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Projective space
in the spirit of Klein's Erlanger Programm

by *Kelly McKennon* at Pullman

Projective space is a medium wherein parallel lines intersect and where there is a perfect duality between minimal and maximal subspaces. There, the infinite exists, and there one can perceive how the infinite affects behavior in the finite. Seeming anomolies, when regarded from the perspective of euclidian space, will sometimes disappear in projective space.

An intuitive way to define a projective space is to take an affine space and, for each maximal family of mutually parallel lines, attach a point "at infinity" through which these lines intersect. The locus of all these new points forms a maximal proper subspace of the projective space.

A shortcoming of this approach is that it presents the subspace of points "at infinity" as more special than it really is. Another shortcoming is that in this setting it is somewhat awkward to describe the "projective group" of permutations of the projective space.

The most common definition, called the "analytic approach", and by many considered to be the most elegant, does not single out any hyperplane. In this definition one considers a linear space L one dimension greater than that of the projective space: the projective space P is defined as the set of all lines through the origin of this linear space. If π is any linear isomorphism of L , and if L is any line through the origin of L , then the set $\{\pi(x) : x \in L\}$ is also a line through the origin: thus π corresponds to a permutation of P . These permutations are said to me the "homographies" of P , and they make up the "projective group" of P .[1]

While smooth — and technically correct — this method requires one to think of a point in the projective space as a line in another space, which some find somewhat cumbersome. Furthermore it involves the concepts of "linear space" and "field", which in some sense are more specialized than the concept of "projective space", and so are, as part of the definition, in some sense superfluous, .

A third definition, known as the "synthetic" approach, is to begin with a family of subsets of a set, known as "lines", and to assign to these lines certain requirements.

An advantage of this method is that it provides a geometric perspective which agrees with natural observation.[2] A disadvantage is that the cases of one and two dimensions must be handled separately.

In 1872 the celebrated mathematician Felix Klein, then a young man working at the University of Erlangen-Nürnberg, suggested that various types of geometries at that time under investigation could be organized with reference to the specific groups of permutations which preserved their characteristic structures.[3] This idea had considerable influence on subsequent mathematics and became known as the "Erlanger Programm". At the time, Klein felt that the groups associated with the majority of important geometries then under consideration were subgroups of the group of homographies of projective space.

In the present paper we develop projective spaces exploiting Klein's approach: we consider a set P on which a group $\mathcal{P}$ of permutations acts, and present a short list of postulates necessary and sufficient for $\mathcal{P}$ to be a projective group of permutations of P .

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[1] This is often done using "homogeneous coordinates" to describe the lines through the origin, which facilitates the use of matrices: see [3] in the bibliography.

[2] See [2] in the bibliography.

[3] See [1] in the bibliography.

Section 1. Projective Space

(1.1) Special Notation and Terminology A **singleton** is a set with precisely one element, a **doubleton** a set with precisely two elements, *etc.*

An element of a set S will be said to be **in** S, but a subset of S will be said to be **within** S. Thus an element s of S is in S but the singleton {s} is within S.

The notation

$$x \equiv y$$

shall mean that "x is defined to be y".

For two sets A and B within a third set C, we adopt the **symmetric difference** notation

$$A \triangle B \equiv \{x \in (A \cup B) : x \notin (A \cap B)\}.$$

In particular, if B is within A, then $A \triangle B$ is the **set complement of** B **within** A.

We shall attempt to distinguish functions by coloring them in blue and arguments of functions by enclosing them in orange parentheses. We shall indicate the definitions of new terms in green.

The notation

$$\phi \big| D \ni x \hookrightarrow E(x) \in R$$

shall mean that ϕ is a function with domain D and range R which takes an element x of D to an element of R of which the construction is indicated by an expression $E(x)$. Of course ϕ is a set of ordered pairs so, if $\mathbb{N}$ denotes the set of natural numbers, the expression

$$\phi \big| \mathbb{N} \ni x \hookrightarrow 2x^2 \in \mathbb{N},$$

just signifies that

$$\phi = \{[1,2], [2,8], [3,18], \ldots\}.$$

We shall employ the universal quantifier $\forall$ and the existential quantifier $\exists$ freely in displays. For instance

$$(\forall\, n \in \mathbb{N} : (\exists\, m \in \mathbb{N} : n = m^2)) \quad n \text{ is not prime.}$$

is read as "for each positive integer n for which there exists a positive integer m such that m^2 equals n, n is not prime."

(1.2) Definitions Let P be a non-void set and $\mathcal{P}$ a non-void group of permutations of P. If P is a singleton, then $\mathcal{P}$ is a singleton as well, and we shall speak of a **trivial projective space** and a **trivial projective group**.

Suppose that P has at least two distinct elements. For any doubleton {a,b} within P and any element x of P, we shall say that x is $\mathcal{P}$-**aligned with** a and b provided that, whenever $\pi \in \mathcal{P}$ interchanges a and b, then

$$(\exists\, y \in (P \triangle \{x\})) \quad \{x,y\} = \{\pi(x), \pi(y)\}. \tag{1}$$

We note that 1 holds, either if π interchanges x and y, or if π fixes both x and y. We shall denote the set of all elements of P which are $\mathcal{P}$-aligned with a and b by

$$\overleftrightarrow{a,b}$$

and say that the $\overleftrightarrow{a,b}$ is **trivial** if it consists only of the doubleton {a,b}.

A subset S of P will be said to be **coaligned** if, for each doubleton $\{a,b\} \subset S$,

$$\overleftrightarrow{a,b} \subset S. \tag{2}$$

The intersection of all coaligned supersets of a subset S of P is itself coaligned, and will be called the **span of** S : we shall denote it by

$$\boxed{\mathrm{S}}\ . \tag{3}$$

A subset S of P will be said to be **independent** if

$$(\forall\, \mathrm{s}\in\mathrm{S})\quad \mathrm{s}\notin \boxed{\mathrm{S}\triangle\{\mathrm{s}\}}\ . \tag{4}$$

A non-independent set is said to be **dependent**.

A **fundamental representative of a dependent set** F is an element a of F such that

$$(\forall\, \mathrm{b}\in\mathrm{F})\quad \mathrm{a}\notin \boxed{\mathrm{F}\triangle\{\mathrm{a,b}\}}\ . \tag{5}$$

When a dependent set has a fundamental representative, it itself will be said to be **fundamental**.[4]

We shall say that an element ϕ of $\mathcal{P}$ is **organic on a subset** S **of** P provided that, whenever $\theta\in\mathcal{P}$ agrees with ϕ on S , then they agree on $\boxed{\mathrm{S}}$ as well.

(1.3) Projective Space Let P be a non-trivial set with a non-trivial group $\mathcal{P}$ of permutations of P . We shall say that P is a **projective space** and $\mathcal{P}$ a **projective group** if the following two axioms hold:

(*Alignment Axiom*) $\quad (\forall\, \{\mathrm{a,b}\}\subset\mathrm{P}$ a doubleton$)\quad \overleftrightarrow{\mathrm{a,b}}$ is non-trivial and coaligned; (1)

and, for any bijection θ from one fundamental set F onto another fundamental set,

(*Fundamental Axiom*) $\quad (\exists\, \phi\in\mathcal{P}$ organic on F$)\quad \phi\big|_{\mathrm{F}}=\theta$. (2)

The elements of a projective group are said to be **homographies** .

When there is an upper bound to the cardinalities of fundamental subsets of a projective space, then the space is said to be **finite dimensional** and the **dimension** is two less than the least upper bound of said cardinalities.

A projective space of dimension 1 will be called here a **meridian**. The usual term is "projective line". However we wish in this presentation to distinguish between lines of a linear space and meridians. We treat meridians specifically in Section (2) *infra*.

A **projective subspace** T of a projective space P with projective group $\mathcal{P}$ is a subset of P equal to the span of itself. The **induced projective group of a subspace** T is just the set

$$\{\phi\big|_{\mathrm{T}}:\ \phi\in\mathcal{P}\ \text{ and }\ (\forall\, \mathrm{t}\in\mathrm{T})\ \phi(\mathrm{t})\in\mathrm{T}\}\ . \tag{3}$$

Any subspace of dimension 1 with its induced projective group, constitutes a projective space in its own right: it is a meridian.

