Basic Search Algorithms

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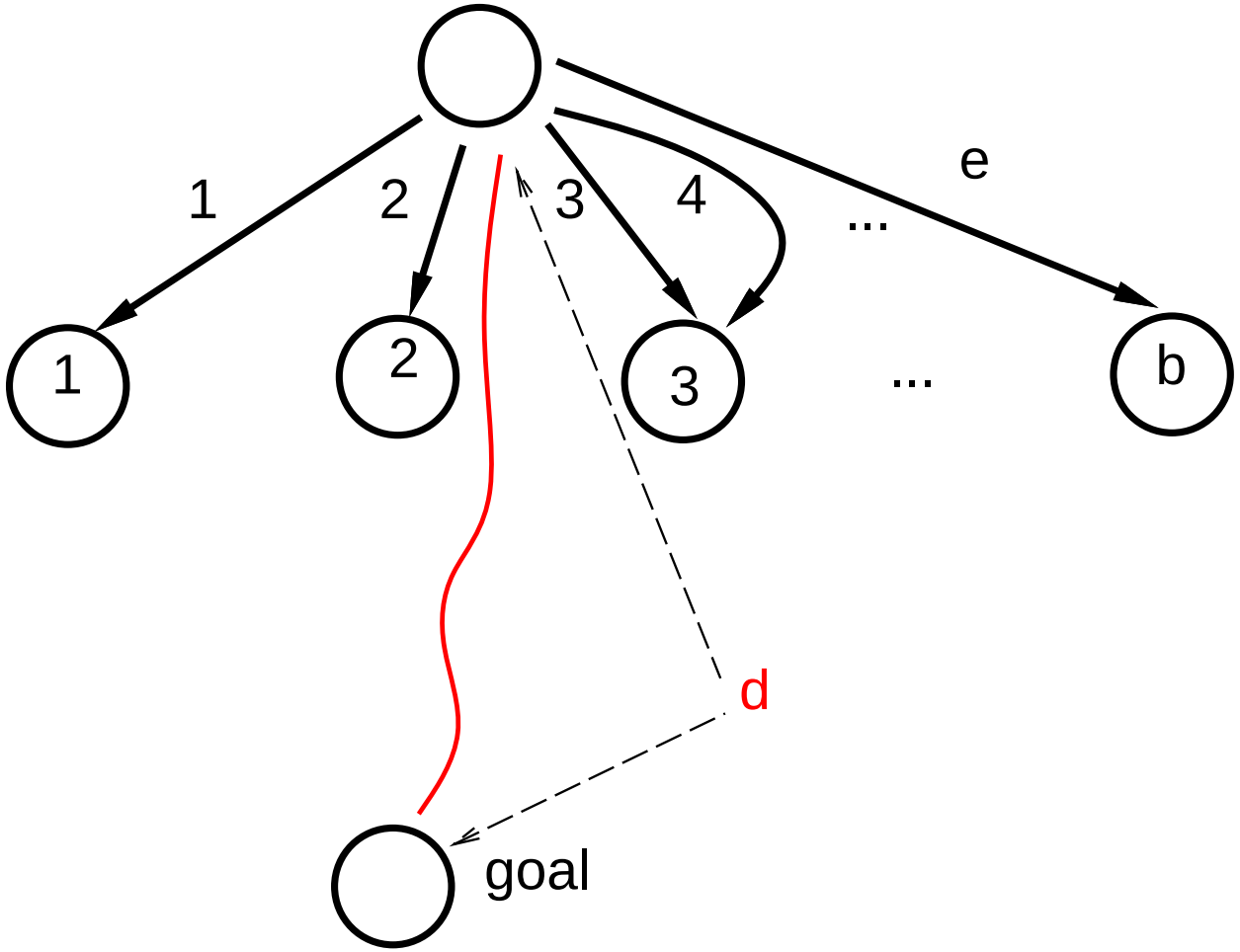
Abstract

- **The complexities of various search algorithms are considered in terms of time, space, and cost of the solution paths.**
  - **Systematic brute-force search**
    - ▷ *Breadth-first search (BFS)*
    - ▷ *Depth-first search (DFS)*
    - ▷ *Depth-first Iterative-deepening (DFID)*
    - ▷ *Bi-directional search*
  - **Heuristic search: best-first search**
    - ▷ A^*
    - ▷ IDA^*
- **The issue of storing information in the DISK instead of the main memory.**
- **Solving 15-puzzle.**

Definitions

- **Node branching factor b: the number of different new states generated from a state.**
  - **Average node branching factor.**
  - **Assumed to be a constant here.**
- **Edge branching factor e: the number of possible new, maybe duplicated, states generated from a state.**
  - **Average node branching factor.**
  - **Assumed to be a constant here.**
- **Depth of a solution d: the shortest length from the initial state to one of the goal states**
  - **The depth of the root is 0.**

Illustration



Single-player Game and Search

- **A single-player game defines a state space in which goals are hidden.**
  - **A pre-defined set of possible configurations.**
  - **An initial configuration and rules of state transitions are given.**
  - **Once an instance of a game is announced or published there is no way to change its configuration or structure.**
  - **The puzzle hidden inside an instance of a game is fixed.**
- **Purpose of a puzzle:**
  - ▷ *Finding the goal that fits given constraints: NoNoGram and Sodoku.*
  - ▷ *Finding a, sometimes best, solution path: 15-puzzle.*
- **A search program finds a goal state starting from the initial state by exploring states in the state space.**
  - **Brute-force search**
    - ▷ *Try each possible state one by one*
    - ▷ *Need better ways to enumerate all possible states*
  - **Heuristic search**
    - ▷ *Use knowledge or heuristics to cut states that cannot be solutions*

Brute-force search

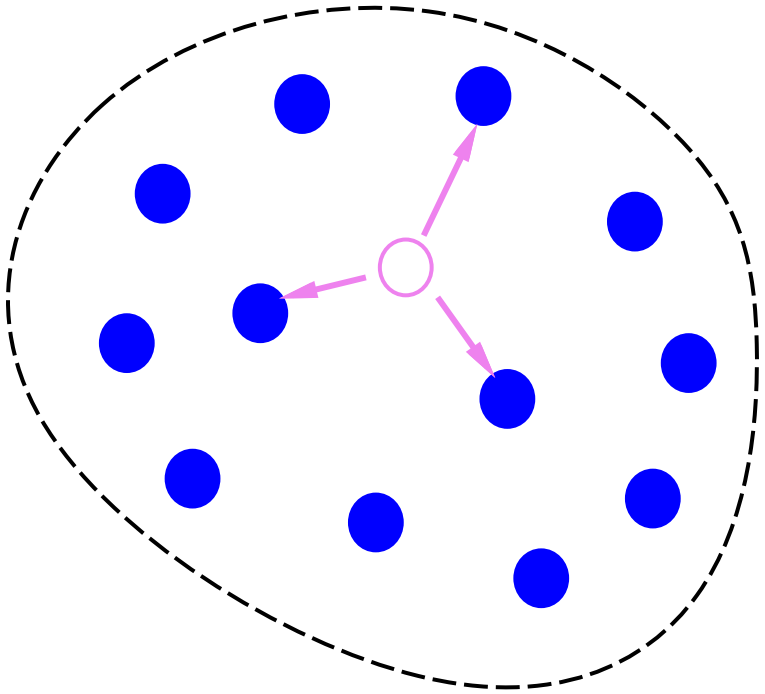
- A brute-force search algorithm is one that uses information about
  - the initial state,
  - operators on finding the states adjacent to a state,
  - and a test function whether a goal is reached.
- A "pure" brute-force search program.
  - A state maybe re-visited many times.
- An "intelligent" brute-force search algorithm.
  - Make sure a state will be eventually visited.
  - Make sure a state will be visited a limited number of times.

A "pure" brute-force search

- **A "pure" brute-force search is a brute-force search algorithm that does not care whether a state to be visited has been visited before or not.**
- **Algorithm Brute-force(N_0)**
  {∗ do brute-force search from the starting state N_0 ∗}
  - $current \leftarrow N_0$
  - **While true do**
    - ▷ ***If*** $current$ ***is a goal, then return success***
    - ▷ $current \leftarrow$ ***a state that*** $current$ ***can reach in one step***
- **Comments**
  - **Very easy to code and use very little memory.**
  - **May take infinite time because there is no guarantee that**
    - ▷ ***a state will be eventually visited.***
  - **If you pick a random next state, then it is called a random walk.**
    - ▷ ***Truly random numbers are hard and expensive to get.***
    - ▷ ***Q: what's the average complexity of a random walk?***

Pure brute-force approach

- **Random forward walk**
  - **From a state, know how to find your neighbors**
    - ▷ *forward neighbors*
    - ▷ *backward neighbors*
  - **Each step moves closer to the goal.**
  - **Branch and cut**



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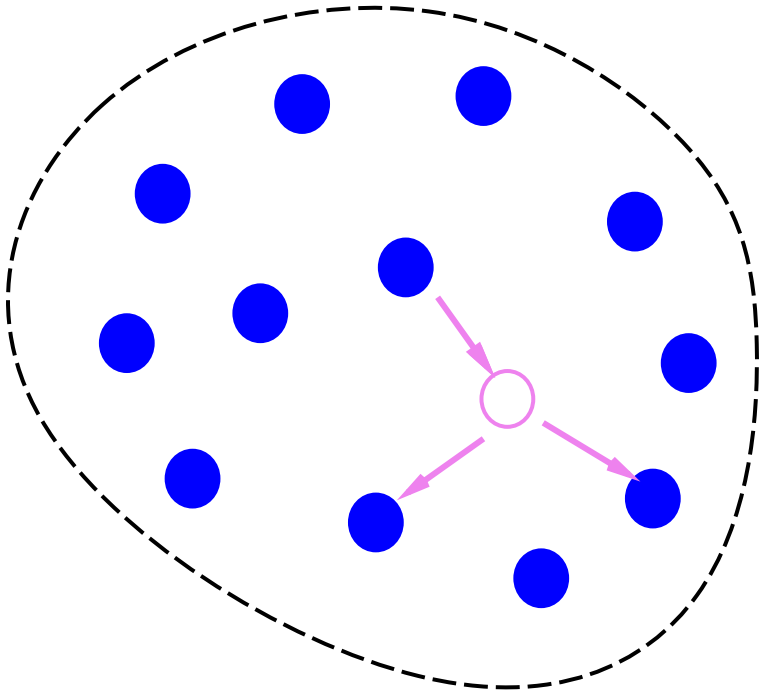
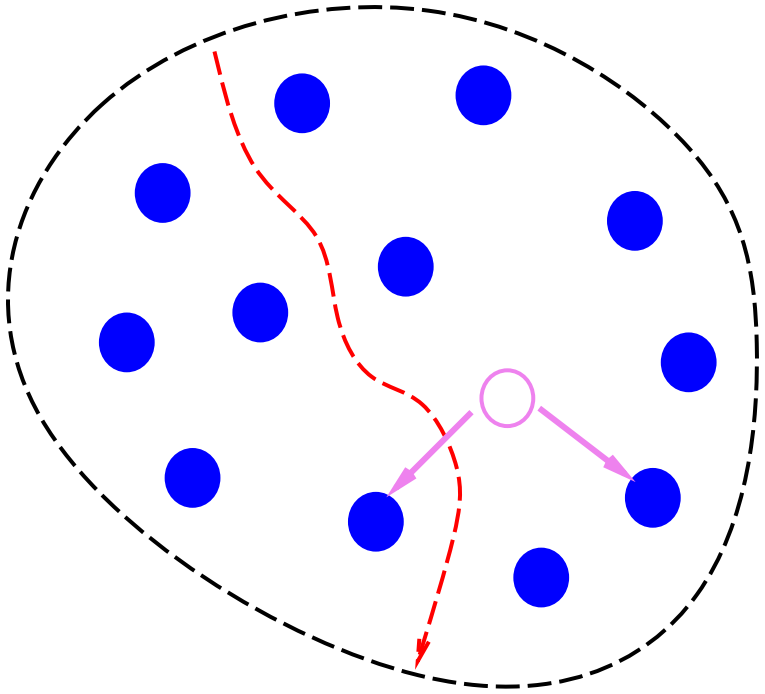


