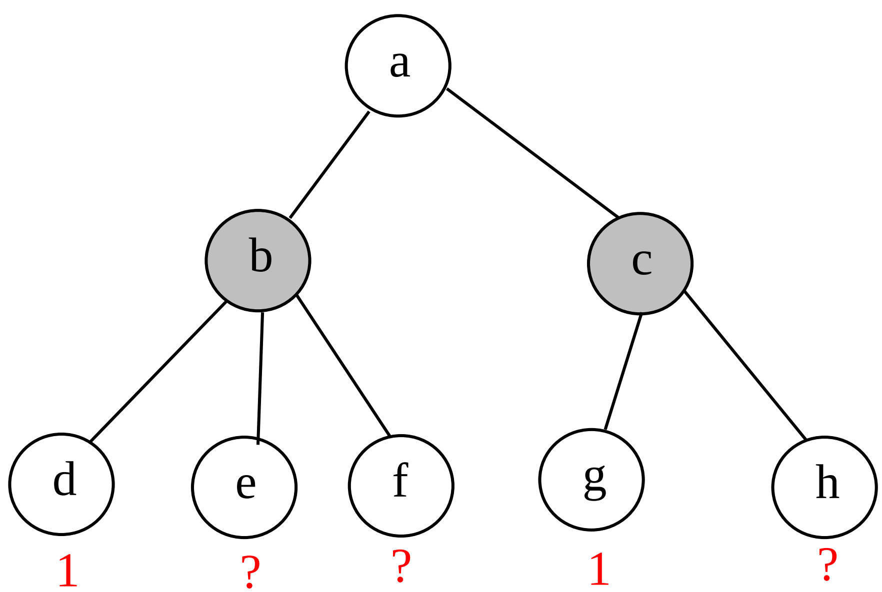
- **The lower and upper bounds of the expected score can be used to do alpha-beta pruning.**
  - **Nicely fit into the alpha-beta search algorithm.**
- **Can do better by not searching the DFS order.**
  - **It is not necessary to search completely the subtree of $x = 1$ first, and then start to look at the subtree of $x = 2$.**
  - **Assume it is a MAX chance node, e.g., the opponent takes a flip.**
    - ▷ ***Knowing some value v'_1 of a subtree for $x = 1$ gives an upper bound, i.e., $score(1) \geq v'_1$.***
    - ▷ ***Knowing some value v'_2 of a subtree for $x = 2$ gives another upper bound, i.e., $score(2) \geq v'_2$.***
    - ▷ ***These bounds can be used to make the search window further narrower.***
- **For Monte-Carlo based algorithm, we need to use a sparse sampling algorithm to efficiently estimate the expected value of a chance node [Kearn et al 2002].**