(1.4) Heuristics We first examine the "analytical" definition of a classical projective space of dimension n and show that it satisfies the axioms of (1.3) . One begins with a linear space L of dimension n+1 over a field F and considers the family $\mathcal{L}$ of all lines through the origin o of L . We shall use $\cdot$ for the multiplicative operator in F and $\bullet$ for scalar multiplication. If λ is a linear automorphism of L , we define

$$\lambda'\big|\mathcal{L}\ni\mathrm{A}\hookrightarrow\{\lambda(\mathrm{x}):\ \mathrm{x}\in\mathrm{A}\}\in\mathcal{L}\ .$$

To obtain a "line" in $\mathcal{L}$, we take a two dimensional linear subspace B of L and consider

$$\mathcal{N}\equiv\ \{\mathrm{A}\in\mathcal{L}:\ \mathrm{A}\subset\mathrm{B}\}\ .$$

Let {m,n} be a doubleton of B such that the origin o is not in the line N determined by m and n . Then N is just $\{\mathrm{x}\bullet\mathrm{m}+\mathrm{n}:\ \mathrm{x}\in\mathrm{F}\}$. For $\mathrm{x}\in(\mathrm{L}\triangle\{\mathrm{o}\})$ we shall denote the line through o and x by $\overleftrightarrow{\mathrm{x}}$.

The most general linear automorphism of B is

[4] Such a subset is also known as a **frame**.

$$\mathrm{B} \ni \mathrm{x} \bullet \mathrm{m} + \mathrm{y} \bullet \mathrm{n} \hookrightarrow (\mathrm{a}\cdot\mathrm{x} + \mathrm{b}\cdot\mathrm{y}) \bullet \mathrm{m} + (\mathrm{c}\cdot\mathrm{x} + \mathrm{d}\cdot\mathrm{y}) \bullet \mathrm{n} \in \mathrm{B}$$

where $\{a,b,c,d\}$ is a subset of F such that $a\cdot d \neq b\cdot c$. It follows that the most general projective automorphism of $\mathcal{N}$ is ϕ where

$$\phi(\overleftrightarrow{\mathrm{m}}) \equiv \overleftrightarrow{\mathrm{a}\bullet\mathrm{m}+\mathrm{b}\bullet\mathrm{n}}$$

and, for $x \in F$,

$$\phi(\overleftrightarrow{\mathrm{x}\bullet\mathrm{m}+\mathrm{n}}) \equiv \overleftrightarrow{(\mathrm{a}\cdot\mathrm{x}+\mathrm{b})\bullet\mathrm{m}+(\mathrm{c}\cdot\mathrm{x}+\mathrm{d})\bullet\mathrm{n}} \, .$$

A necessary and sufficient condition for ϕ to interchange the lines $\overleftrightarrow{\mathrm{m}}$ and $\overleftrightarrow{\mathrm{n}}$ is for both a and d to be 0. In this case we have for non-zero x,

$$\phi(\overleftrightarrow{\mathrm{x}\bullet\mathrm{m}+\mathrm{n}}) = \overleftrightarrow{\mathrm{b}\bullet\mathrm{m}+(\mathrm{c}\cdot\mathrm{x})\bullet\mathrm{n}} = \overleftrightarrow{(\frac{\mathrm{b}}{\mathrm{c}\cdot\mathrm{x}})\bullet\mathrm{m}+\mathrm{n}} \, .$$

Composing ϕ with itself, we obtain

$$\phi\circ\phi(\overleftrightarrow{\mathrm{x}\bullet\mathrm{m}+\mathrm{n}}) = \phi(\overleftrightarrow{(\frac{\mathrm{b}}{\mathrm{c}\cdot\mathrm{x}})\bullet\mathrm{m}+\mathrm{n}}) = \overleftrightarrow{\mathrm{b}\bullet\mathrm{m}+(\mathrm{c}\cdot\frac{\mathrm{b}}{\mathrm{c}\cdot\mathrm{x}})\bullet\mathrm{n}} = \overleftrightarrow{\mathrm{x}\bullet\mathrm{m}+\mathrm{n}}$$

so ϕ is self-inverse. If $\frac{b}{c}$ has a square root r, then $r\bullet m+n$ and $-r\bullet m+n$ are the only fixed points for ϕ: if $\frac{b}{c}$ has no square roots, ϕ has no fixed points.

Referring back to 1.2.1, we can take y to be $\overleftrightarrow{\mathrm{b}\bullet\mathrm{m}+\mathrm{c}\cdot\mathrm{x}\bullet\mathrm{n}}$ when $x \neq r\bullet m+n$ and take y to be $\overleftrightarrow{-\mathrm{r}\bullet\mathrm{m}+\mathrm{n}}$ when $x = r\bullet m+n$.

If p is an element of L not in the plane B, then it is not difficult to find a linear isomorphism λ of L such that λ' interchanges $\overleftrightarrow{\mathrm{m}}$ and $\overleftrightarrow{\mathrm{n}}$ but $\lambda'\circ\lambda'(\overleftrightarrow{\mathrm{p}}) \neq \overleftrightarrow{\mathrm{p}}$. This suggests the *alignment axiom* (1.3.1) and actually shows that the *alignment axiom* (1.3.1) holds for the "analytic" definition of a projective space.

That the *fundamental axiom* (1.3.2) holds, is what is known as the **fundamental theorem of projective geometry**.

We shall show *infra* that the converse holds: that a projective space as defined in (1.3) is a projective space defined in a classical way, over a field of characteristic differing from 2.

(1.5) Theorem I *Let $\mathcal{P}$ be a projective group of permutations on a projective space* P*. Let* $\{a,b\}$ *be a doubleton within* P *and let ϕ be an element of $\mathcal{P}$. Then*

$$\{\phi(\mathrm{x}) : \mathrm{x} \in \overleftrightarrow{\mathrm{a,b}}\} = \overleftrightarrow{\phi(\mathrm{a}),\phi(\mathrm{b})} \, . \tag{1}$$

Proof: Let x be any element of $\overleftrightarrow{\mathrm{a,b}}$. By definition 1.2.1

$$(\forall\, \theta\in\mathcal{P}:\ \theta(\mathrm{a}) = \mathrm{b} \ \text{ and } \ \theta(\mathrm{b}) = \mathrm{a})(\exists\, \mathrm{y}\in(\mathrm{P}\triangle\{\mathrm{x}\})) \qquad \{\mathrm{x,y}\} = \{\theta(\mathrm{x}), \theta(\mathrm{y})\} \, . \tag{2}$$

Let ρ be any element of $\mathcal{P}$ which interchanges $\phi(a)$ and $\phi(b)$. Then $\phi^{-1}\circ\rho\circ\phi$ interchanges a and b. It follows from 2 that there exists $y \in (P\triangle\{x\})$ such that

$$\{\mathrm{x,y}\} = \{\phi^{-1}\circ\rho\circ\phi(\mathrm{x}), \phi^{-1}\circ\rho\circ\phi(\mathrm{y})\} \, .$$

and so

$$\{\phi(\mathrm{x}), \phi(\mathrm{y})\} = \{\rho(\phi(\mathrm{x})), \rho(\phi(\mathrm{y}))\} \, .$$

Consequently $\phi(x)$ is in $\overleftrightarrow{\phi(\mathrm{a}),\phi(\mathrm{b})}$. We have shown that

$$\{\phi(\mathrm{x}) : \mathrm{x} \in \overleftrightarrow{\mathrm{a,b}}\} \subset \overleftrightarrow{\phi(\mathrm{a}),\phi(\mathrm{b})} \, . \tag{3}$$

Let u be any element of $\overleftrightarrow{\phi(\mathrm{a}),\phi(\mathrm{b})}$. By the definition 1.2.1 there exists v in P such that, for each θ which interchanges $\phi(a)$ and $\phi(b)$,

$$\{u,v\} = \{\theta(u), \theta(v)\}\,. \tag{4}$$

Let γ be any element of $\mathcal{P}$ which interchanges a and b. We have

$$\phi\circ\gamma\circ\phi^{-1}(\phi(a)) = \phi(b) \quad \text{and} \quad \phi\circ\gamma\circ\phi^{-1}(\phi(b)) = \phi(a)$$

and so it follows from 4 that there exists v in P such that

$$\{u,v\} = \{\phi\circ\gamma\circ\phi^{-1}(u), \phi\circ\gamma\circ\phi^{-1}(v)\}\,. \tag{5}$$

Let z be the element of P such that $\phi(z) = v$ and let w be the element of P such that $\phi(w) = u$.. We have

$$\{w,z\} = \{\phi^{-1}(u), \phi^{-1}(v)\} \overset{\text{by } 5}{=\!=} \{\phi^{-1}(\phi\circ\gamma\circ\phi^{-1}(u)), \phi^{-1}(\phi\circ\gamma\circ\phi^{-1}(v))\} = \{\gamma(w), \gamma(z)\}\,.$$

We have shown that the inequality reverse to that in 3 holds, which establishes 1. Q.E.D.