Pitfall of Brute-Force approach

- **When you need to cross a bridge, which is something you need to do immediately to reach the next level.**
  - **Example: In Chinese Chess, you need to un-check immediately.**
  - **Sometimes you do not know that is a bridge and/or how to cross a bridge.**



Example:bridge

Intelligent brute-force search

- **An "intelligent" brute-force search algorithm.**
  - **Assume S is the set of all possible states**
  - **Use a systematic way to examine each state in S one by one so that**
    - ▷ *a state is not examined too many times — does not have too many duplications;*
    - ▷ *it is efficient to find an unvisited state in S.*
- **Need to know whether a state has been previously visited efficiently.**
  - **Need some mechanism to "remember" the past behaviors.**
    - ▷ *Store previously visited states in memory*
    - ▷ *Use a smart visiting order, say assign a unique index from 0 to $S-1$, to avoid visiting a state twice where S is the number of distinct states.*
- **Some notable algorithms.**
  - **Breadth-first search (BFS).**
  - **Depth-first search (DFS) and its variations.**
  - **Depth-first Iterative deepening (DFID).**
  - **Bi-directional search.**

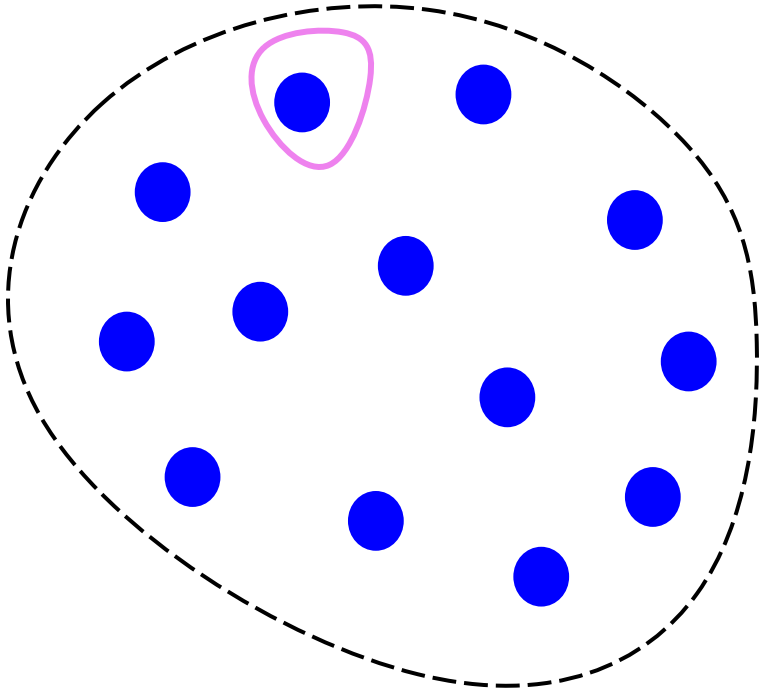
Breadth-first search (BFS)

- $deeper(N)$: **gives the set of all possible states that can be reached from the state N.**
  - **It takes at least $O(e)$ time to compute $deeper(N)$.**
  - **The number of distinct elements in $deeper(N)$ is b.**
- **Algorithm BFS(N_0) {$*$ do BFS from the starting state N_0 $*$}**
  - **If the starting state N_0 is a goal state, then return success**
  - QUEUE_INIT(Q)
  - ENQUEUE(Q,N_0);
  - **While** QUEUE_EMPTY(Q) **is FALSE do**
    - ▷ $N \leftarrow$ DEQUEUE(Q)
    - ▷ ***for each state Z in $deeper(N)$ do***
      if Z is a goal state then return success
      else ENQUEUE(Q,Z)
  - **Return fail**

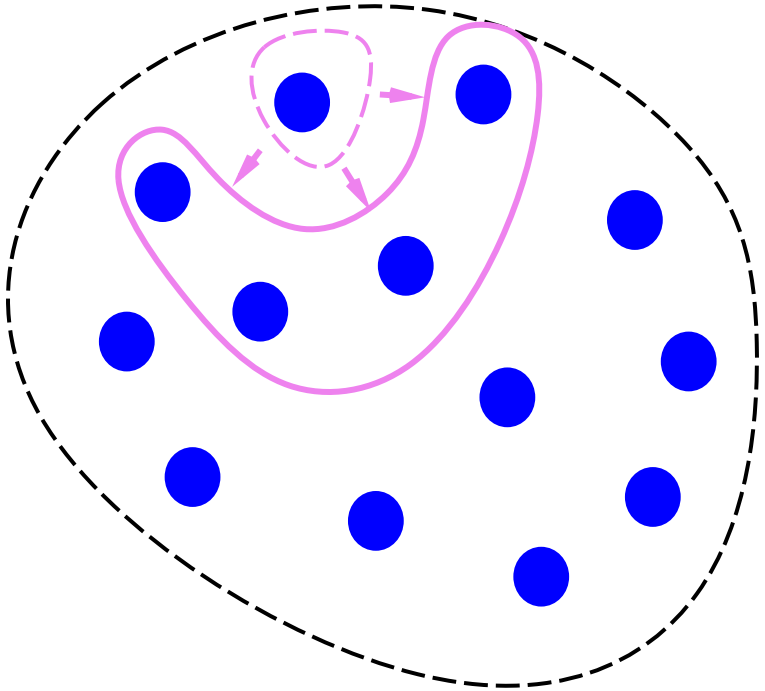
Comments

- $neighbor(N) = deeper(N) + previous(N)$
  - **The neighbors of N includes the ones go forward and backward, for example, Maze solver.**
    - ▷ ***Undirected graphs.***
  - **Here we assume we know how to only go forward, namely $deeper$, as in Sudoku.**
    - ▷ ***Directed graphs.***

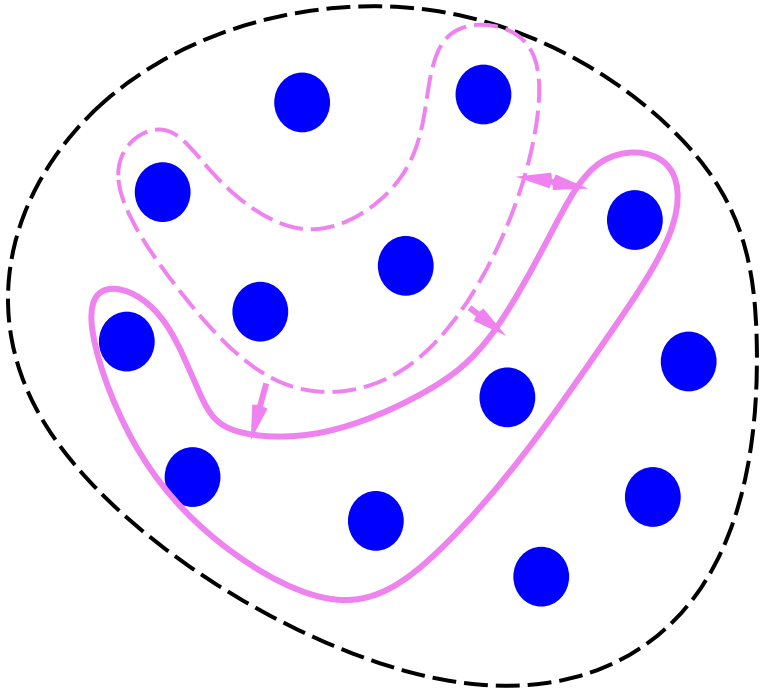
BFS



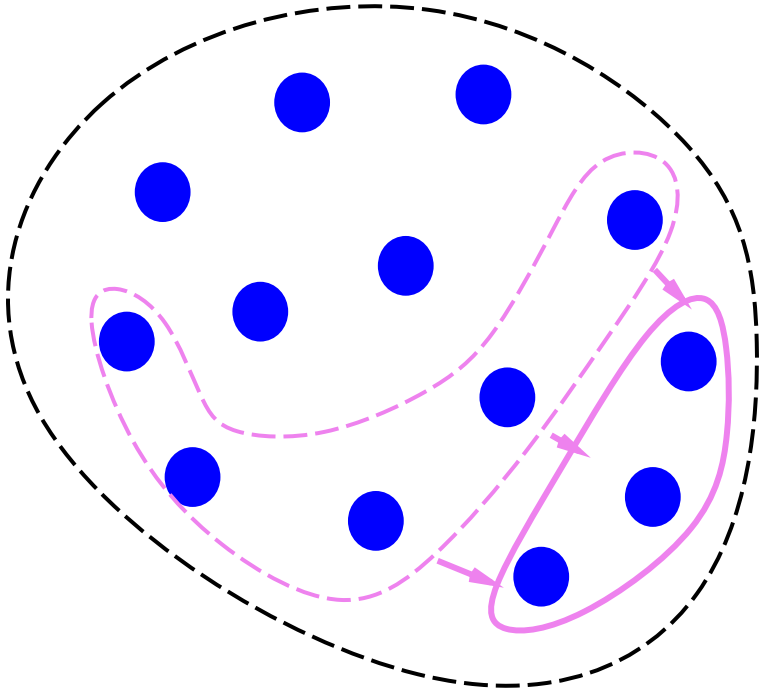
BFS



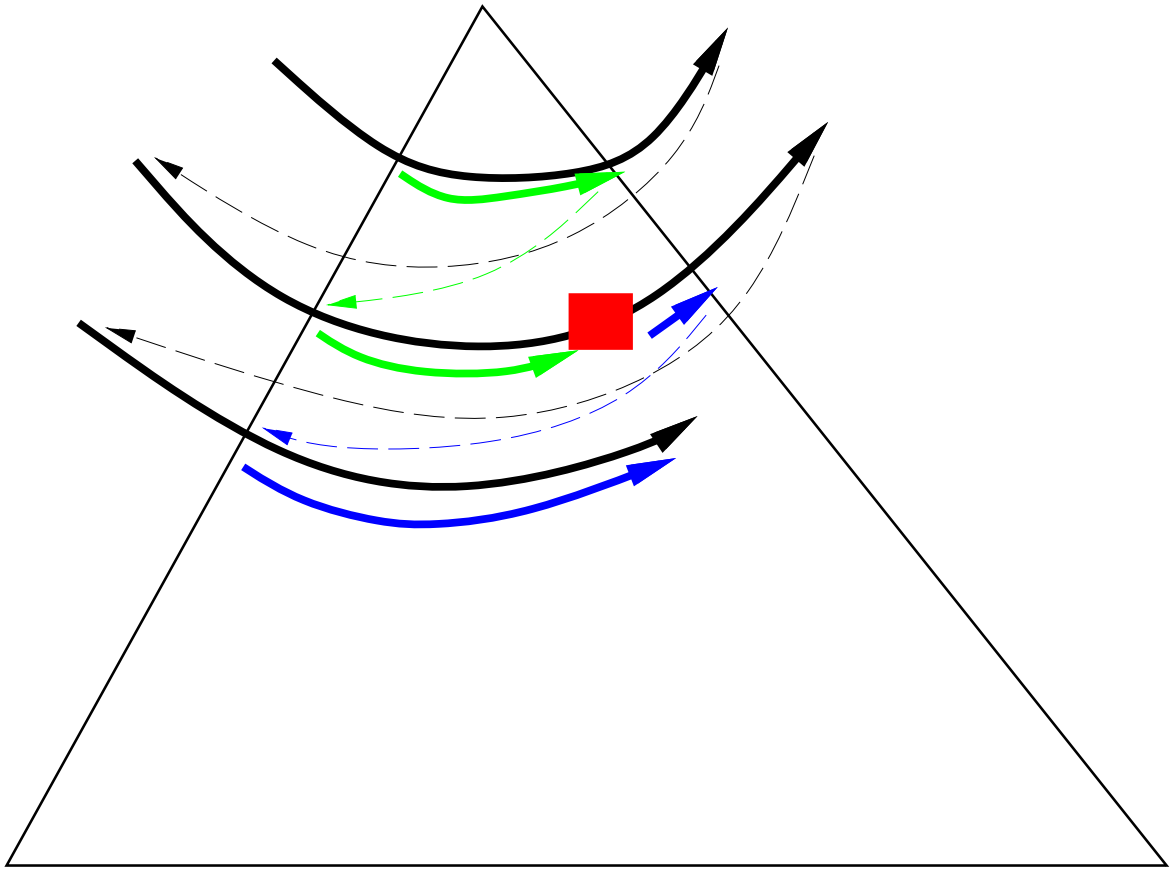
BFS



BFS



BFS: Illustration



bfs ordering

current node

closed list

open list

BFS: analysis (1/2)