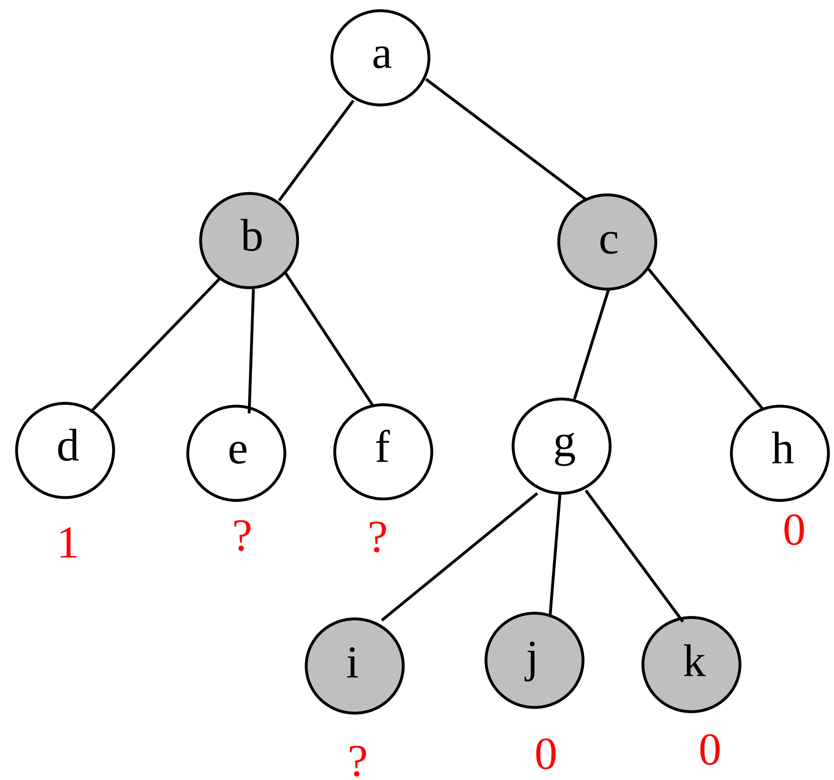
Proof number search

- **Consider the case of a 2-player game tree with either 0 or 1 on the leaves.**
  - **win, or not win which is lose or draw;**
  - **lose, or not lose which is win or draw;**
  - **Call this a binary valued game tree.**
- **If the game tree is known as well as the values of some leaves are known, can you make use of this information to search this game tree faster?**
  - **The value of the root is either 0 or 1.**
  - **If a branch of the root returns 1, then we know for sure the value of the root is 1.**
  - **The value of the root is 0 only when all branches of the root returns 0.**
  - **An AND-OR game tree search.**

Which node to search next?

- A **most proving node** for a node u: a descendent node if its value is 1, then the value of u is 1.
- A **most disproving node** for a node u: a descendent node if its value is 0, then the value of u is 0.





Proof or Disproof Number

- **Assign a proof number and a disproof number to each node u in a binary valued game tree.**
  - $proof(u)$**: the minimum number of leaves needed to visited in order for the value of u to be 1.**
  - $disproof(u)$**: the minimum number of leaves needed to visited in order for the value of u to be 0.**
- **The definition implies a bottom-up ordering.**

Proof Number: Definition

- u **is a leaf:**
  - **If** $value(u)$ **is unknown, then** $proof(u)$ **is the cost of evaluating** u**.**
  - **If** $value(u)$ **is 1, then** $proof(u) = 0$**.**
  - **If** $value(u)$ **is 0, then** $proof(u) = \infty$**.**
- u **is an internal node with all of the children** $u_1, \ldots, u_b$**:**
  - **if** u **is a MAX node,**

$$proof(u) = \min_{i=1}^{i=b} proof(u_i);$$

  - **if** u **is a MIN node,**

$$proof(u) = \sum_{i=1}^{i=b} proof(u_i).$$

Disproof Number: Definition

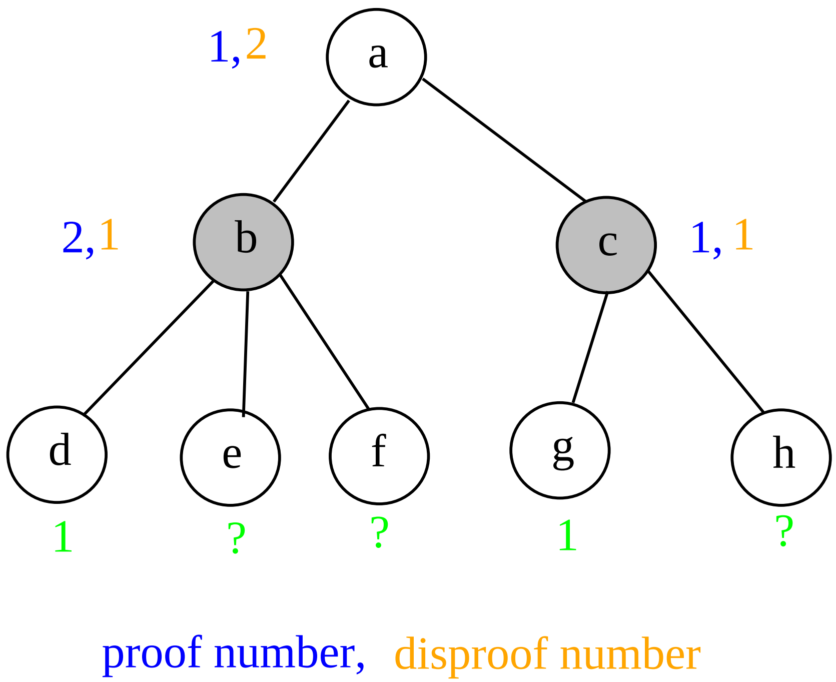
- u **is a leaf:**
  - **If** $value(u)$ **is unknown, then** $disproof(u)$ **is cost of evaluating** u**.**
  - **If** $value(u)$ **is 1, then** $disproof(u) = \infty$**.**
  - **If** $value(u)$ **is 0, then** $disproof(u) = 0$**.**
- u **is an internal node with all of the children** $u_1, \ldots, u_b$**:**
  - **if** u **is a MAX node,**