We now look at the structure of a projective space and how it may be built from a fundamental set.

(1.6) Definitions and Notation The **facade generated by an independent subset** S of a projective space is the union of the spans of the proper subsets of S. The span of a maximal proper subset of S will be said to be a **face of the facade**. The S-**interior of the span of** S is the complement in $\boxed{S}$ of the facade of S. We shall denote it by[5]

$$\overset{\circ}{S}\,. \tag{1}$$

As a simple but important example, we note that for any doubleton {a,b} with P,

$$\overset{\circ}{\{a,b\}} = (\overleftrightarrow{a,b} \triangle \{a,b\})\,. \tag{2}$$

(1.7) Corollary I *Let ϕ be an element of the projective group of permutations on a projective space. Then ϕ preserves coaligned sets, spans of sets, independent sets, dependent sets, facades, faces, fundamental sets and S-interiors.*

Proof: Follows directly from Theorem I (1.5).

(1.8) Theorem II *Let $\mathcal{P}$ be a projective group of permutations on a projective space P. Let I be an independent subset of P of cardinality exceeding one. Then*

$$(\forall\, p \in (P \triangle \boxed{I}))\quad \{p\} \cup I \text{ is independent.} \tag{1}$$

Proof: Assume that 1 did not hold for some independent set I. Then there were an element p of $(P \triangle \boxed{I})$ such that $\{p\} \cup I$ were dependent. The element p not being in $\boxed{I}$, there would exist q in I such that

$$q \in \boxed{\{p\} \cup (I \triangle \{q\})}\,. \tag{2}$$

The element p is evidently a fundamental member of the dependent set $(\{p\} \cup I)$. It follows from the *fundamental axiom* (1.3.2) that there would exist ϕ in $\mathcal{F}$ such that $\phi(q) = p$, $\phi(p) = q$ and $\phi(x) = x$ for all x in $I \triangle \{q\}$. We would have

$$\{\phi(x) : x \in \boxed{\{p\} \cup (I \triangle \{q\})}\} \overset{\text{by (1.7)}}{=\!=} \boxed{\{\phi(p)\} \cup (I \triangle \{q\})} = \boxed{I}\,. \tag{3}$$

From 2 we know that $\phi(q)$ would be in the set on the left side of 3. But $\phi(q)$ is just p, and p could not be in the right side of 3: an absurdity. Q.E.D.

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[5] We emphasize that the term S-interior $\overset{\circ}{S}$ as defined here is dependent not only on the span $\boxed{S}$, but also on the set S from which it is derived. Furthermore it is **not** a subset of S — it has void intersection with S.

(1.9) Theorem III *Let* F *be a fundamental subset of a projective space* P. *Then*

$$\textit{every element of } \mathrm{F} \textit{ is a fundamental representative of } F \tag{1}$$

and

$$\textit{each maximal proper subset of } F \textit{ is independent.} \tag{2}$$

Proof*:* Let c be a fundamental representative of F and let a be any other element of F. It follows from the *fundamental axiom* (1.3.2) that there exists $\phi \in \mathcal{P}$ such that $\phi(\mathrm{a}) = \mathrm{c}$, $\phi(\mathrm{c}) = \mathrm{a}$ and $\phi(\mathrm{x}) = \mathrm{x}$ for all x in $(\mathrm{F} \triangle \{\mathrm{a},\mathrm{c}\})$. If there were an element b of F such that a were in $(\mathrm{F} \triangle \{\mathrm{a},\mathrm{b}\})$, it follows from (1.7) that c would be in $(\mathrm{F} \triangle \{\mathrm{c},\phi(\mathrm{b})\})$, with $\phi(\mathrm{b})$ also being an element of F. That would be absurd, which implies that c must be a fundamental representative of F. This establishes 1.

Let G is any maximal proper subset of F. If G were dependent, then there would be an element b of G such that b were in $\boxed{\mathrm{G} \triangle \{\mathrm{b}\}}$. If s is the element of the singleton $(\mathrm{F} \triangle \mathrm{G})$, then b would be in $(\mathrm{F} \triangle \{\mathrm{s},\mathrm{b}\})$, whence would follow that b could not be a representative element of F. It would follow from (1) that F could not be fundamental: an absurdity. Q.E.D.

(1.10) Lemma *Let* I *be a independent subset of a projective space such that the cardinality of* I *is at least* 2. *Then*

$$\overset{\square}{\mathrm{I}} \textit{ is non-void.} \tag{1}$$

Furthermore, if the cardinality of I *is at least* 3, a *is an element of* I *and* c *is any point of* $(\boxed{\mathrm{I}} \triangle \{\mathrm{a}\})$, *then*

$$\left(\overleftrightarrow{\mathrm{a},\mathrm{c}} \cap \boxed{\mathrm{I} \triangle \{\mathrm{a}\}} \right) \textit{ is non-void.} \tag{2}$$

Proof: If the cardinality of I is 2, then 1 holds as a result of the *alignment axiom* (1.3.1). We shall proceed by induction and suppose that n is a positive integer at least 3 and that 1 and 2 have been verified for all independent sets of cardinality less than n. It follows from our induction hypothesis that there exists an element e of $\overset{\square}{(\mathrm{I} \triangle \{\mathrm{a}\})}$. Let d be any element of $\overset{\square}{\{\mathrm{a},\mathrm{e}\}}$. Evidently $(\mathrm{I} \cup \{\mathrm{d}\})$ is dependent. Assume that d were not a fundamental element of $(\mathrm{I} \cup \{\mathrm{d}\})$. Then there would exist an element b of I such that d would be in $\boxed{(\mathrm{I} \cup \{\mathrm{d}\}) \triangle \{\mathrm{d},\mathrm{b}\}}$. If b were a, then d would be in $\boxed{\mathrm{I} \triangle \mathrm{a}}$ and since e is in $\boxed{\mathrm{I} \triangle \mathrm{a}}$, a would be as well, which would be absurd. If follows that d would be in the span of a proper subset of $(\mathrm{I} \triangle \{\mathrm{a}\})$. Since e is in $\boxed{\mathrm{I} \triangle \{\mathrm{a}\}}$, it would follow that a would be as well, which would violate the independence of I. It follows that

$$\mathrm{d} \text{ is a fundamental element of } (\mathrm{I} \cup \{\mathrm{d}\}). \tag{3}$$

This also establishes (2) because if d were not in the I-interior of the span of I, then it could not be a fundamental element of $(\mathrm{I} \cup \{\mathrm{d}\})$.

Let c be any element of $(\boxed{\mathrm{I}} \triangle \{\mathrm{a}\})$. Evidently $(\mathrm{I} \cup \{\mathrm{c}\})$ is a dependent set. If (2) did not hold for c, we assert that

$$\mathrm{c} \text{ would be a fundamental element of } (\mathrm{I} \cup \{\mathrm{c}\}). \tag{4}$$

If not, there would exist an element b of I distinct from a such that c would be in $\boxed{(\mathrm{I} \cup \{\mathrm{c}\}) \triangle \{\mathrm{c},\mathrm{b}\}} = \boxed{\mathrm{I} \triangle \{\mathrm{b}\}}$. But then by the induction hypothesis, $\overleftrightarrow{\mathrm{a},\mathrm{c}}$ would have to intersect $\boxed{\mathrm{I} \triangle \{\mathrm{b}\}}$: an absurdity. Thus, both (3) and (4) would hold and so it follows from the *fundamental axiom* (1.3.2) that there would exist a permutation ϕ in $\mathcal{P}$ such that $\phi(\mathrm{d}) = \mathrm{c}$ and $\phi(\mathrm{x}) = \mathrm{x}$ for all x in I. Since e is in $\boxed{\mathrm{I} \triangle \{\mathrm{a}\}}$, it follows from (1.7) that $\phi(\mathrm{e})$ would be in the same set. But $\phi(\mathrm{e})$ would also be in $\overleftrightarrow{\phi(\mathrm{a}),\phi(\mathrm{d})} = \overleftrightarrow{\mathrm{a},\mathrm{c}}$: that is, $\overleftrightarrow{\mathrm{a},\mathrm{c}}$ would intersect $\boxed{\mathrm{I} \triangle \{\mathrm{a}\}}$ at $\phi(\mathrm{e})$. That would be absurd, whence follows that (2) holds. Q.E.D.