- **How to find the path from the starting state to the goal after BFS return success?**
  - **When a state, other than N_0, is added, record its parent state N in this state.**
  - **We can then back trace the path by tracing the parent pointers.**
- **Space complexity:**
  - $O(b^d)$
    - ***The average number of distinct elements at depth d is b^d.***
    - ***We may need to store all distinct elements at depth d in the Queue.***
- **Time complexity:**
  - $1 * e + b * e + b^2 * e + b^3 * e + \cdots + b^{d-1} * e = (b^d - 1) * e/(b-1) = O(b^{d-1} * e)$, **if b is a constant.**
    - ***For each element N in the Queue, it takes at least $O(e)$ time to find $deeper(N)$.***
    - ***It is always true that $e \geq b$.***

BFS: analysis (2/2)

- **Nodes to be considered:**
  - **Open list: the set of nodes that are in the queue, namely, those to be explored later.**
  - **Closed list (optional): the set of nodes that have been explored.**
  - **During searching, a node in the open list is first selected, and then explored, and finally placed into the closed list.**
- **A smart mechanism for the closed list is needed if you want to make sure each node is visited at most once.**
  - **It needs to keep track of all visited nodes.**
    - $1 + b + b^2 + b^3 + \cdots + b^d = (b^{d+1} - 1)/(b - 1) = O(b^d)$.
  - **Need a good algorithm to check for states in $deeper(N)$ have been visited or not.**
    - ***Hash***
    - ***Binary search***
    - $\cdots$
  - **This is not really needed since it won't guarantee to improve the performance because of the extra cost to maintain and compare states in the closed list under the assumption that a goal is reachable!**

BFS: comments

- **Always finds an optimal solution, i.e., one with the smallest possible depth d.**
  - **Do not need to worry about falling into loops as long as there exists a goal.**
    - ▷ ***Need to store nodes that are already visited (closed list) if it is possible to have no solution.***
- **Using distance to the root to partition the search space and to make sure each node will be visited some time.**
  - **We need to use $length = i$ states in order to find $length = i + 1$ states.**
  - **In order not to go backward, we only need to store $length = (i - 1)$ states in the closed list.**
- **Need extra effort to find the solution path.**
- **Most critical drawback: huge space requirement.**
  - **It is tolerable for an algorithm to be 100 times slower, but not so for one that is 100 times larger.**

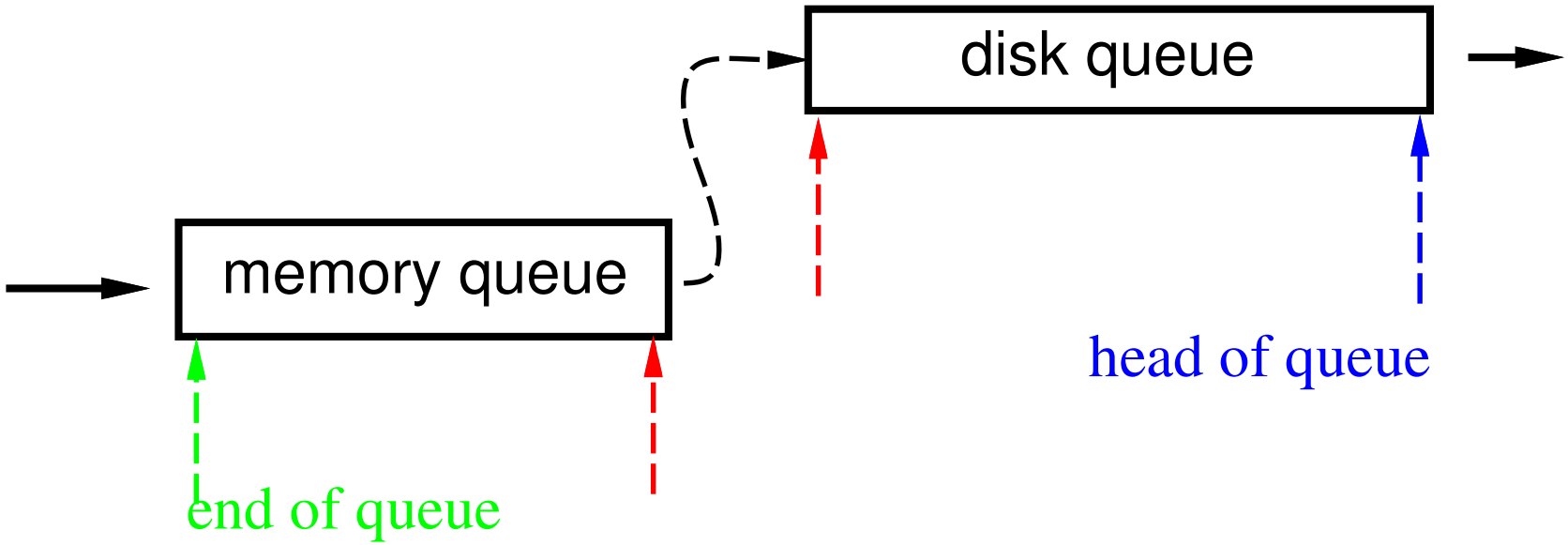
BFS: ideas when there is little memory

- **What can be done when you do not have enough main memory?**
  - **DISK**
    - ▷ *Store states that has been visited before into DISK and maintain them as sorted ⇒ closed list.*
    - ▷ *Store the QUEUE into DISK ⇒ open list.*
  - **Memory: buffers**
    - ▷ *Most recently visited nodes ⇒ closed list.*
    - ▷ *Candidates of possible newly explored nodes ⇒ open list.*
  - **Merge closed list in memory with the one in DISK when memory is full**
  - **Append the buffer of newly explored nodes (open list) to the QUEUE in DISK when memory is full or QUEUE in DISK is empty.**
    - ▷ *We only need to know when a newly explored node has been visited or not when it is about to be removed from the QUEUE.*
    - ▷ *The decision of whether it has been visited or not can be delayed.*

BFS: disk based

- **Algorithm BFS$_{disk}(N_0)$**
  {∗ do disk based BFS from the starting state N_0 ∗}
  {∗ only show maintaining of open list ∗}
  - **If the starting state N_0 is a goal state, then return success**
  - **QUEUE_INIT(Q_d) for nodes to visit in DISK**
  - **QUEUE_INIT(Q_m) for nodes to visit in main memory**
  - **ENQUEUE(Q_d,N_0);**
  - **While (QUEUE_EMPTY(Q_d) AND QUEUE_EMPTY(Q_m)) is FALSE do**
    - ▷ ***If*** QUEUE_EMPTY(Q_d), ***then*** {
      Append states in Q_m ***to*** Q_d; ***Empty*** Q_m
      }
    - ▷ $N \leftarrow$ DEQUEUE(Q_d)
    - ▷ ***for each state*** Z ***in*** $deeper(N)$ ***do***
      if Z ***is a goal state then return success***
      else if Z ***is not visited before then*** ENQUEUE(Q_m,Z)
    - ▷ ***If*** QUEUE_FULL(Q_m), ***then*** {
      Append states in Q_m ***to*** Q_d; ***Empty*** Q_m
      }
  - **Return fail**

Open lists



Disk based algorithms

- **When data cannot be loaded into the memory, you need to re-invent algorithms even for tasks that may look simple.**
  - **Batched processing.**
    - ▷ *Accumulate tasks and then try to perform these tasks when they need to.*
    - ▷ *Combine tasks into one to save disk I/O time.*
    - ▷ *Ordered disk accessing patterns.*
- **Main ideas:**
  - **It is not too slow to read all records of a large file in sequence.**
  - **It is very slow to read every record in a large file in a random order.**
  - **Sorting of data stored on the DISK can be done relatively efficient.**
  - **When two files are sorted, it is cost effective to**
    - ▷ *compare the difference of them;*
    - ▷ *merge them.*

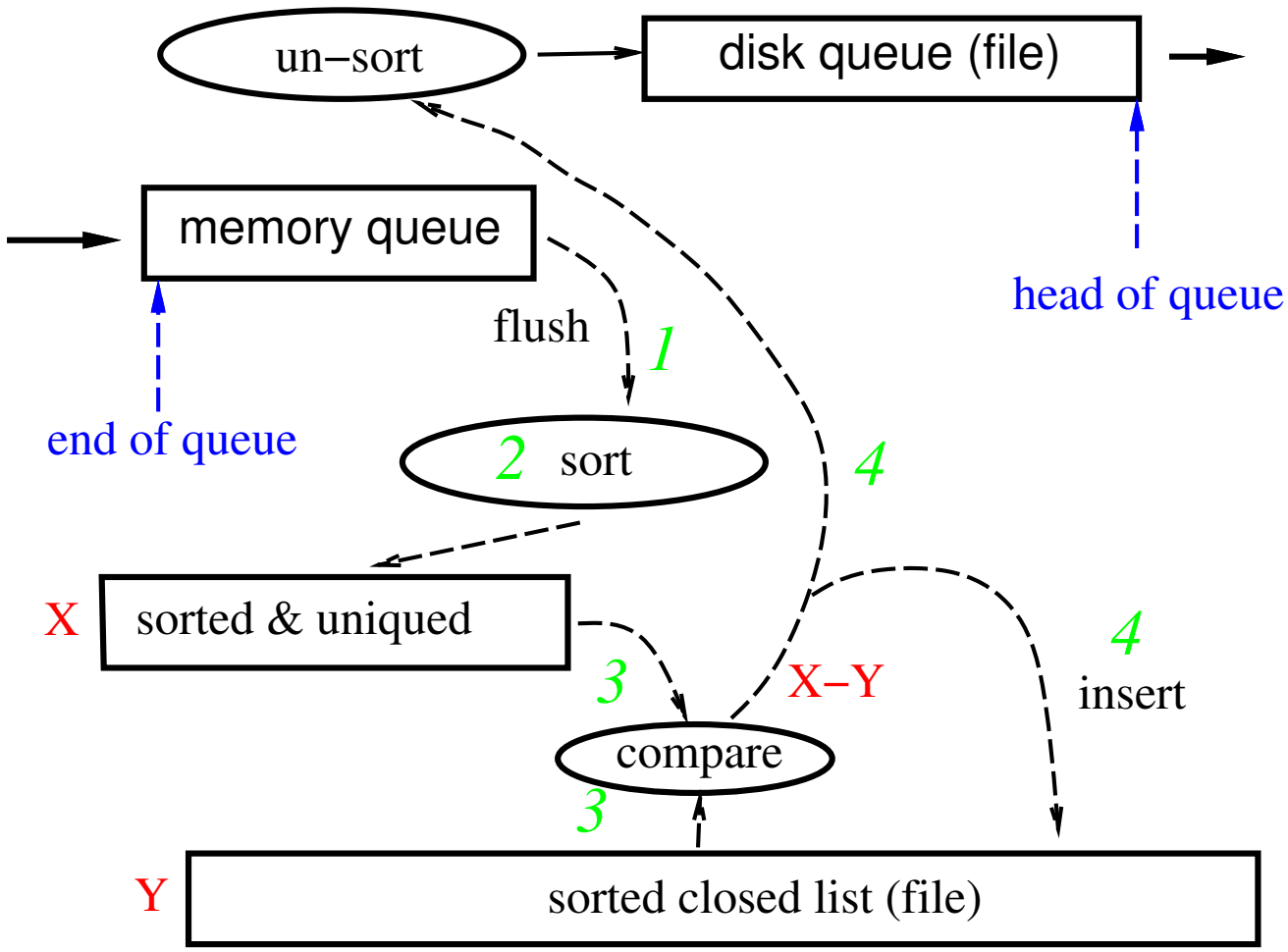
Disk based BFS (1/2)

- States to be visited (open list) are already sorted using their depths in ascending order.
  - No extra work is needed.
  - The states are appended according to their depths.
- Implementation of the QUEUE.
  - QUEUE can be stored in one disk file.
  - Newly explored ones are appended at the end of the file.
    - *fopen("filename","a+");*
  - Always retrieve the one at the head of the disk queue.
    - *lseek() can be used to mark the current head of queue.*
    - *Can periodically move the content of the disk queue to the beginning of the file.*
    - *Can move the content of the disk queue to the beginning of the file when the disk queue is empty.*
- A newly explored node will be explored after the current QUEUE is empty.
  - Property of BFS.