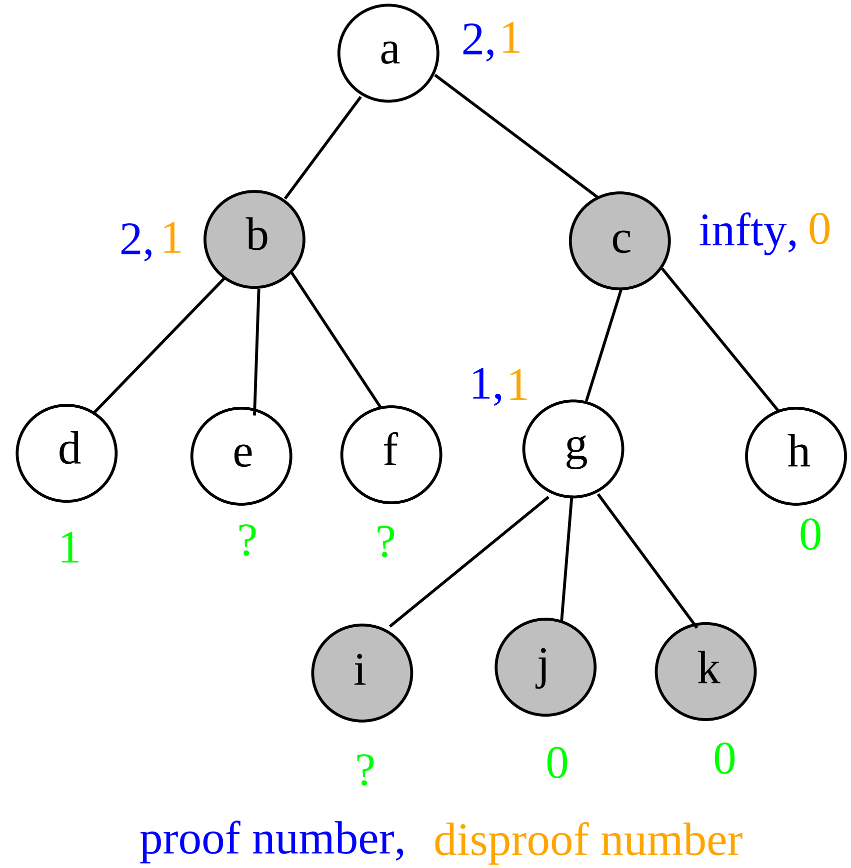
$$disproof(u) = \sum_{i=1}^{i=b} disproof(u_i);$$

  - **if** u **is a MIN node,**

$$disproof(u) = \min_{i=1}^{i=b} disproof(u_i).$$

Illustrations





How to use these numbers

- **If the numbers are known in advance, then from the root, we search a child u with the value equals to $\min\{proof(root), disproof(root)\}$.**
  - **Find a path from the root towards a leaf recursively as follows.**
    - ▷ ***If we try to prove it, then pick a child with the least proof number for a MAX node, and pick any node that has a chance to be proved for a MIN node.***
    - ▷ ***If we try to disprove it, then pick a child with the least disproof number for a MIN node, and pick any node that has a chance to be disproved for a MAX node.***
- **Assume each leaf takes a lot of time to evaluate.**
  - **For example, the game tree represents an open game tree or an endgame tree.**
  - **Depends on the results we have so far, pick the next leaf to prove or disprove.**
- **Need to be able to update these numbers on the fly.**