(1.11) Theorem IV *Let* $\mathcal{P}$ *be a projective group of permutations on a projective space* P. *Let* I *be an independent subset of* P *of cardinality exceeding one. Then*

$$(\forall\, a \in I) \quad I^{\square} = \bigcup_{e \in (I \triangle \{a\})^{\circ}} \{a,e\}^{\square}. \tag{1}$$

Furthermore, the union given on the right-hand side of 1 *is a partition of* $I^{\square}$.

Proof: Let a be an element of I. To establish 1 we shall first show that

$$\bigcup_{e \in (I \triangle \{a\})^{\circ}} \{a,e\}^{\square} \subset \; I^{\square}. \tag{2}$$

and then that

$$I^{\square} \; \subset \bigcup_{e \in (I \triangle \{a\})^{\circ}} \{a,e\}^{\square}. \tag{3}$$

Let e be any element of $(I \triangle \{a\})^{\square}$. It follows from the second half of the **proof** of Lemma 1.10 that any point d of $\{a,e\}^{\square}$ is in $I^{\square}$. This establishes (2).

Let x be any element of $I^{\square}$. It follows from (1.10.2) that $\overleftrightarrow{a,x}$ intersects $\boxed{I \triangle \{a\}}$ at some element y. If y were not in $(I \triangle \{a\})^{\square}$, then y would be the span of a proper subset of $(I \triangle \{a\})$ and so x would be in the span of a proper subset of I. That would be absurd. This establishes (3). Q.E.D.

(1.12) Corollary II *Let $\mathcal{P}$ be a projective group of permutations on a projective space* P. *Let* I *be an independent subset of* P *of cardinality exceeding one. Then*

$$(\forall\, a \in I) \qquad \boxed{I} = \bigcup_{c \in \boxed{I \triangle \{a\}}} \overleftrightarrow{a,c}\,. \tag{1}$$

(1.13) Corollary III *Let* C *be a maximal proper coaligned subset of a non-trivial projective space* P. *Then a line within* P *is either entirely contained within* C *or it intersects* C *in exactly one point.*

Proof: Suppose that L were a line within P disjoint from C, and let {a,b} be a doubleton within L. If I is any maximal independent subset of C, then Theorem II would imply that $\{a\} \cup I$ were independent, and so $I \cup \{a\}$ would be a maximal independent subset of P: in particular, $\boxed{I \cup \{a\}} = P$. It would follow from Corollary II that b would be on a line containing a and an element of $\boxed{I}$: an absurdity. Q.E.D.

Section 2. Meridians

(2.1) Definitions and Notation Let M be a meridian with projective group $\mathcal{M}$. According to their definitions, M and $\mathcal{M}$ are characterized by the following properties:

$$\text{M has at least three elements;} \tag{1}$$

for all ϕ in $\mathcal{M}$ which interchanges the elements of some doubleton within M,

$$(\forall\, x\in M)(\exists\, y\in M\triangle\{x\}) \quad \{x,y\} = \{\phi(x),\phi(y)\}\,; \tag{2}$$

and, for each pair {a,b,c} and {x,y,z} of tripletons of M,

(*Fundamental Property*)
$$(\exists!\, \phi\in\mathcal{M}) \quad \phi(a) = x\,, \quad \phi(b) = y \quad \text{and} \quad \phi(c) = z\,. \tag{3}$$

These three are explained as follows: M is a projective space which equals the span of just two of its elements. Thus (1) and (2) together are just an expression of the *alignment axiom* (1.3.1) above. Since a fundamental subset of a 1-dimensional projective space has precisely three elements, (3) is an expression of *fundamental axiom* (1.3.2).

The equality of (2) has two separate cases: when x is just $\phi(x)$ — and when it isn't. The second case can be reformulated as:

(*Involution Property*)
$$\text{if } \phi\in\mathcal{M} \text{ interchanges two elements of M, then } \phi = \phi^{-1} \tag{4}$$

whereas the first case resolves into

(*Fixed Point Property*)
$$\text{an involution in } \mathcal{M} \text{ either has no fixed points or two.} \tag{5}$$

We use the term **involution** here for a self-inverse function which is not equal to the identity function. It is a consequence of the *fundamental property* (2.1.3) that if an involution in $\mathcal{M}$ has two fixed points, it has precisely two, for otherwise it would be the identity function.

The function ϕ of (3) often will be written as

$$\boxed{\begin{smallmatrix} x & y & z \\ a & b & c \end{smallmatrix}}\,. \tag{6}$$

(2.2) Fixed Point Theorem *Let* {a,b} *be a doubleton within a meridian* M *with meridian group* $\mathcal{M}$. *Then*

$$(\exists!\, \phi\in\mathcal{M}) \quad \phi^{-1} = \phi\,, \quad \phi(a) = a \quad \textit{and} \quad \phi(b) = b\,. \tag{1}$$

Proof: Let c be an element of M distinct from a and b. It follows from the *fixed point property* (2.1.4) that the permutation $\boxed{\begin{smallmatrix} a & b & c \\ b & a & c \end{smallmatrix}}$ has a fixed point d distinct from c. Direct calculation shows that $\boxed{\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}}\circ\boxed{\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}}\circ\boxed{\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}}$ leaves a and b fixed. Since $\boxed{\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}}$ interchanges a and c, it follows from the *involution property* (2.1.5) that it is self-inverse. Since $\boxed{\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}}$ interchanges a and b, it too is self-inverse. It follows that $\boxed{\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}}\circ\boxed{\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}}\circ\boxed{\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}}$ is self-inverse, and so the "existence part" of (1) has been established.

Suppose that ϕ is any element of $\mathcal{P}$ which is self-inverse and which fixes both a and b. Let e denote $\phi(c)$: then $\phi(e) = c$. Direct calculation shows that $\boxed{\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}}\circ\phi\circ\boxed{\begin{smallmatrix} c & a & e \\ a & c & b \end{smallmatrix}}$ fixes c and interchanges a and b. It follows from the *fundamental property* (2.1.3) that it must be identical with $\boxed{\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}}$, which in turn implies that $\phi = \boxed{\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}}\circ\boxed{\begin{smallmatrix} b & a & c \\ a & b & c \end{smallmatrix}}\circ\boxed{\begin{smallmatrix} c & a & d \\ a & c & b \end{smallmatrix}}$. This establishes the "uniqueness part" of (1). Q.E.D.

(2.3) Notation In view of the uniqueness part of the *fixed point theorem* (2.2), an involution is determined by its values at any two elements of M which are not interchanged by that involution. In the sequel, notation such as

$$\boxed{\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}} \tag{1}$$

will signify that the function in question is the involution sending a to c and b to d. In particular, the involution fixing two points a and b of M will be denoted by

$$\boxed{\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}}\,. \tag{2}$$

(2.4) Harmonic Pairs Let {a,b,c,d} be a quadrupleton within a meridian. We shall say that {{a,c},{b,d}} is a **harmonic pair** if

$$\boxed{\begin{smallmatrix} b & c & d \\ a & b & c \end{smallmatrix}}(d) = a\,. \tag{1}$$

If {{a,c},{b,d}} is a harmonic pair, then $\boxed{\begin{smallmatrix} b & d \\ a & c \end{smallmatrix}}\circ\boxed{\begin{smallmatrix} b & c & d \\ a & b & c \end{smallmatrix}}$ interchanges b and d and keeps a and c fixed. From the *involution property* (2.1.5) we know that it is an involution. Conversely, if ϕ is an involution fixing distinct elements a and c of M and interchanging elements b and d, then $\boxed{\begin{smallmatrix} b & d \\ a & c \end{smallmatrix}}\circ\phi$ equals $\boxed{\begin{smallmatrix} b & c & d \\ a & b & c \end{smallmatrix}}$ and (1) holds. *It follows that {{a,c},{b,d}} is a harmonic pair if, and only if,*

$$(\exists\, \phi\in\mathcal{M}) \quad \phi(a) = a\,, \quad \phi(c) = c\,, \quad \phi(b) = d \quad \textit{and} \quad \phi(d) = b\,. \tag{2}$$

Thus, *if {{a,c},{b,d}} is a harmonic pair, then*

$$\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}} = \boxed{\begin{smallmatrix} a & c \\ a & c \end{smallmatrix}}\,. \tag{3}$$

Given any tripleton {a,b,c} in M, the involution $\boxed{\begin{smallmatrix} b & c & a \\ b & a & c \end{smallmatrix}}$ has exactly one fixed point d other than b. *It follows that*

$$(\forall\, \{a,b,c\} \textit{ a tripleton in } M)(\exists!\, d\in M) \quad \{\{a,c\},\{b,d\}\} \quad \textit{is a harmonic pair.} \tag{4}$$

(2.5) Translation Theorem translation theorem *A* **translation** *of a meridian is an element of the meridian group having a single fixed point.*

If α and β are distinct involutions fixing a common point p, *then*

$$\alpha\circ\beta \quad \textit{is a translation fixing } p\,. \tag{1}$$

Conversely, if τ is a translation which fixes a point a *of* M, *and if* $c = \tau(b)$ *for* $\{b,c\}\subset(M\triangle\{a\})$, *then*

$$\boxed{\begin{smallmatrix} a & c \\ a & c \end{smallmatrix}}\circ\boxed{\begin{smallmatrix} a & c & b \\ a & b & c \end{smallmatrix}} = \tau\,. \tag{2}$$

Proof: We prove (1) first. If $\alpha\circ\beta$ were not a translation, then there would be a point q distinct from p at which α and β would agree, but this would violate the *fixed point theorem* (2.2) since α and β are distinct.