Disk based BFS (2/2)

- **How to find out a newly explored node has been visited before or not if this is desired?**
  - **Maintain the list of visited nodes (closed list) on DISK sorted according to some index function on ID's of the nodes.**
    - ▷ *When the memory buffer is full, sort it according to their indexes.*
    - ▷ *Merge the sorted list of newly visited nodes in the memory buffer into the one stored on DISK.*
  - **We can easily compare two sorted lists and find out the intersection or difference of the two.**
    - ▷ *We can easily remove the ones that are already visited before once Q_m is sorted.*
    - ▷ *To revert items in Q_m back to its the original BFS order, which is needed for persevering the BFS search order, we need to sort again using the original BFS ordering.*
- **Why we can delay the decision of whether a newly explored node has been visited or not?**
  - **We only need to know when a newly explored node has been visited or not when it is about to be removed from the QUEUE.**
  - **The decision of whether it has been visited or not can be delayed or batched for processing.**

Closed lists

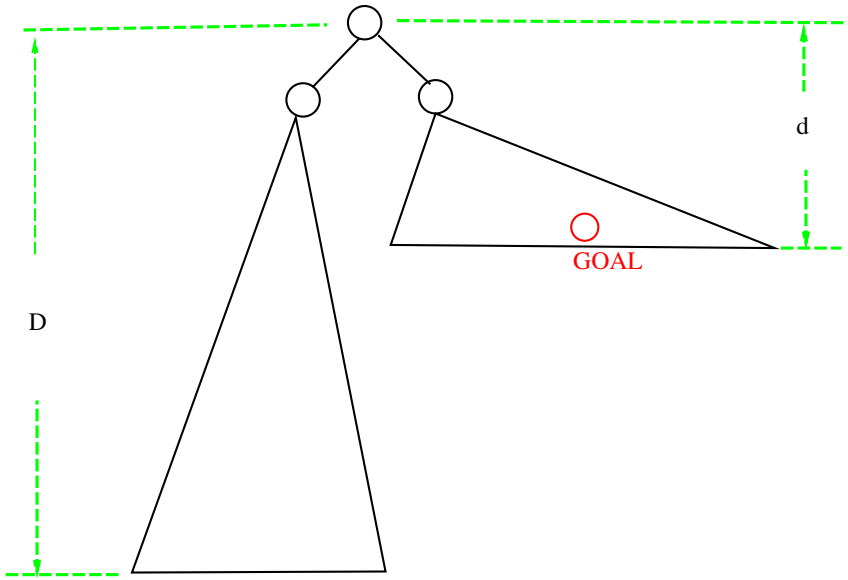


Depth-first search (DFS)

- $next(current, N)$: **returns the state next to the state "**$current$**" in** $deeper(N)$**.**
  - **Assume states in** $deeper(N)$ **are given a linear order with dummy first and last elements both being** $null$**, and assume** $current \in deeper(N)$**.**
  - **Assume we can efficiently generate** $next(current, N)$ **based on "**$current$**" and** N**.**
- **Algorithm DFS(**N_0**)** {∗ **do DFS from the starting state** N_0 ∗}
  - STACK_INIT(S)
  - PUSH(S,$(null, N_0)$)
  - **While** STACK_EMPTY(S) **is FALSE do**
    - ▷ $(current, N) \leftarrow$ POP(S)
    - ▷ $R \leftarrow next(current, N)$
    - ▷ ***If*** R ***is*** $null$***, then continue*** {∗ ***visited all*** N***'s children; backtrack!!*** ∗}
    - ▷ ***If*** R ***is a goal, then return success***
    - ▷ PUSH(S,(R, N))
    - ▷ ***If*** R ***is already in*** S***, then continue*** {∗ ***to avoid loops*** ∗}
    - ▷ ***Can introduce some cut-off depth here in order not to go too deep***
    - ▷ PUSH(S,$(null, R)$) {∗ ***search deeper*** ∗}
  - **Return fail**

DFS: analysis (1/2)

- **Time complexity:**
  - $O(e^d)$
    - ▷ *The number of possible branches at depth d is e^d.*
  - **This is only true when the game tree searched is not skewed.**
    - ▷ *The leaves of the game tree are all of $O(d)$.*
  - **It can be as bad as $O(e^D)$ where D is the maximum depth of the tree.**



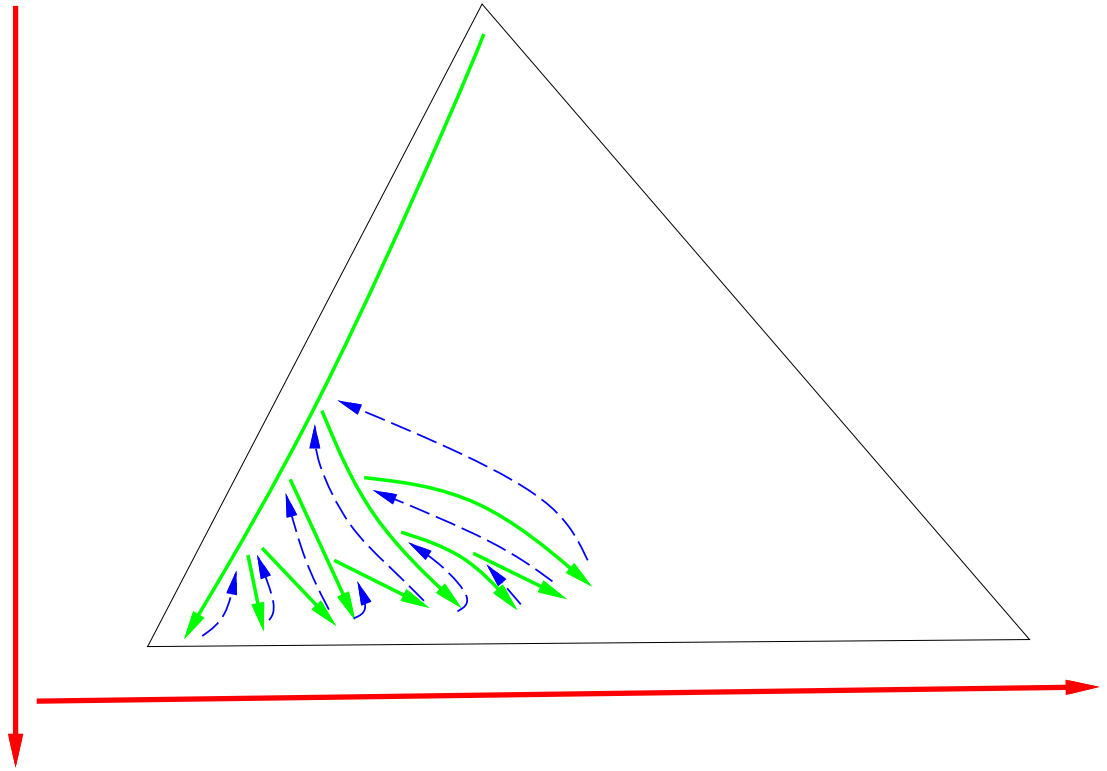
- **Space complexity:**
  - $O(d)$
    - ▷ *Only need to store the current path in the Stack.*
  - **This is also only true when the tree is not skewed.**
  - **It can be as bad of $O(D)$ where D is the maximum depth of the tree.**

DFS: analysis (2/2)

- **open list: STACK**
- **closed list: visited nodes.**
- **To avoid revisit a node too many times which is is a real issue in order to get out of a long and wrong branch as fast as you can.**
  - **undirected graphs: only need to store the current searching path in the closed list.**
    - ▷ *Can only have backward edges.*
  - **directed graphs: need to store all visited nodes.**
    - ▷ *Can have cross, forward and backward edges.*
- **Data structures for the closed list.**
  - ▷ *Hash table*
  - ▷ *Sorted list and then use binary search*
  - ▷ *Balanced search tree*
  - ▷ . . .
- **Solution found may not be optimal.**

DFS: Illustration

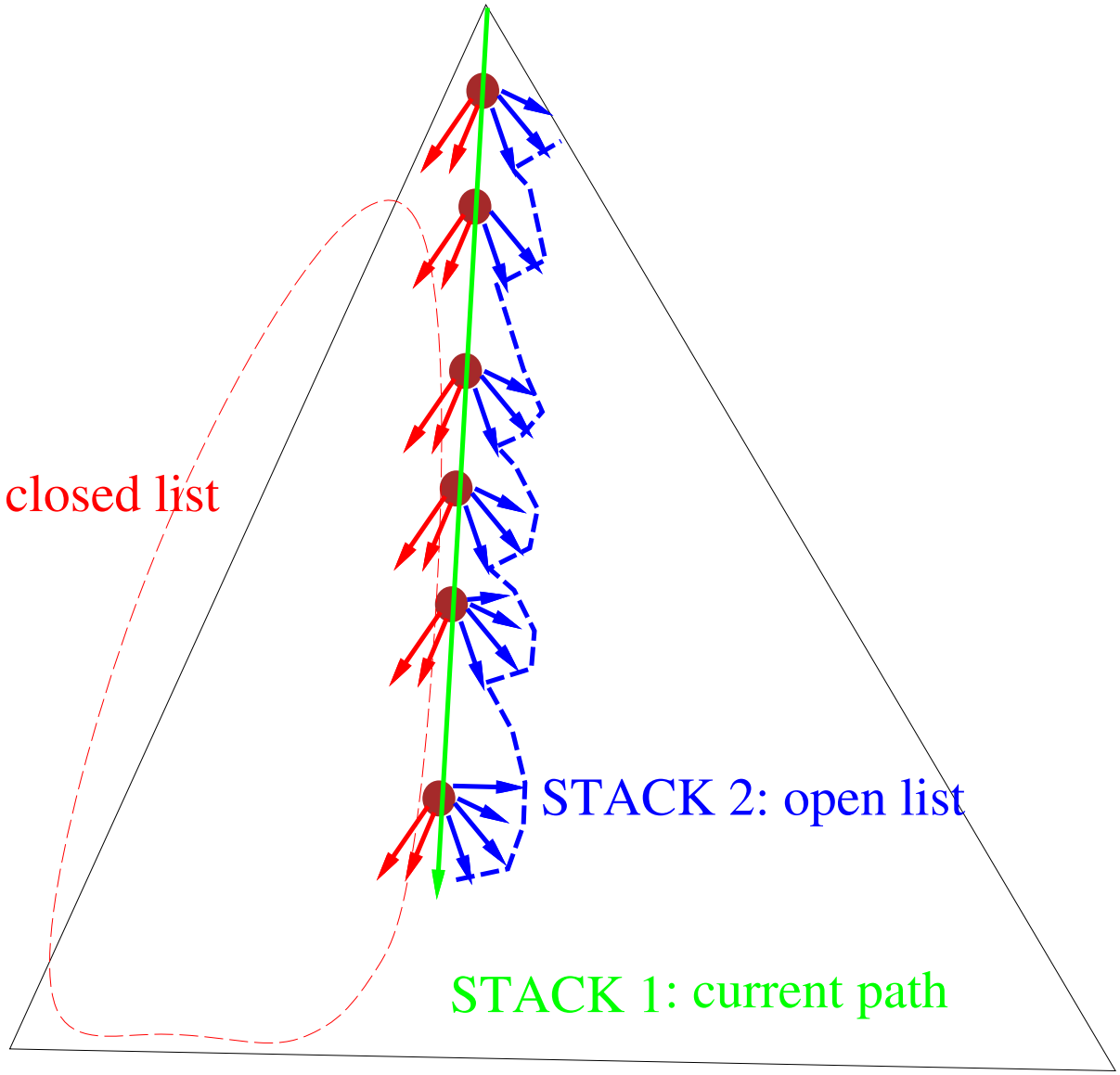
- **Layout the nodes in a 2-D plane, and sweep the plane from "shallow to deep" and from "left to right".**