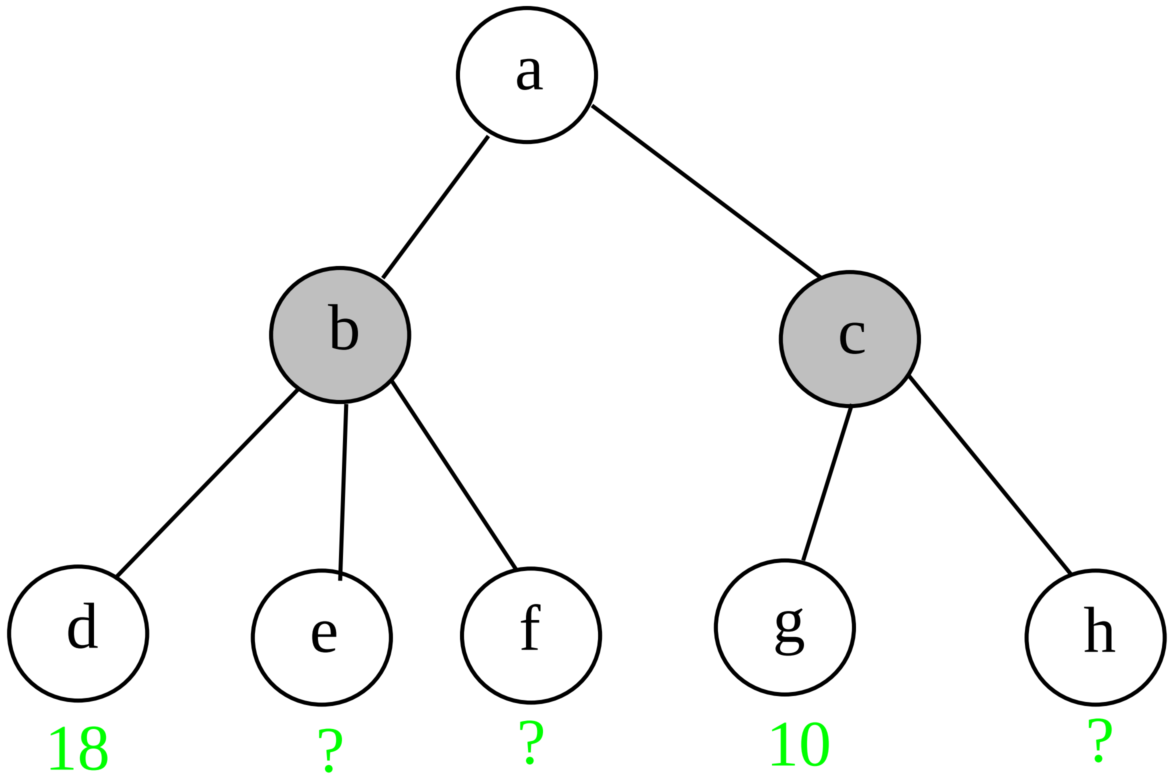
PN-search: algorithm

- *loop:* **Compute or update proof and disproof numbers for each node in a bottom up fashion.**
  - **If** $proof(root) = 0$ **or** $disproof(root) = 0$**, then we are done, otherwise**
    - ▷ $proof(root) \leq disproof(root)$: ***we try to prove it.***
    - ▷ $proof(root) > disproof(root)$: ***we try to disprove it.***
- $u \leftarrow root$; $\{*$ **find the leaf to prove or disprove** $*\}$
  - **if we try to prove, then**
    - ▷ ***while*** u ***is not a leaf do***
    - ▷ ***if*** u ***is a MAX node, then***
      $u \leftarrow$ ***leftmost child of*** u ***with the smallest non-zero proof number;***
    - ▷ ***if current is a MIN node, then***
      $u \leftarrow$ ***leftmost child of*** u ***with a non-zero proof number;***
  - **if we try to disprove, then**
    - ▷ ***while*** u ***is not a leaf do***
    - ▷ ***if*** u ***is a MAX node, then***
      $u \leftarrow$ ***leftmost child of*** u ***with a non-zero disproof number;***
    - ▷ ***if current is a MIN node, then***
      $u \leftarrow$ ***leftmost child of*** u ***with the smallest non-zero disproof number;***
- **Prove or disprove** u**; go to** $loop$**;**