We turn to (2). Let τ be a translation with fixed point a, let b be any point of M distinct from a, let c be $\tau(b)$ and let d be $\tau(c)$. The permutation $\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}\circ\tau$ interchanges b and c, and so by the *involution property* (2.1.5) is an involution. Consequently

$$a = (\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}\circ\tau)\circ(\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}\circ\tau)(a) = \boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}\circ\tau\circ\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}(a)$$

whence follows that

$$\tau(\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}(a)) = \boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}(a)\,. \tag{3}$$

If $\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}$ did not fix a, then the translation τ would have more than one fixed point, which were absurd. Since $\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}$ fixes a, we know from (2.4.2) that {{a,c},{b,d}} is a harmonic pair. It follows from (2.4.3) that

$$\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}} = \boxed{\begin{smallmatrix} a & c \\ a & c \end{smallmatrix}} \,. \tag{4}$$

The permutations $\boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}\circ\tau$ and $\boxed{\begin{smallmatrix} a & c & b \\ a & b & c \end{smallmatrix}}$ agree at a , b and c and so, by the *fundamental property* (2.1.3) must be identical. We have

$$\boxed{\begin{smallmatrix} a & c & b \\ a & b & c \end{smallmatrix}} = \boxed{\begin{smallmatrix} c & d & b \\ c & b & d \end{smallmatrix}}\circ\tau \overset{\text{by (4)}}{=\!=\!=} \boxed{\begin{smallmatrix} a & c \\ a & c \end{smallmatrix}}\circ\tau$$

whence follows that (2) holds. Q.E.D.

(2.6) Libras Latent within a meridian are numerous structures which we shall call here "libras". By a **function libra** $\mathcal{L}$ we shall mean a family of bijections from one set A , called the **common domain**, onto another set B , called the **common range**, such that

$$(\forall\, \{\alpha,\beta,\gamma\} \subset \mathcal{L}) \quad \alpha\circ\beta^{-1}\circ\gamma \in \mathcal{L}\,. \tag{1}$$

It is sometimes useful to adopt the following notation:

$$(\forall\, \{\alpha,\beta,\gamma\} \subset \mathcal{L}) \quad \llbracket \alpha,\beta,\gamma \rrbracket \equiv \alpha\circ\beta^{-1}\circ\gamma\,. \tag{2}$$

An **abstract libra** is a set L with ternary operation $L \ni [x,y,z] \hookrightarrow \lfloor x,y,z \rfloor \in L$ which, for all $\{v,w,x,y,z\} \subset L$ satisfies the equalities

$$\lfloor v,v,w \rfloor = \lfloor w,v,v \rfloor = w \tag{3}$$

and

$$\lfloor \lfloor w,v,x \rfloor ,y,z \rfloor = \lfloor w,v,\lfloor x,y,z \rfloor \rfloor\,. \tag{4}$$

It follows from (3) and (4) that the equality

$$\lfloor v,\lfloor w,x,y \rfloor ,z \rfloor = \lfloor \lfloor v,y,x \rfloor ,w,z \rfloor \tag{5}$$

holds as well, for

$$v \overset{\text{by (3)}}{=\!=\!=} \lfloor v,w,w \rfloor \overset{\text{by (3)}}{=\!=\!=} \lfloor v,w,\lfloor x,x,w \rfloor \rfloor \overset{\text{by (3)}}{=\!=\!=} \lfloor v,w,\lfloor \lfloor x,y,y \rfloor ,x,w \rfloor \rfloor \overset{\text{by (4)}}{=\!=\!=}$$
$$\lfloor \lfloor v,w,\lfloor x,y,y \rfloor \rfloor ,x,w \rfloor \overset{\text{by (4)}}{=\!=\!=} \lfloor \lfloor \lfloor v,w,x \rfloor ,y,y \rfloor ,x,w \rfloor \overset{\text{by (4)}}{=\!=\!=} \lfloor \lfloor v,w,x \rfloor ,y,\lfloor y,x,w \rfloor \rfloor$$

whence follows that

$$\lfloor v,\lfloor y,x,w \rfloor ,z \rfloor = \lfloor \lfloor \lfloor v,w,x \rfloor ,y,\lfloor y,x,w \rfloor \rfloor ,\lfloor y,x,w \rfloor ,z \rfloor \overset{\text{by (4)}}{=\!=\!=}$$
$$\lfloor \lfloor v,w,x \rfloor ,y,\lfloor \lfloor y,x,w \rfloor ,\lfloor y,x,w \rfloor ,z \rfloor \rfloor \overset{\text{by (3)}}{=\!=\!=} \lfloor \lfloor v,w,x \rfloor ,y,z \rfloor\,.$$

A bijection from one libra onto another is a **libra isomorphism** if it preserves the ternary libra operator. A libra is said to be **abelian** provided that

$$(\forall\, \{x,y,z\} \subset L) \quad \lfloor x,y,z \rfloor = \lfloor z,y,x \rfloor . \tag{6}$$

A group G with binary operation $\cdot$ is a libra relative to the ternary operation $G \ni (x,y,z) \hookrightarrow (x\cdot y^{-1}\cdot z) \in G$. Conversely, a libra L , relative to any element e of L, is a group relative to the operation

$$(\forall\, \{x,y\} \subset L) \quad x\cdot y \equiv \lfloor x,e,y \rfloor\,. \tag{7}$$

In this case we speak of the **libra group with identity** e .

Evidently a libra is abelian if, and only if, any one of the groups associated with it is abelian. In this case, all the groups associated with the libra are abelian.

(2.7) Definition Let M be a meridian relative to a meridian group $\mathcal{M}$, and let a and b be elements of M. We define[6]

$$\mathcal{M}_{[a,b]} \equiv \{\phi \in \mathcal{M} : \ \ \phi \text{ is an involution}, \ \ \phi(a) = b \ \text{ and } \ \phi(b) = a\}\,. \tag{1}$$

We note, in view of the *involution property* (2.1.5), that the requirement in (1) that ϕ be an involution is superfluous if $a \neq b$.

(2.8) Theorem II *Let* M *be a meridian relative to a meridian group* $\mathcal{M}$, *and let* a *and* b *be elements of* M. *Then* $\mathcal{M}_{[a,b]}$ *is an abelian libra.*

Proof: We need first to show that the libra operation of (2.6.2) returns an involution, for it evidently interchanges a with b. For the case $a \neq b$, this follows directly from the *involution property* (2.1.5).

We take up the case in which a and b are identical. Let $\{\alpha,\beta,\gamma\}$ be within $\mathcal{M}_{[a,a]}$, and define δ to be $[\![\alpha,\beta,\gamma]\!]$. Since all three arguments are involutions, we have

$$\delta = \alpha\circ\beta\circ\gamma\,.$$

It follows from (2.5.1) that $\beta\circ\gamma$ is a translation. From the *fixed point property* (2.1.4) we know that α has another fixed point c distinct from a. Let b be the element of M such that $\beta\circ\gamma(b) = c$. By (2.5.2), we know that $\boxed{\substack{a\,c\\a\,c}}\circ\boxed{\substack{a\,c\,b\\a\,b\,c}} = \beta\circ\gamma$. But $\boxed{\substack{a\,c\\a\,c}}$ is just α. We have

$$\alpha\circ\delta = \beta\circ\gamma = \boxed{\substack{a\,c\\a\,c}}\circ\boxed{\substack{a\,c\,b\\a\,b\,c}} = \alpha\circ\boxed{\substack{a\,c\,b\\a\,b\,c}}\,,$$

whence follows that $\delta = \boxed{\substack{a\,c\,b\\a\,b\,c}}$, which by the *involution property* (2.1.5) is an involution.