DFS with two Stacks

- **May be complex to implement $next(current, N)$.**
- **Uses two stacks for DFS.**
  - **Stack 1: to keep track of the branches to be searched.**
    - ***When a new node is visited, push all of its children to the stack plus a "null" symbol as a separator.***
    - ***Pop one if you want to visit the next state.***
    - ***When a "null" symbol is popped, then we know it is to backtrack from the current node.***
    - ***Needs $O(d \cdot b)$ space.***
  - **Stack 2: to keep track of the current path.**
    - ***When a new node is visited, push it into the stack.***
    - ***When a "null" symbol is found, then we pop a node to indicate "backtrack".***
    - ***Needs $O(d)$ space.***

DFS with two Stacks: Illustration



DFS using two STACKs

Algorithm DFS′(N_0) {∗ do DFS from the starting state N_0 ∗}
{∗ uses two stacks and takes $O(d \cdot b + d)$ space ∗}

- STACK_INIT(S) {∗ **open list** ∗}
- STACK_INIT(P) {∗ **the current path** ∗}
- PUSH(S,N_0)
- **While** STACK_EMPTY(S) **is FALSE do**
  - $N \leftarrow$ POP(S)
  - **if** N **is** $Null$ **then** $N \leftarrow$ POP(P)**; continue** {∗ **backtrack** ∗}
    else PUSH(P,N) {∗ **search deeper** ∗}
  - **if** N **is a goal state, then return success**
  - PUSH(S,$Null$) {∗ **maker for end of search siblings** ∗}
  - **for each state** Z **in** $deeper(N)$ **do**
    ▷ ***if*** Z ***is not in*** P ***then*** PUSH(S,Z) {∗ ***to avoid loops*** ∗}
  - **Can introduce some cut-off depth here in order not to go too deep**
- **Return fail**

DFS: comments

- If it needs to find the path leading to the goal, you have to store the parent node of each node being visited.
- Without a good cut-off depth, it may not be able to find a solution in time.
- May not find an optimal solution at all.
- Heavily depends on the move ordering.
  - Which one to search first when you have multiple choices for your next move?
- A node can be searched many times.
  - Need to do something, e.g., hashing, to avoid researching too much.
  - Need to balance the effort to memorize and the effort to research.
- Very easy to construct the solution path from the STACK once the goal is found.
- Most critical drawback: huge and unpredictable time complexity.

DFS: when there is little memory

- **Difficult to implement a STACK on a DISK so far if the STACK is too large to be fit into the main memory.**
  - **The size of a stack (open list) won't be too large normally.**
  - **The size of the closed list can be huge.**
- **We need to decide instantly whether a node has been visited before or not.**
  - **The decision of whether a node has been visited or not cannot be delayed.**
    - ▷ ***Batch processing is not working here.***
    - ▷ ***It may take too much time to handle a disk based hash table.***
- **Use data compression and/or bit-operation techniques to store as many visited nodes as possible.**
  - **Some nodes maybe visit again and again.**
  - **Need a good heuristic to store the most frequently visited nodes.**
    - ▷ ***Avoid swapping too often.***

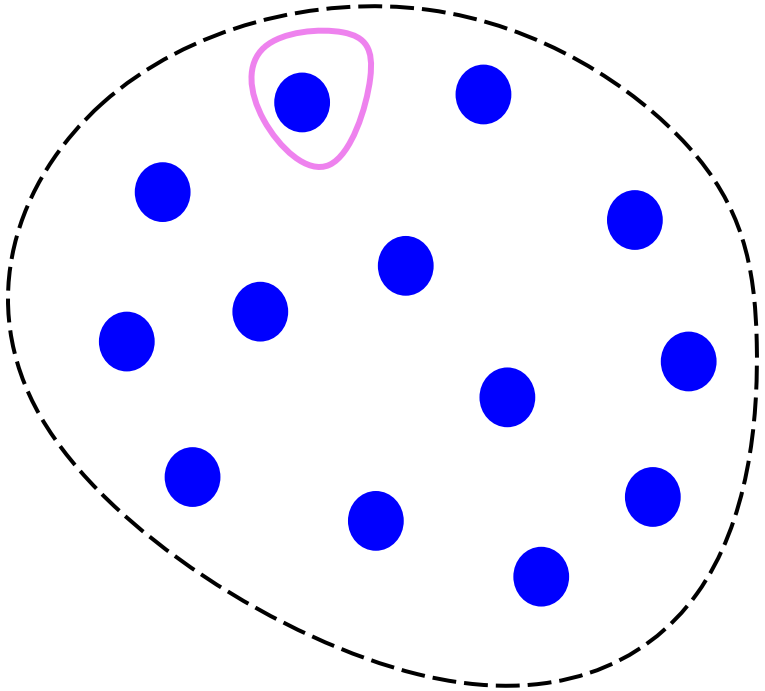
DFS with a depth limit

- **Do DFS from the starting state N_0 without exceeding a given depth $limit$.**
  - **length$(root, y)$: the number of edges visited from the root node $root$ to the node y during DFS searching.**
- **Algorithm DFS$_{depth}(N_0, limit)$**
  - STACK_INIT(S)
  - PUSH$(S,(null, N_0))$ **where N_0 is the initial state**
  - **While** STACK_EMPTY(S) **is FALSE do**
    - ▷ $(current, N) \leftarrow$ POP(S)
    - ▷ $R \leftarrow next(current, N)$
    - ▷ ***If R is a goal, then return success***
    - ▷ ***If R is $null$, then continue {$*$ visited all N's children; backtrack!! $*$}***
    - ▷ PUSH$(S,(R, N))$
    - ▷ ***If length$(N_0, R) > limit$, then continue {$*$ cut off $*$}***
    - ▷ ***If R is already in S, then continue {$*$ to avoid loops $*$}***
    - ▷ PUSH$(S,(null, R))$ ***{$*$ search deeper $*$}***
  - **Return fail**

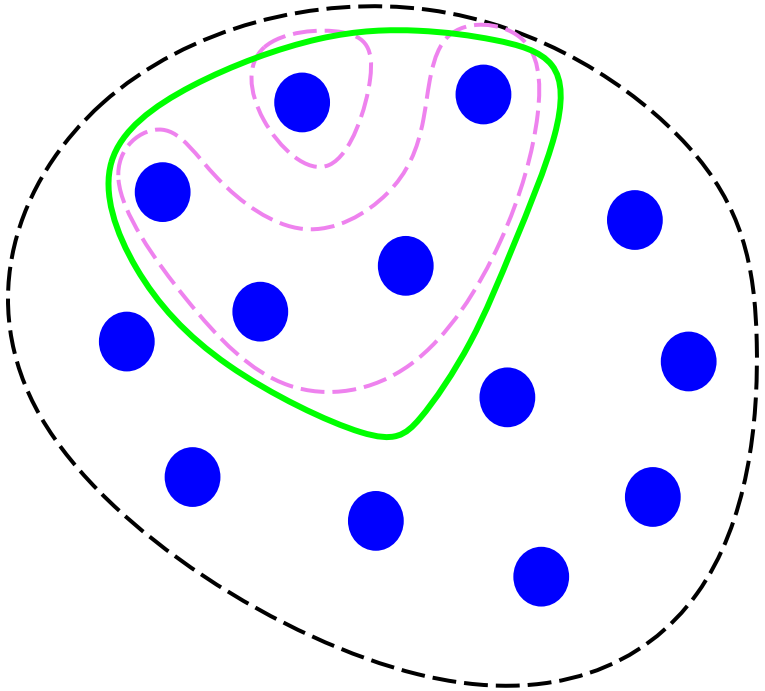
Depth-first iterative-deepening (DFID)

- **DFS$_{depth}(N, current_limit)$: DFS from the starting state N and with a depth cut off at the depth $current\_limit$.**
- **Algorithm DFID(N_0,cut_off_depth) {$*$ do DFID from the starting state N_0 with a depth limit cut_off_depth $*$}**
  - $current\_limit \leftarrow 0$
  - **While** $current\_limit < cut\_off\_depth$ **do**
    - ▷ ***If DFS$_{depth}(N_0, current_limit)$ finds a goal state g, then return g as the found goal state***
    - ▷ $current\_limit \leftarrow current\_limit + 1$
  - **Return fail**
- **Space complexity:**
  - $O(d)$