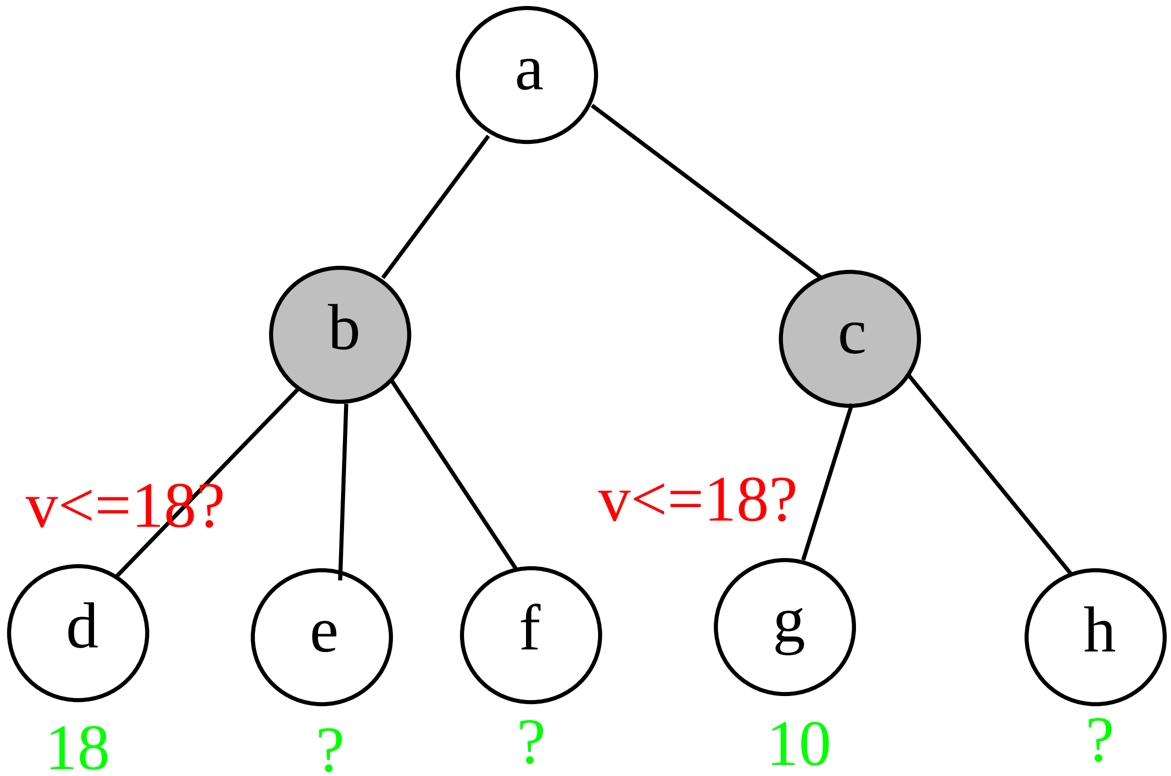
Multi-Valued game Tree

- **The values of the leaves may not be binary.**
  - **Assume the values are non-negative integers.**
  - **Note: it can be in any finite countable domain.**
- **Revision of the proof and disproof numbers.**
  - $proof_v(u)$: **the minimum number of leaves needed to visited in order for the value of u to $\geq v$.**
    - $proof(u) \equiv proof_1(u)$.
  - $disproof_v(u)$: **the minimum number of leaves needed to visited in order for the value of u to $< v$.**
    - $disproof(u) \equiv disproof_1(u)$.