Finally we show that the libra is abelian. Let $\{\alpha,\beta,\gamma\}$ be within $\mathcal{M}_{[a,b]}$. We have

$$[\![\alpha,\beta,\gamma]\!] = \alpha\circ\beta\circ\gamma = (\alpha\circ\beta\circ\gamma)^{-1} = \gamma^{-1}\circ\beta^{-1}\circ\alpha^{-1} = \gamma\circ\beta\circ\alpha = [\![\gamma,\beta,\alpha]\!]\,.$$

Q.E.D.

(2.9) Theorem III *Let* $\mathcal{L}$ *be a function libra of self-inverse permutations of a set* L *such that*

$$(\forall\,\{u,v\}\subset L)(\exists!\,\phi\in\mathcal{L}) \quad \phi(u) = v\,. \tag{1}$$

For $\{u,v\}\subset L$, *we shall denote the element* ϕ *of* (1) *by*

$$\boxed{\substack{v\\u}}\,.$$

Let $\mathcal{T}$ *be the family of all* $\alpha\circ\beta$ *such that* $\{\alpha,\beta\}\subset\mathcal{L}$. *Then*

$$(\forall\,\{\rho,\sigma\}\subset\mathcal{T}) \quad \rho = \sigma \ \text{ if they agree at any point}, \tag{2}$$

$$(\forall\,\{u,v\}\subset L)(\exists!\,\theta\in\mathcal{T}) \quad \theta(u) = v \tag{3}$$

and

$$\mathcal{T} \ \text{ is an abelian group.} \tag{4}$$

For $\{u,v\}\subset L$, *we shall denote the element* θ *of* (3) *by*

$$\boxed{\boxed{\substack{v\\u}}}\,.$$

For $\{x,y,z\}\subset L$ *we define*

$$\lfloor x,y,z \rfloor \equiv \boxed{\substack{z\\x}}(y)\,. \tag{5}$$

Then L *is a libra and*

$$(\forall\, e\in L) \qquad \text{the function } \ \mathcal{L}\ni\theta\hookrightarrow\theta(e)\in L \ \ \text{is a libra isomorphism.} \tag{6}$$

Proof: If $\{\alpha,\beta,\gamma,\delta\}$ is within $\mathcal{L}$ and $\alpha\circ\beta(s) = \gamma\circ\delta(s)$ for some $s\in L$, then $\gamma\circ\alpha\circ\beta(s) = \delta(s)$ and so (1) implies that $\gamma\circ\alpha\circ\beta = \delta$. It follows that $\alpha\circ\beta = \gamma\circ\delta$, which proves (2).

[6] The use of these meridian libras to form fields was first exploited by Tits in [4], so far as we know.

For $\{u,v\}$ within L, $\boxed{\substack{v\\u}}\circ\boxed{\substack{u\\u}}(u)=v$ and, in view of (2), $\boxed{\substack{v\\u}}\circ\boxed{\substack{u\\u}}$ is the only element of $\mathcal{T}$ which does this. This establishes (3).

Let θ,τ be within $\mathcal{T}$. Then there exists $\{\alpha,\beta,\gamma,\delta\}$ within $\mathcal{L}$ such that $\theta=\alpha\circ\beta$ and $\tau=\gamma\circ\delta$. We have

$$\theta\circ\tau=\alpha\circ\beta\circ\gamma\circ\delta=(\alpha\circ\beta\circ\gamma)^{-1}\circ\delta=\gamma\circ\beta\circ\alpha\circ\delta=\gamma\circ(\beta\circ\alpha\circ\delta)^{-1}=\gamma\circ\delta\circ\alpha\circ\gamma=\tau\circ\theta$$

which proves (4).

For any u and v in L we have

$$\lfloor u,u,v\rfloor \stackrel{\text{by }(5)}{=\!=} \boxed{\substack{u\\v}}(u) \stackrel{\text{by }(1)}{=\!=} v \quad\text{and}\quad \lfloor u,v,v\rfloor \stackrel{\text{by }(5)}{=\!=} \boxed{\substack{u\\v}}(v) \stackrel{\text{by }(1)}{=\!=} u$$

which establishes (2.6.3).

Let $\{a,b,c,d,e\}$ be within L. We have

$$\boxed{\substack{c\\a}}\circ\boxed{\substack{c\\b}}(b)=a \quad\text{and}\quad \boxed{\substack{e\\c}}\circ\boxed{\substack{d\\c}}(d)=e$$

which, by (2) implies that

$$\boxed{\substack{c\\a}}\circ\boxed{\substack{c\\b}}=\boxed{\boxed{\substack{a\\b}}} \quad\text{and}\quad \boxed{\substack{e\\c}}\circ\boxed{\substack{d\\c}}=\boxed{\boxed{\substack{e\\d}}}. \tag{7}$$

Letting $u\equiv\boxed{\boxed{\substack{a\\b}}}(c)$ and $v\equiv\boxed{\boxed{\substack{e\\d}}}(c)$, we have

$$\lfloor\lfloor a,b,c\rfloor,d,e\rfloor \stackrel{\text{by }(5)}{=\!=} \lfloor\boxed{\substack{c\\a}}(b),d,e\rfloor = \lfloor\boxed{\substack{c\\a}}\circ\boxed{\substack{c\\b}}(c),d,e\rfloor \stackrel{\text{by }(7)}{=\!=} \lfloor\boxed{\boxed{\substack{a\\b}}}(c),d,e\rfloor = \lfloor u,d,e\rfloor \stackrel{\text{by }(5)}{=\!=}$$

$$\boxed{\substack{e\\u}}(d) = \boxed{\substack{e\\u}}\circ\boxed{\substack{d\\u}}(u) \stackrel{\text{by }(7)}{=\!=} \boxed{\boxed{\substack{e\\d}}}(u) = \boxed{\boxed{\substack{e\\d}}}\circ\boxed{\boxed{\substack{a\\b}}}(c) \stackrel{\text{by }(4)}{=\!=} \boxed{\boxed{\substack{a\\b}}}\circ\boxed{\boxed{\substack{e\\d}}}(c) = \boxed{\boxed{\substack{a\\b}}}(v) \stackrel{\text{by }(2)}{=\!=} \boxed{\substack{v\\a}}\circ\boxed{\substack{v\\b}}(v) =$$

$$\boxed{\substack{v\\a}}(b) \stackrel{\text{by }(5)}{=\!=} \lfloor a,b,v\rfloor = \lfloor a,b,\boxed{\boxed{\substack{e\\d}}}(c)\rfloor \stackrel{\text{by }(7)}{=\!=} \lfloor a,b,\boxed{\substack{e\\c}}\circ\boxed{\substack{d\\c}}(c)\rfloor = \lfloor a,b,\boxed{\substack{e\\c}}(d)\rfloor \stackrel{\text{by }(5)}{=\!=} \lfloor a,b,\lfloor c,d,e\rfloor\rfloor$$

which establishes (2.6.4).

Finally, we prove (6). Let x be any element of L and let $\{\alpha,\beta,\gamma\}$ be a subset of $\mathcal{L}$. We set $c\equiv\gamma(x)$, $b\equiv\beta(c)$ and $a\equiv\alpha(b)$. Consequently $\alpha=\boxed{\substack{b\\a}}$, $\beta=\boxed{\substack{c\\b}}$ and $\gamma=\boxed{\substack{x\\c}}$. We have

$$[\![\alpha,\beta,\gamma]\!](x) \stackrel{\text{by }(2.6.2)}{=\!=} \alpha\circ\beta\circ\gamma(x) = \boxed{\substack{b\\a}}\circ\boxed{\substack{c\\b}}\circ\boxed{\substack{x\\c}}(x) = a \stackrel{\text{by }(2.6.3)}{=\!=} \lfloor a,x,\lfloor b,b,x\rfloor\rfloor \stackrel{\text{by }(2.6.4)}{=\!=} \lfloor\lfloor a,x,b\rfloor,b,x\rfloor$$

$$\stackrel{\text{by }(2.6.3)}{=\!=} \lfloor\lfloor\lfloor a,x,b\rfloor,b,x\rfloor,c,\lfloor c,x,x\rfloor\rfloor \stackrel{\text{by }(2.6.5)}{=\!=} \lfloor\lfloor a,x,b\rfloor,\lfloor c,x,b\rfloor,\lfloor c,x,x\rfloor\rfloor = \lfloor\alpha(x),\beta(x),\gamma(x)\rfloor.$$

Q.E.D.