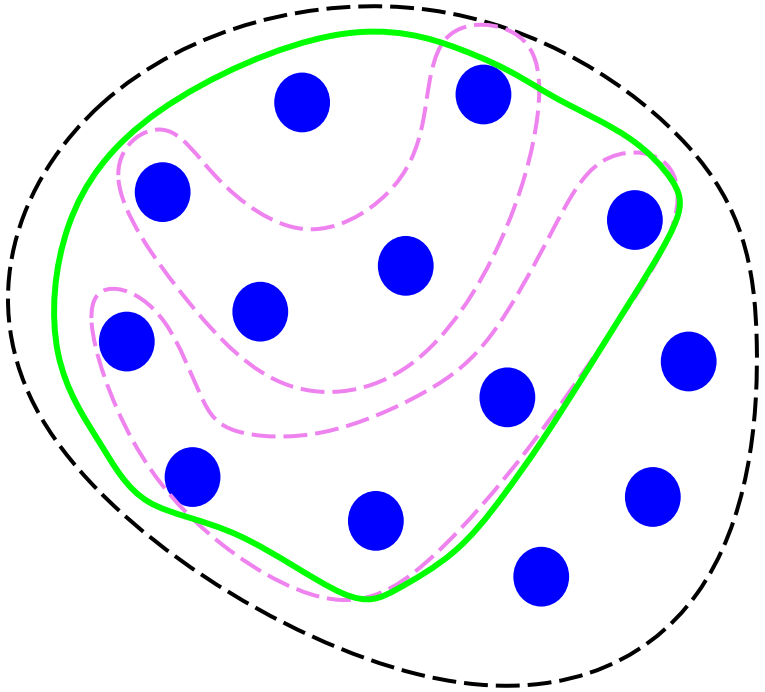
DFID



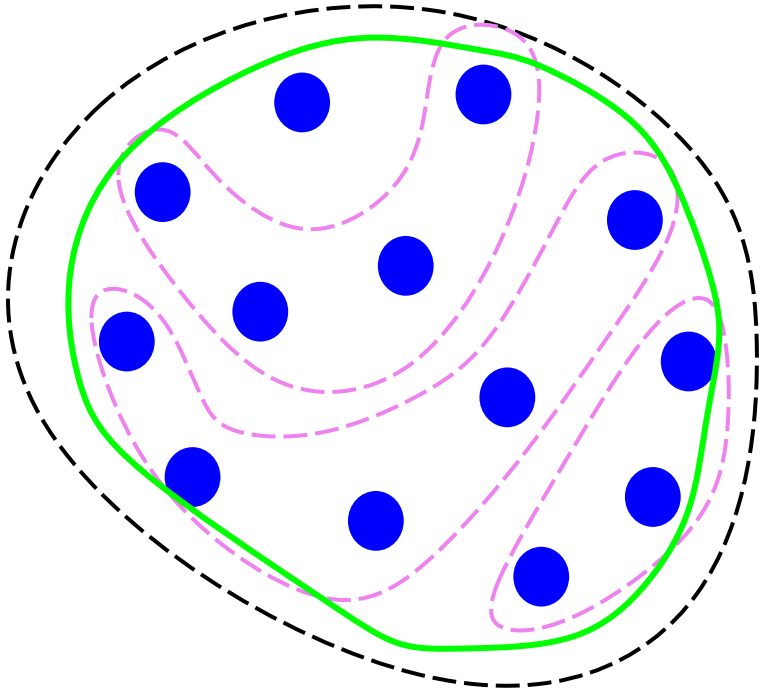
DFID



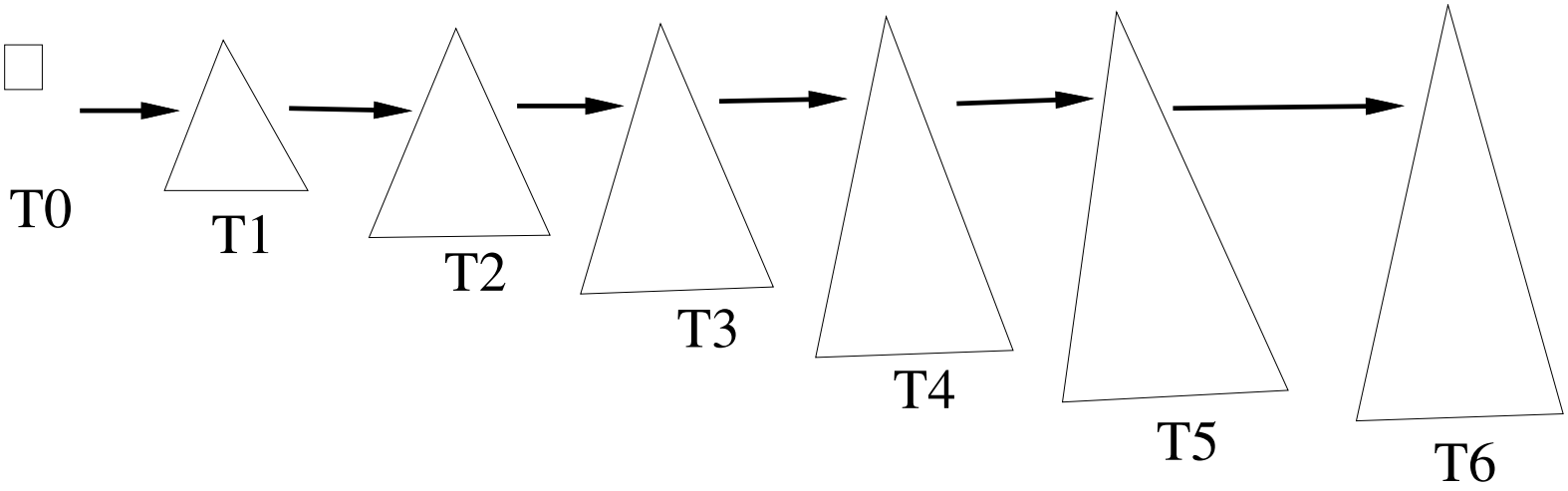
DFID



DFID



DFID: Illustration



Time complexity of DFID (1/2)

- **The branches at depth i are generated $d - i + 1$ times.**
  - **There are e^i branches at depth i.**
- **Total number of branches visited $M(e, d)$ is**

$$\begin{aligned}
&(d+1)e^0 + de^1 + (d-1)e^2 + \cdots + 2e^{d-1} + e^d \\
&= e^d(1 + 2e^{-1} + 3e^{-2} + \cdots + (d+1)e^{-d}) \\
&\leq e^d(1 - 1/e)^{-2} \text{ if } e > 1
\end{aligned}$$

- **Analysis:**
  - $(1-x)^{-2} = 1/(1-2x+x^2) = 1 + 2x + 3x^2 + \cdots + kx^{k-1} + (k+1)x^k + \cdots$.
  - ***if*** $x \geq 0$, $(k+1)x^k + (k+2)x^{k+1} \cdots \geq 0$.
  - ***Hence*** $1 + 2x + 3x^2 + \cdots + kx^{k-1} \leq (1-x)^{-2}$, ***if*** $0 \leq x$.

Time complexity of DFID (2/2)

- **Let $M(e, d)$ be the total number of branches visited by DFID with an edge branching factor of e and depth d.**
- **Examples:**
  - **When $e = 2$, $M(e, d) \leq 4e^d$.**
  - **When $e = 3$, $M(e, d) \leq 9/4e^d$.**
  - **When $e = 4$, $M(e, d) \leq 16/9e^d$.**
  - **When $e = 5$, $M(e, d) \leq 25/16e^d < 1.57e^d$.**
  - $\cdots$
  - **When $e = 30$, $M(e, d) \leq 900/841e^d < 1.071e^d$.**
- **$M(e, d) = O(e^d)$ with a small constant factor when e is sufficiently large.**

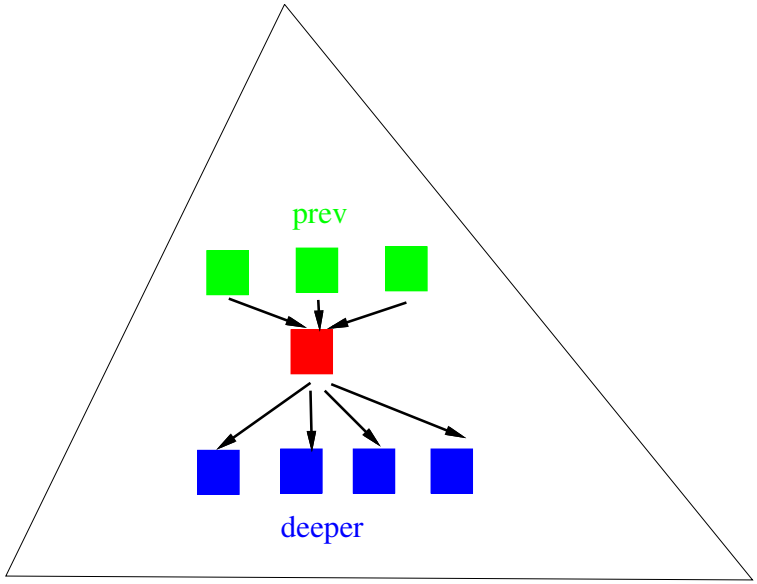
DFID: comments

- **No need to worry about a good cut-off depth as in DFS.**
- **We can find $length \leq i$ states without storing too much additional information.**
  - **However, it is really difficult to only find $length == i$ states.**
- **Trade time with space.**
- **May still need a mechanism to decide instantly whether a node has been visited before or not.**
- **Good for a tournament situation where each move needs to be made in a limited amount of time.**
- **Q:**
  - ***Does DFID always find an optimal solution?***
  - ***How about BFID?***

DFS with depth limit and direction (1/2)

- **Two refined service routines when direction of the search is considered:**
  - **DFS**$_{dir}(B, G, successor, i)$**: DFS with the set of starting states** B**, goal states** G**,** $successor$ **function and depth limit** i**.**
  - $next_{dir}(current, successor, N)$**: returns the state next to the state "**$current$**" in** $successor(N)$**.**
- **In the above two routines:**
  - $successor$ **is** $deeper$ **for forward searching**
  - $successor$ **is** $prev$ **for backward searching**
- **Note:**
  - **Given a state** N**,** $prev(N)$ **gives all states that can reach** N **in one step.**
  - **Given a state** N**,** $deeper(N)$ **gives the set of all possible states that** N **can reach in one step.**

Search directions



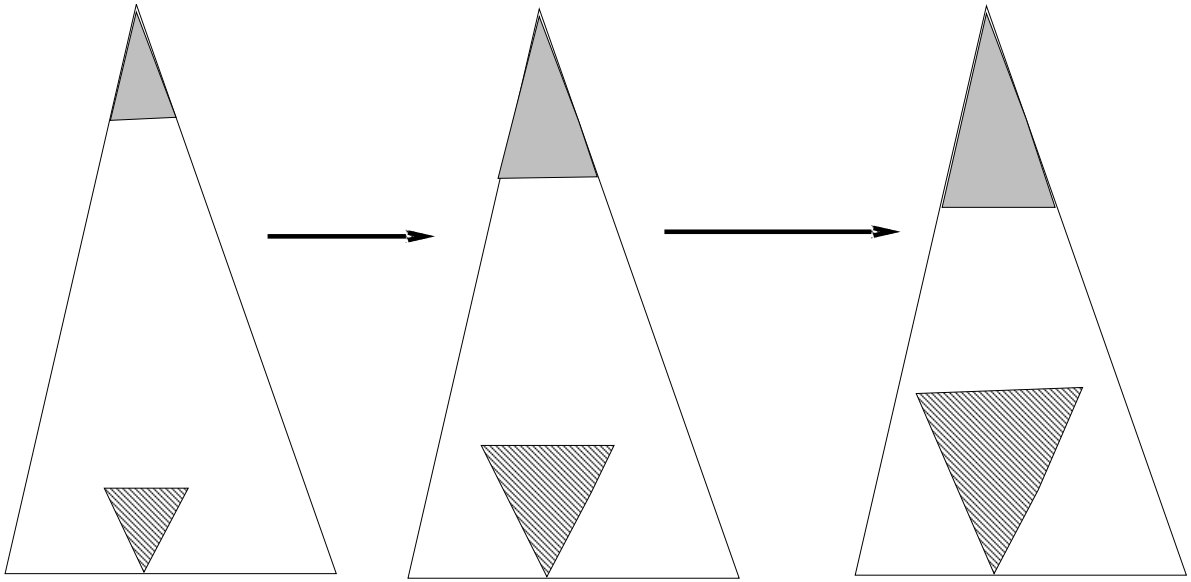
DFS with depth limit and direction (2/2)

- **DFS$_{dir}(B, G, successor, i)$: DFS with the set of starting states B, goal states G, $successor$ function and depth limit i.**
- **Algorithm DFS$_{dir}(B, G, successor, limit)$**
  - STACK_INIT(S)
  - **For each possible starting state t in B do**
    - ▷ PUSH(S,$(null, t)$)
  - **While** STACK_EMPTY(S) **is FALSE do**
    - ▷ $(current, N) \leftarrow$ POP(S)
    - ▷ $R \leftarrow next_{dir}(current, successor, N)$
    - ▷ ***If R is a goal in G, then return success***
    - ▷ ***If R is $null$, then continue {$*$ visited all N's children; backtrack!! $*$}***
    - ▷ PUSH(S,(R, N))
    - ▷ ***If length$(B, R) > limit$, then continue {$*$ cut off $*$}***
    - ▷ ***If R is already in S, then continue {$*$ to avoid loops $*$}***
    - ▷ PUSH(S,$(null, R)$) ***{$*$ search deeper $*$}***
  - **Return fail**
- **Note length(B, x) is the length of a shortest path between the state x and a state in B.**