Illustration



Illustration



Multi-Valued proof number

- u **is a leaf:**
  - **If** $value(u)$ **is unknown, then** $proof_v(u)$ **is cost of evaluating** u**.**
  - **If** $value(u) \geq v$**, then** $proof_v(u) = 0$**.**
  - **If** $value(u) < v$**, then** $proof_v(u) = \infty$**.**
- u **is an internal node with all of the children** $u_1, \ldots, u_b$**:**
  - **if** u **is a MAX node,**

$$proof_v(u) = \min_{i=1}^{i=b} proof_v(u_i);$$

  - **if** u **is a MIN node,**

$$proof_v(u) = \sum_{i=1}^{i=b} proof_v(u_i).$$

Multi-Valued disproof number

- u **is a leaf:**
  - **If** $value(u)$ **is unknown, then** $disproof_v(u)$ **is cost of evaluating** u**.**
  - **If** $value(u) \geq v$**, then** $disproof_v(u) = \infty$**.**
  - **If** $value(u) < v$**, then** $disproof_v(u) = 0$**.**
- u **is an internal node with all of the children** $u_1, \ldots, u_b$**:**
  - **if** u **is a MAX node,**

$$disproof_v(u) = \sum_{i=1}^{i=b} disproof_v(u_i);$$

  - **if** u **is a MIN node,**

$$disproof_v(u) = \min_{i=1}^{i=b} disproof_v(u_i).$$

Revised PN-search(v): algorithm

- *loop:* **Compute or update proof$_v$ and disproof$_v$ numbers for each node in a bottom up fashion.**
  - **If** $proof_v(root) = 0$ **or** $disproof_v(root) = 0$**, then we are done, otherwise**
    - ▷ $proof_v(root) \leq disproof_v(root)$: ***we try to prove it.***
    - ▷ $proof_v(root) > disproof_v(root)$: ***we try to disprove it.***
- $u \leftarrow root$; $\{*$ **find the leaf to prove or disprove** $*\}$
  - **if we try to prove, then**
    - ▷ ***while*** u ***is not a leaf do***
    - ▷ ***if*** u ***is a MAX node, then***
      $u \leftarrow$ ***leftmost child of*** u ***with the smallest non-zero proof$_v$ number;***
    - ▷ ***if current is a MIN node, then***
      $u \leftarrow$ ***leftmost child of*** u ***with a non-zero proof$_v$ number;***
  - **if we try to disprove, then**
    - ▷ ***while*** u ***is not a leaf do***
    - ▷ ***if*** u ***is a MAX node, then***
      $u \leftarrow$ ***leftmost child of*** u ***with a non-zero disproof$_v$ number ;***
    - ▷ ***if current is a MIN node, then***
      $u \leftarrow$ ***leftmost child of*** u ***with the smallest non-zero disproof$_v$ number;***
- **Prove or disprove** u**; go to** $loop$**;**

Multi-valued PN-search: algorithm

- **When the values of the leaves are not binary, use an open value binary search to find an upper bound of the value.**
  - **Set the initial value of v to be 1.**
  - $loop$**: PN-search(**v**)**
    - ▷ ***Prove the value of the search tree is*** $\geq v$ ***or disprove it by showing it is*** $< v$.
  - **If it is proved, then double the value of v and go to $loop$ again.**
  - **If it is disproved, then the true value of the tree is between $\lfloor v/2 \rfloor$ and $v-1$.**
  - **{$*$ Use a binary search to find the exact returned value of the tree. $*$}**
  - $low \leftarrow \lfloor v/2 \rfloor$; $high \leftarrow v-1$;
  - **while** $low \leq high$ **do**
    - ▷ ***if*** $low = high$, ***then return*** low ***as the tree value***
    - ▷ $mid \leftarrow \lfloor (low + high)/2 \rfloor$
    - ▷ ***PN-search(***mid***)***
    - ▷ ***if it is disproved, then*** $high \leftarrow mid - 1$
    - ▷ ***else if it is proved, then*** $low \leftarrow mid$

Comments

- Can be used to construct opening books.
- Appear to be good for searching certain types of game trees.
  - Find the easiest way to prove or disprove a conjecture.
  - A dynamic strategy depends on work has been done so far.
- Performance has nothing to do with move ordering.
  - Performances of most previous algorithms depend heavily on whether good move orderings can be found.
- Searching the “easiest” branch may not give you the best performance.
  - Performance depends on the value of each internal node.
- Commonly used in verifying conjectures, e.g., first-player win.
  - Partition the opening moves in a tree-like fashion.
  - Try to the “easiest” way to prove or disprove the given conjecture.
- Take into consideration the fact that some nodes may need more time to process than the other nodes.

References and further readings (1/2)

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