(2.10) Corollary *Let $\mathcal{M}$ be a meridian group and let* M *be the corresponding meridian. For* $\{a,b\}$ *within* M *we adopt the notation*

$$M_{[a,b]} \equiv M\triangle\{a,b\}. \tag{1}$$

For $\{x,y,z\}$ *within* $M_{[a,b]}$ *we define*

$$\lfloor x,y,z\rfloor \equiv \boxed{\substack{z\ a\\x\ b}}(y). \tag{2}$$

Then $M_{[a,b]}$ *is a libra relative to* (2) *and, for each* $u\in M_{[a,b]}$,

$$\textit{the function}\quad \mathcal{M}_{[a,b]}\ni\phi\hookrightarrow\phi(u)\in M_{[a,b]}\quad \textit{is a libra isomorphism from } \mathcal{M}_{[a,b]} \textit{ onto } M_{[a,b]}. \tag{3}$$

Proof: Letting $\mathcal{L}\equiv\mathcal{M}_{[a,b]}$ and $L\equiv M_{[a,b]}$, this follows from Theorem (2.8) and Theorem (2.9). Q.E.D.

(2.11) Dilations Let $\{a,b\}$ be a subset of a meridian M, let $\mathcal{L}$ be the libra $M_{[a,b]}$ of (2.10) and let

$\mathcal{T}$ be as in (2.9). If a and b are identical, it follows from the *translation theorem* ((2.5)) that the elements of $\mathcal{T}$ are just the translations which fix a.

If a and b are distinct, we say that the elements of $\mathcal{T}$ are the **dilations with fixed points** a **and** b. Every non-identity permutation in $\mathcal{M}$ which fixes a and b is such a dilation,

(2.12) Meridian Fields Let {o,l,∞} be a tripleton within a meridian M with meridian group $\mathcal{M}$ of permutations. For {x,y} within $M_{[\infty,\infty]}$ we define

$$x{+}y \equiv \boxed{\begin{smallmatrix}\infty & y\\ \infty & x\end{smallmatrix}}(o)\,. \tag{1}$$

For {x,y} within $M_{[o,\infty]}$ we define

$$x\cdot y \equiv \boxed{\begin{smallmatrix}\infty & y\\ o & x\end{smallmatrix}}(l) \tag{2}$$

and, for all z in $M_{[\infty,\infty]}$ we define

$$o\cdot z \equiv z\cdot o \equiv o\,. \tag{3}$$

We shall say that $M_{[\infty,\infty]}$ is the **field generated by** [∞,o,1] and that + and · are the **field operators**.

(2.13) Theorem IV *Relative to* + *and* ·, $M_{[\infty,\infty]}$ *is a field of characteristic different than 2. Furthermore, the additive and multiplicative inverses of elements* $x\in M_{[\infty,\infty]}$ *and* $y\in M_{[o,\infty]}$, *respectively, are given by*

$$-x \equiv \boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}(x) \quad \textit{and} \quad y^{-1} \equiv \boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}(y)\,, \quad \textit{respectively.} \tag{1}$$

Proof: Everything but the distributive law follows directly from Corollary (2.10) and the fact that we are dealing with groups associated with the libras $M_{[\infty,\infty]}$ and $M_{[o,\infty]}$.

Let {x,y,z} be within $M_{[\infty,\infty]}$ such that x and y+z are distinct from o. We define

$$\theta \equiv \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & z\\ \infty & y\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\,. \tag{2}$$

Evidently

$$\theta \text{ is an element of } \mathcal{M}_{[\infty,\infty]}\,. \tag{3}$$

We note that, due to the fact that $\mathcal{M}_{[\infty,o]}$ is a libra and that both of these involutions below agree at y+z,

$$\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & y+z\\ o & l\end{smallmatrix}} = \boxed{\begin{smallmatrix}\infty & y+z\\ o & x\end{smallmatrix}}\,. \tag{4}$$

We have

$$\theta(o) = \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & z\\ \infty & y\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}(o) = \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & z\\ \infty & y\end{smallmatrix}}(o) \stackrel{\text{by (2.12.1)}}{=\!=\!=} \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}(y{+}z) =$$

$$\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & y+z\\ o & l\end{smallmatrix}}(l) \stackrel{\text{by (4)}}{=\!=\!=} \boxed{\begin{smallmatrix}\infty & y+z\\ o & x\end{smallmatrix}}(l) \stackrel{\text{by (2.12.2)}}{=\!=\!=} x\cdot(y{+}z)\,. \tag{5}$$

Moreover, the first pair below agree at o and y and the second pair agree at o and z, which implies that,

$$\boxed{\begin{smallmatrix}\infty & y\\ o & x\end{smallmatrix}} = \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & y\end{smallmatrix}} \quad \text{and} \quad \boxed{\begin{smallmatrix}\infty & z\\ o & x\end{smallmatrix}} = \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & z\end{smallmatrix}}\,. \tag{6}$$

We have

$$\theta(x\cdot y) \stackrel{\text{by (2)}}{=\!=\!=} \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & z\\ \infty & y\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}(x\cdot y) \stackrel{\text{by (2.12.2)}}{=\!=\!=} \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & z\\ \infty & y\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & y\\ o & x\end{smallmatrix}}(l) \stackrel{\text{by (6)}}{=\!=\!=}$$

$$\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & z\\ \infty & y\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & y\end{smallmatrix}}(l) = \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & z\\ \infty & y\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & y\end{smallmatrix}}(l) =$$

$$\boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}(z) = \boxed{\begin{smallmatrix}\infty & l\\ o & x\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & z\end{smallmatrix}}(l) \overset{\text{by (6)}}{=\!=\!=} \boxed{\begin{smallmatrix}\infty & z\\ o & x\end{smallmatrix}}(l) \overset{\text{by (2.12.2)}}{=\!=\!=\!=} x{\cdot}z\,.$$

From this and (2) follows that $\theta = \boxed{\begin{smallmatrix}\infty & x{\cdot}z\\ \infty & x{\cdot}y\end{smallmatrix}}$, whence follows that

$$\theta(o) \;=\; \boxed{\begin{smallmatrix}\infty & x{\cdot}z\\ \infty & x{\cdot}y\end{smallmatrix}}(o) \overset{\text{by (2.12.2)}}{=\!=\!=\!=} x{\cdot}y+x{\cdot}z\,.$$

This last, along with (5), proves that the distributive law holds.

If the field were of characteristic 2, then for any x in $M_{[\infty,\infty]}$,

$$\boxed{\begin{smallmatrix}\infty & x\\ \infty & x\end{smallmatrix}}(o) \overset{\text{by (1)}}{=\!=\!=} x+x = o = \boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}(o)\,.$$

It would follow that $\boxed{\begin{smallmatrix}\infty & x\\ \infty & x\end{smallmatrix}}$ and $\boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}$, fixing two distinct points, would have to be identical involutions. Thus both would fix three points and so be the identity permutation, which is not in $\mathcal{M}_{[\infty,\infty]}$ by definition: an absurdity.

Let x be any element of $M_{[\infty,\infty]}$. We have

$$o = x+(-x) \overset{\text{by (1)}}{=\!=\!=} \boxed{\begin{smallmatrix}\infty & x\\ \infty & \text{-}x\end{smallmatrix}}(o)$$

whence follows that $\boxed{\begin{smallmatrix}\infty & x\\ \infty & \text{-}x\end{smallmatrix}}$ must equal $\boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}$. Thus $-x = \boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}(x)$. A similar argument shows that, for y in $M_{[\infty,o]}$, $\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}(y) = y^{-1}$. Q.E.D.

(2.14) Addition *The definition for the binary operator on* $M_{[\infty,\infty]}$ *has an alternate form which, since it extends readily to higher dimensional projective space, we shall mention here:*

$$(\forall\,\{x,y\}\subset M_{[\infty,\infty]})\quad x+y = \boxed{\begin{smallmatrix}\infty & m\\ \infty & m\end{smallmatrix}}(o) \quad \textit{where} \quad m \equiv \boxed{\begin{smallmatrix}x & y\\ x & y\end{smallmatrix}}(\infty)\,. \tag{1}$$

Proof: From (2.4.3) we know that $\{\{x,y\},\{m,\infty\}\}$ is a harmonic pair. It follows from (2.4.2) that $\boxed{\begin{smallmatrix}\infty & m\\ \infty & m\end{smallmatrix}}$ interchanges x and y. Consequently $\boxed{\begin{smallmatrix}\infty & m\\ \infty & m\end{smallmatrix}}$ and $\boxed{\begin{smallmatrix}\infty & y\\ \infty & x\end{smallmatrix}}$, agreeing on three points, must be identical permutations. This proves (1). Q.E.D.