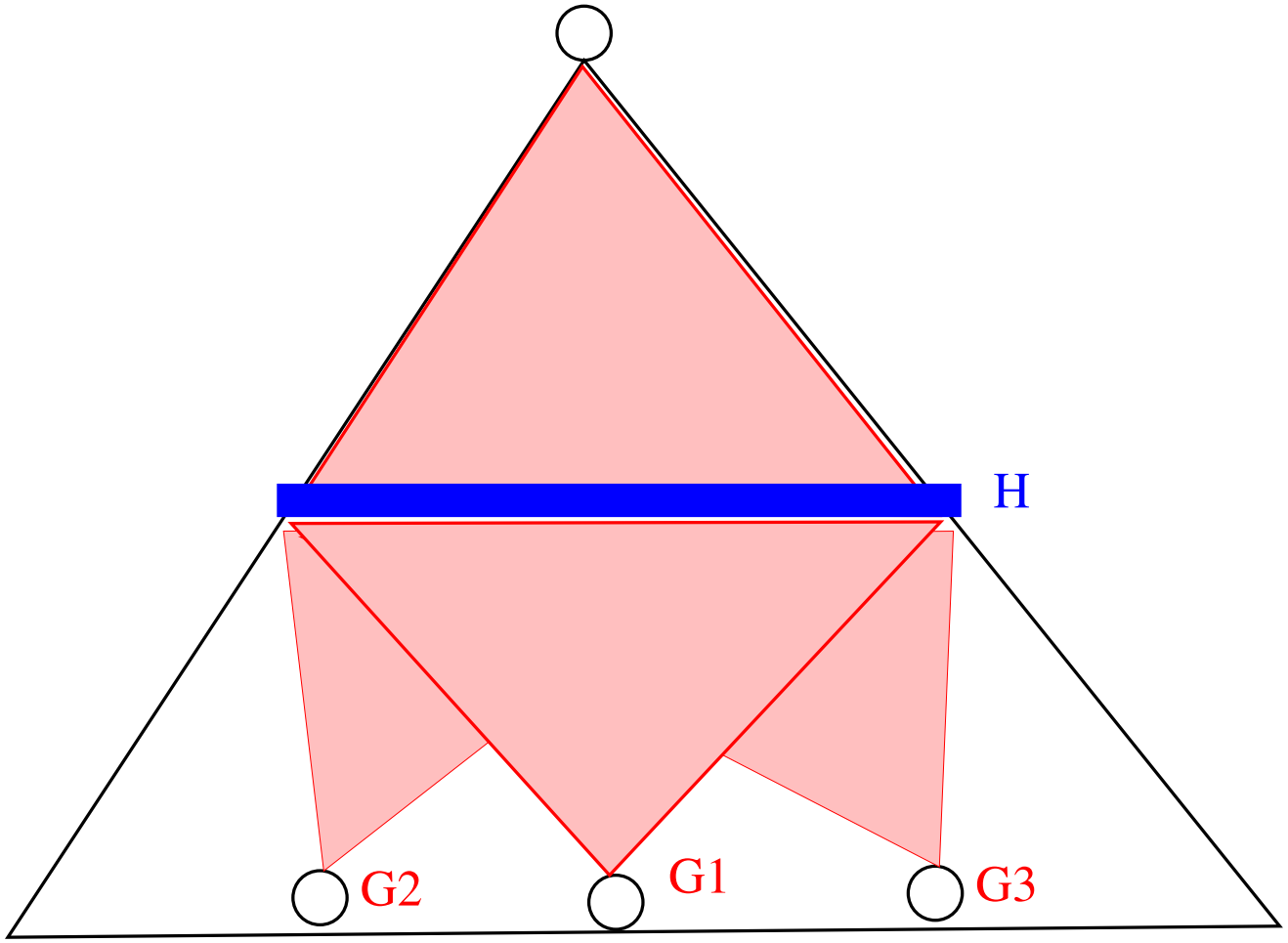
Bi-directional search

- **Combined with iterative-deepening.**
- **DFS$_{dir}(B, G, successor, i)$: DFS with the set of starting states B, goal states G, $successor$ function and depth limit i.**
  - $successor$ **is** $deeper$ **for forward searching**
  - $successor$ **is** $prev$ **for backward searching**
    - ▷ ***Given a state S_i, $prev(S_i)$ gives all states that can reach S_i in one step.***
- **Algorithm BDS(N_0,cut_off_depth)**
  - $current\_limit \leftarrow 0$
  - **while** $current\_limit < cut\_off\_depth$ **do**
    - ▷ ***if DFS$_{dir}(\{N_0\}, G, deeper, current_limit)$ returns success, then return success {∗ forward searching ∗} else store all states at depth $= current\_limit$ in an area H***
    - ▷ ***if DFS$_{dir}(G, H, prev, current_limit)$ returns success, then return success {∗ backward searching ∗}***
    - ▷ ***if DFS$_{dir}(G, H, prev, current_limit + 1)$ returns success, then return success {∗ in case the optimal solution is odd-lengthed ∗}***
    - ▷ $current\_limit \leftarrow current\_limit + 1$
  - **return fail**
- **Backward searching at depth $= current\_limit + 1$ is needed to find odd-lengthed optimal solutions.**

BDIR: Illustration



Bi-directional search: Example



Bi-directional search: analysis

- **Time complexity:**
  - $O(e^{d/2})$
- **Space complexity:**
  - $O(e^{d/2})$: **needed to store the half-way meeting points H.**
- **Comments:**
  - **Run well in practice.**
  - **Depth of the solution is expected to be the same for a normal uni-directional search, however the number of nodes visited is greatly reduced.**
  - **Pay the price of storing solutions at half depth.**
  - **Need to know how to enumerate the set of goals.**
  - **Trade off between time and space.**
    - ▷ ***What can be stored on DISK?***
    - ▷ ***What operations can be batched?***
  - **Q:**
    - ▷ ***How about using BFS in forward searching?***
    - ▷ ***How about using BFS in backward searching?***
    - ▷ ***How about using BFS in both directions?***

Heuristic search

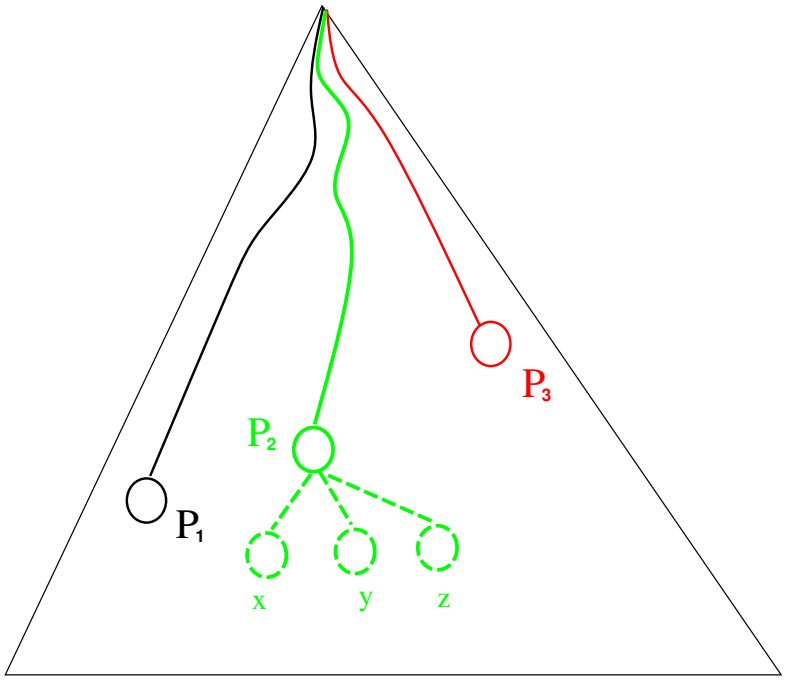
- **Heuristics**: criteria, methods, or principles for deciding which among several alternative courses of actions promises to be the most effective in order to achieve some goal [Judea Pearl 1984].
  - Need to be simple and effective in discriminate correctly between good and bad choices.
- A **heuristic search** is a search algorithm that uses information about
  - the initial state,
  - operators on finding the states adjacent to a state,
  - a test function whether a goal is reached, and
  - heuristics to pick the next state to explore.
- A "good" heuristic search algorithm:
  - States that are not likely leading to the goals will not be explored further.
    - ▷ *A state is cut or pruned.*
  - States are explored in an order that are according to their likelihood of leading to the goals $\rightarrow$ **good move ordering**.

Heuristic search: A*

- **A* search: best first heuristic search with branch and bound, and with a lower-bound estimation.**
- **Algorithm A*(N_0)**
  - PRIORITY_QUEUE_INIT(PQ) **to store partial paths with keys being the costs of the paths.**
    - ▷ ***Paths in PQ are sorted according to their current costs plus a lower bound on the remaining distances.***
  - ENPRIORITY_QUEUE(PQ,P_0) **where P_0 is the path from N_0 to N_0.**
  - **While** PRIORITY_QUEUE_EMPTY(PQ) **is FALSE do**
    - ▷ $P \leftarrow$ DEPRIORITY_QUEUE(PQ)
    - ▷ ***11: If P reaches a goal, then return success***
    - ▷ ***12: Find extended paths from P by extending one step***
    - ▷ ***for each path P' formed by adding a state N reachable from P do***
    - ▷ ***If N has not been visited before, then*** ENPRIORITY_QUEUE(PQ,P')
    - ▷ ***15: else if N has been visited from a path P'' with a larger cost,*** PRIORITY_QUEUE_REMOVE(PQ,P'') ENPRIORITY_QUEUE(PQ,P')
  - **Return fail**

A*: Illustration

- **The current PQ contains P_1, P_2 and P_3.**
  - **Assume the cost of P_2 is the smallest in PQ.**
    - ▷ ***"Cost" of a partial path will be defined later.***
- **Explore the children, x, y and z, of P_2 first.**
- **Now the PQ has 5 items, $P_2 + x$, $P_2 + y$, $P_2 + z$, P_1 and P_3.**



A^* algorithm: discussions

- When a path is inserted, namely at Line 15, check for whether it has reached some nodes that have been visited before.
  - It may take a huge space and a clever algorithm to implement an efficient Priority Queue.
  - It may need a clever data structure to efficiently check for possible duplications.
    - ▷ *Open list: a PQ to store those partial paths, with costs, that can be further explored.*
    - ▷ *Closed list: a data structure to store all visited nodes with the least cost leading to it from the starting state.*
    - ▷ *Check for duplicated visits in the closed list only.*
    - ▷ *A newly expanded node is inserted only if either it has never been visited before, or being visited, but along a path of larger cost.*
- Checking of the termination condition:
  - We need to check for whether a goal is found only when a path is popped from the PQ, i.e., at Line 11.
  - We cannot check for whether a goal is found when a path is generated and inserted into the PQ, i.e., at Line 12.
    - ▷ *We will not be able find the optimal solution if we do the checking at Line 12.*

Cost function (1/2)

- **Can be an unweighted graph or a weighted graph.**
- **Cost function:**
  - **Given a path P,**
    - ▷ *let $g(P)$ be the current cost of P;*
    - ▷ *let $h(P)$ be the estimation of remaining, or heuristic cost of P;*
    - ▷ *$f(P) = g(P) + h(P)$ is the cost function.*
  - **How to find a good $h()$ is the key of an A* algorithm?**
  - **It is known that if $h()$ never overestimates the actual cost to the goal (this is called admissible), then A* always finds an optimal solution.**
    - ▷ ***Q: How to prove this?***
  - **Note: If $h()$ is admissible and P reaches the goal, then $h(P) = 0$ and $f(P) = g(P)$.**
  - **Need an lower bound estimation that is as large as possible.**
  - **Can design the cost function so that A* emulates the behavior of other search routines.**

Cost function (2/2)

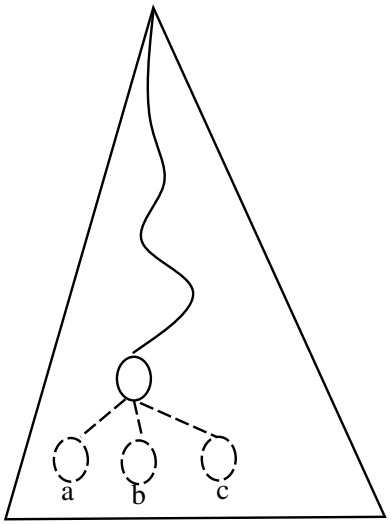
- **Assume all costs are positive, there is no need to check for falling into a loop.**
  - **A node may be visited several times through different paths.**
- **It consumes a lot of memory and time to store and compare the set of visited nodes (closed list) which is needed to improve the efficiency.**
- **It also consume a lot of memory and time to store and process the PQ, namely open list.**
- **Q:**
  - ***What disk based techniques can be used?***
  - ***Why do we need a non-trivial $h(P)$ that is admissible?***
  - ***How to design an admissible cost function?***

DFS with costs and a threshold

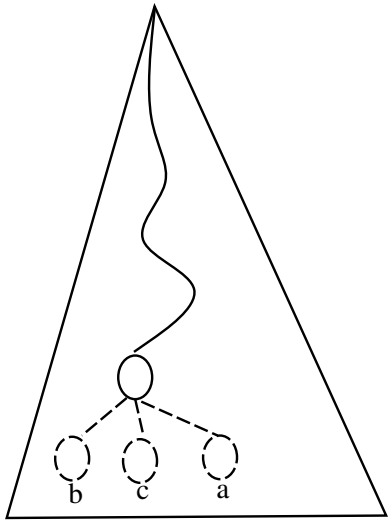
- **DFS$_{cost}(N, f, threshold)$ is a version of DFS with a starting state N and a cost function f that cuts off a path when its cost is more than a given $threshold$.**
  - **DFS$_{depth}(N, cut_off_depth)$ is a special version of DFS$_{cost}(N, f, threshold)$.**
- **Algorithm DFS$_{cost}(N_0, f, threshold)$**
  - STACK_INIT(S)
  - PUSH$(S,(null, N_0))$ **where N_0 is the initial state**
  - **While** STACK_EMPTY(S) **is FALSE do**
    - $(current, N) \leftarrow$ POP(S)
    - $R \leftarrow next(current, N)$ {$*$ *pick a good move ordering here* $*$}
    - ***If*** $R = null$, ***then continue*** {$*$ ***visited all*** N***'s children; backtrack!!*** $*$}
    - PUSH$(S,(R, N))$
    - ***Let*** P ***be the path from*** N_0 ***to*** R
    - ***If*** $f(P) > threshold$, ***then continue*** {$*$ ***cut off*** $*$}
    - ***If*** R ***is a goal, then return success*** {$*$ ***goal is found*** $*$}
    - ***If*** R ***is already in*** S, ***then continue*** {$*$ ***to avoid loops*** $*$}
    - PUSH$(S,(null, R))$ {$*$ ***search deeper*** $*$}
  - **Return fail**

How to pick a good move ordering ?

- **Instead of just using $next(current, N)$ to find the next unvisited neighbors of N with the information of the last visited node being $current$, we do the followings.**
  - **Use a routine to order the neighbors of N so that it is always the case the neighbors are visited from low cost to high cost.**
    - ▷ ***Let this routine be*** $next1(current, N)$***.***
  - **Note we still need dummy first and last elements which are represented as $null$.**



next() follows an arbitrary, but fixed order



next1() follows a sorted order

DFS with a greedy move ordering

- **Algorithm DFS1$_{cost}$(**N_0**,**f**,**$threshold$**)**
  - STACK_INIT(S)
  - PUSH(S,$(null, N_0)$) **where** N_0 **is the initial state**
  - **While** STACK_EMPTY(S) **is FALSE do**
    - ▷ $(current, N) \leftarrow$ POP(S)
    - ▷ $R \leftarrow next1(current, N)$
    - ▷ ***If*** $R = null$***, then continue*** {$*$ ***visited all*** N***'s children; backtrack!!*** $*$}
    - ▷ PUSH(S,(R, N))
    - ▷ ***Let*** P ***be the path from*** N_0 ***to*** R
    - ▷ ***If*** $f(P) > threshold$***, then continue*** {$*$ ***cut off*** $*$}
    - ▷ ***If*** R ***is a goal, then return success*** {$*$ ***Goal is found!*** $*$}
    - ▷ ***If*** R ***is already in*** S***, then continue*** {$*$ ***to avoid loops*** $*$}
    - ▷ PUSH(S,$(null, R)$) {$*$ ***search deeper*** $*$}
  - **Return fail**

How to in-cooperate ideas from A*

- **Instead of using a stack in DFS$_{cost}$, use a priority queue.**
- **Algorithm DFS2$_{cost}$(N_0,f,$threshold$)**
  - PRIORITY_QUEUE_INIT(PQ) **with keys** $f(P)$ **where** P **is the path from** N_0 **to the state stored**
  - ENPRIORITY_QUEUE(PQ,$(null, N_0)$)
  - **While** PRIORITY_QUEUE_EMPTY(PQ) **is FALSE do**
    - ▷ $(current, N) \leftarrow$ DEPRIORITY_QUEUE(PQ)
    - ▷ $R \leftarrow next1(current, N)$
    - ▷ ***If*** $R = null$***, then continue*** {∗ ***visited all*** N***'s children; backtrack!!*** ∗}
    - ▷ ENPRIORITY_QUEUE(PQ,(R, N))
    - ▷ ***Let*** P ***be the path from*** N_0 ***to*** R
    - ▷ ***If*** $f(P) > threshold$***, then continue*** {∗ ***cut off*** ∗}
    - ▷ ***If*** R ***is a goal, then return success*** {∗ ***Goal is found!*** ∗}
    - ▷ ***If*** R ***is already in*** PQ***, then continue*** {∗ ***to avoid loops*** ∗}
    - ▷ ENPRIORITY_QUEUE(PQ,$(null, R)$) {∗ ***search deeper*** ∗}
  - **Return fail**

DFS1() and DFS2()

- **DFS1()**
  - **Using a locally best-first or greedy approach to pick the next child to explore.**
- **DFS2()**
  - **It may be costly to maintain a priority queue as in the case of A*.**
  - **Similar to A*, globally pick the next path to explore.**
    - ▷ ***DFS2 extends one child at a time.***
    - ▷ ***A* expands all children at once.***
  - **Similar to DFS1, using a locally best-first or greedy approach to pick the next child to explore.**

IDA* = DFID + A*

- **DFS$_{cost}(N, f, threshold)$ is a version of DFS with a starting state N and a cost function f that cuts off a path when its cost is more than a given $threshold$.**
- **IDA*: iterative-deepening A***
- **Algorithm IDA$^*(N_0, threshold)$**
  - $threshold \leftarrow h(null)$
  - **While $threshold$ is reasonable do**
    - ▷ ***DFS$_{cost}(N_0, g + h(), threshold)$***
      {$*$ Can also use either DFS1$_{cost}()$ or DFS2$_{cost}()$ here $*$}
    - ▷ ***If the goal is found, then return success***
    - ▷ $threshold \leftarrow$ ***the least $g(P) + h(P)$ cost among all paths P being cut***
  - **Return fail**

IDA*: comments

- **IDA* does not need to use a priority queue as in the case of A* if DFS1() is used.**
  - **IDA* using DFS1() is optimal in terms of solution cost, time, and space over the class of admissible best-first searches on a tree.**
- **Issues in updating** $threshold$**.**
  - **Increase too little: re-search too often.**
  - **Increase too large: cut off too little.**
  - **Q: How to guarantee optimal solutions are not cut?**
    - ▷ ***It can be proved, as in the case of A*, that given an admissible cost function, IDA* will find an optimal solution, i.e., one with the least cost, if one exists.***
- **Cost function is the knowledge used in searching.**
- **Combine knowledge and search!**
- **Need to balance the amount of time spent in realizing knowledge and the time used in searching.**
  - ▷ ***A tradeoff issue.***

15 puzzle (1/2)

- **Introduction of the game:**
  - 15 tiles in a 4*4 square with numbers from 1 to 15.
  - One empty cell.
  - A tile can be slid horizontally or vertically into an empty cell.
  - From an initial position, slide the tiles into the goal position.
- **Example:**
  - Initial position:

10	8		12
3	7	6	2
1	14	4	11
15	13	9	5

  - Goal position:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

15 puzzle (2/2)

- **Total number of positions:** $16! = 20,922,789,888,000 \leq 2.1 * 10^{13}$.
  - **It is feasible, in terms of computation time, to enumerate all possible positions, since about the year 2007.**
    - ▷ *Can use DFS or DFID now.*
    - ▷ *Need to avoid falling into loops or re-visit a node too many times.*
  - **It is still too large to store all possible positions in the main memory of a PC now (namely, the year 2020). However, some servers do have the capacity.**
    - ▷ *Cannot use BFS efficiently even now.*
    - ▷ *It takes efforts to find an optimal solution.*
    - ▷ *Disk based BFS may help.*

Solving 15 puzzles (1/2)

- **Using DEC 2060 a 1-MIPS machine (year 1985): solved the 15 puzzle problem within 30 CPU minutes for all testing positions, and in average generates over 1.5 million nodes per minute.**
  - **Note:**
    - ***Intel Core i3 7350K has 2 cores (year 2017) and is rated at 13,204 MIPS.***
    - ***ARM Cortex A7 has upto 4 cores (year 2011) and is rated at 2,850 MIPS.***
    - ***Apple A11 has 6 cores (year 2017) is at least 3.3 times more powerful than A7 per core.***
    - ***Apple A12 has 6 cores (year 2018) is at least 15% more powerful than A11 per core.***
    - ***Apple A13, in 7nm, has 6 cores (year 2019) is at least 20% more powerful than A12 per core (roughly 13,000 MIPS).***
    - ***Apple A14, in 5nm, has 6 cores (year 2020) is at least 15% more powerful than A13 per core (roughly 15,000 MIPS), and is 30% power efficient.***
  - **Further note: More GPU and AI engines are seen in later Apple CPU's.**

Solving 15 puzzles (2/2)

- **Result:**
  - **The average solution length was 53 moves.**
  - **The maximum was 66 moves.**
  - **IDA* generated more nodes than A*, but ran faster due to less overhead per node.**
- **Heuristics used:**
  - **$g(P)$: the number of moves made so far.**
  - **$h(P)$: the Manhattan distance between the current and the goal position.**
    - ***Suppose a tile is currently at (i, j) and its goal is at (i', j'), then the Manhattan distance for this tile is $|i - i'| + |j - j'|$.***
    - ***The Manhattan distance between a position and a goal position is the sum of the Manhattan distance of every tile.***
    - ***$h(P)$ is admissible.***
    - ***$f(P) = g(P) + h(P)$ may not be monotone.***
    - ***Q: What kinds of $f()$ are monotone?***

What else can be done?

- Bi-directional search and IDA*?
  - How to design a good and non-trivial heuristic function?
- How to find an optimal solution?
- How to get a better move ordering in DFS?
- Balancing in resource allocation:
  - The efforts to memorize past results versus the amount of efforts to search again.
  - The efforts to compute a better heuristic, i.e., the cost function.
  - The amount of resources spent in implementing a better heuristic and the amount of resources spent in searching.
- Search in parallel.
- More techniques for disk based algorithms.
- Q: Can these techniques be applied to two-person games?

References and further readings

- Judea Pearl. Heuristics: Intelligent search strategies for computer problem solving. Addison-Wesley, 1984.

* R. E. Korf. Depth-first iterative-deepening: An optimal admissible tree search. *Artificial Intelligence*, 27:97–109, 1985.

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