(2.15) Lemma *Let* F *be the field associated with a tripleton* $\{\infty,o,l\}$ *within a meridian* M. *Let* a *be an element of* $M_{[\infty,\infty]}$ *and let* b *be an element of* $M_{[o,\infty]}$. *Then there exist* α *and* β *in* $\mathcal{M}$ *such that*

$$(\forall\, x\in M_{[\infty,\infty]})\quad \alpha(x) = a+x \quad\text{and}\quad (\forall\, x\in M_{[\infty,o]})\quad \beta(x) = b{\cdot}x\,. \tag{1}$$

Proof: For each x in $M_{[\infty,\infty]}$ we have

$$\boxed{\begin{smallmatrix}\infty & x\\ \infty & a\end{smallmatrix}} = \boxed{\begin{smallmatrix}\infty & o\\ \infty & a\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & x\\ \infty & o\end{smallmatrix}}$$

whence follows that

$$a+x \overset{\text{by (2.12.1)}}{=\!=\!=\!=} \boxed{\begin{smallmatrix}\infty & x\\ \infty & a\end{smallmatrix}}(o) = \boxed{\begin{smallmatrix}\infty & o\\ \infty & a\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & x\\ \infty & o\end{smallmatrix}}(o) = \boxed{\begin{smallmatrix}\infty & o\\ \infty & a\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}(x)\,.$$

For each x in $M\triangle\{o,\infty\}$ we have

$$\boxed{\begin{smallmatrix}\infty & x\\ o & b\end{smallmatrix}} = \boxed{\begin{smallmatrix}\infty & l\\ o & b\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}}$$

whence follows that

$$b{\cdot}x \overset{\text{by (2.12.2)}}{=\!=\!=\!=} \boxed{\begin{smallmatrix}\infty & x\\ \infty & b\end{smallmatrix}}(l) = \boxed{\begin{smallmatrix}\infty & l\\ o & b\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & x\\ o & l\end{smallmatrix}}(l) = \boxed{\begin{smallmatrix}\infty & l\\ o & b\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}\infty & l\\ o & l\end{smallmatrix}}(x)\,.$$

Q.E.D.

(2.16) Linear Fractional Transformations *Let* $M_{[\infty,\infty]}$ *be a meridian field generated by a triple as in (2.12), and let + and · be the corresponding field operations. Then the elements of* $\mathcal{M}$ *can be described in terms of these field operations as follows: if* $\{a,b,c,d\}$ *is a subset of* $M_{[\infty,\infty]}$ *such that* $a{\cdot}d \neq b{\cdot}c$,

$$
\mathrm{M} \ni \mathrm{x} \hookrightarrow \begin{cases} \frac{\mathrm{a}\cdot\mathrm{x}+\mathrm{b}}{\mathrm{c}\cdot\mathrm{x}+\mathrm{d}} & \text{if } \mathrm{x} \neq \infty \text{ and } \mathrm{c}\cdot\mathrm{x}+\mathrm{d} \neq \mathrm{o}\,; \\ \frac{\mathrm{a}}{\mathrm{c}} & \text{if } \mathrm{x} = \infty \text{ and } \mathrm{c} \neq \mathrm{o}\,; \\ \infty & \text{if } \mathrm{c}\cdot\mathrm{x}+\mathrm{d} = \mathrm{o} \text{ or if } \mathrm{x} = \infty \text{ and } \mathrm{c} = \mathrm{o}\,. \end{cases} \quad . \tag{1}
$$

Proof: We know from (2.15) that translations and dilations are homographies. The function

$$
\boxed{\begin{smallmatrix}\infty & 1\\ \infty & 1\end{smallmatrix}} \Big| \mathrm{M}_{[\infty,\mathrm{o}]} \ni \mathrm{x} \hookrightarrow \frac{1}{\mathrm{x}} \in \mathrm{M}_{[\infty,\mathrm{o}]}
$$

is called an **inversion** and it is obviously the restriction to $\mathrm{M}_{[\infty,\mathrm{o}]}$ of a homography. If we can describe the function in (1) as a composition of translations, dilations and inversions, it will follow that that function itself is a homography. We define

$$
\alpha(\mathrm{x}) \equiv \frac{\mathrm{a}}{\mathrm{c}}+\mathrm{x}\,, \quad \beta(\mathrm{x}) \equiv (\mathrm{b}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{d})\cdot\mathrm{x}\,, \quad \gamma(\mathrm{x}) \equiv \frac{1}{\mathrm{x}}\,, \quad \delta(\mathrm{x}) \equiv (\mathrm{c}\cdot\mathrm{c})\cdot\mathrm{x}\,, \quad \text{and} \quad \epsilon(\mathrm{x}) \equiv \frac{\mathrm{d}}{\mathrm{c}}+\mathrm{x}
$$

and check that $\alpha\circ\beta\circ\gamma\circ\delta\circ\epsilon$ is just the function in (1). Q.E.D.

Section 3. The Meridian of Throws

(3.1) Definitions and Notation Let P be a finite dimensional projective space, and let $\mathcal{P}$ be its homography group of permutations. If A and B are lines within P , if {a,b,c} is a tripleton within A , and if {x,y,z} is a tripleton within B , then by the *fundamental axiom* (1.3.2), there exists an element of $\mathcal{P}$ locally unique on A which sends a to x , b to y and c to z : we shall denote the restriction of that permutation to A by

$$\boxed{\begin{smallmatrix} x & y & z \\ a & b & c \end{smallmatrix}}\,. \tag{1}$$

Such functions will be called **projective meridian isomorphisms**.

In this sense, each ordered coaligned tripleton in a projective space is equivalent to each other such coaligned tripleton. Such is not the case for ordered coaligned quadrupletons however. If {a,b,c,d} and {w,x,y,z} are coaligned quadrupletons, there may, or may not, be an element ϕ of the homography group $\mathcal{P}$ such that

$$\phi(a) = w\,, \quad \phi(b) = x\,, \quad \phi(c) = y \quad \text{and} \quad \phi(d) = z\,.$$

The mathematician Karl von Staudt looked into this nearly two hundred years ago and his study lead to the discovery of a meridian isomorphic to each of the line meridians of a projective space, but with this difference: this new meridian had three points objectively different from all the other points. To obtain these results however, he had to consider not just quadrupletons, but ordered quadruples with repeated elements. To make this idea more precise we introduce an index set

$$\mathbf{III} \equiv \{\heartsuit,\spadesuit,\clubsuit,\diamondsuit\} \tag{2}$$

and the family

$$\Omega \tag{3}$$

consisting of all functions $f\,\big|\,\mathbf{III} \hookrightarrow P$ such that the set $\{f(x) : x \in \mathbf{III}\}$ is coaligned and of cardinality at least 3 . For such a coaligned set {w,x,y,z} within P , we shall adopt the notation

$$\boxed{\begin{smallmatrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}} \tag{4}$$

for the function on Ω which sends $\heartsuit$ to w , $\spadesuit$ to x , $\clubsuit$ to y and $\diamondsuit$ to z .

Besides the identity permutation of $\mathbf{III}$, there are three other permutations of particular interest here: the one that interchanges $\heartsuit$ with $\diamondsuit$ and $\spadesuit$ with $\clubsuit$, the one that interchanges $\spadesuit$ with $\diamondsuit$ and $\clubsuit$ with $\heartsuit$, and the one that interchanges $\clubsuit$ with $\diamondsuit$ and $\spadesuit$ with $\heartsuit$: we shall denote them, respectively, by

$$\boxed{\begin{smallmatrix} \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{smallmatrix}}\,, \quad \boxed{\begin{smallmatrix} \spadesuit & \clubsuit \\ \diamondsuit & \heartsuit \end{smallmatrix}}\,, \quad \text{and} \quad \boxed{\begin{smallmatrix} \clubsuit & \spadesuit \\ \diamondsuit & \heartsuit \end{smallmatrix}}\,, \tag{5}$$

respectively. We shall denote that group of four permutations by[7]

$$K. \tag{6}$$

If two elements f and g of Ω have quadrupleton images and are equivalent in the sense that there exists some homography ϕ of P such that $f = \phi \circ g$, then, for each element σ of K, f is equivalent to $g \circ \sigma$ as well. This is evident from the identities:

$$\boxed{\begin{smallmatrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} z & x \\ w & y \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}}\,, \quad \boxed{\begin{smallmatrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \spadesuit & \clubsuit \\ \diamondsuit & \heartsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} y & x \\ w & z \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}}\,, \quad \text{and} \quad \boxed{\begin{smallmatrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \clubsuit & \spadesuit \\ \diamondsuit & \heartsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} z & x \\ y & w \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} w & x & y & z \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}}\,.$$

The same is not true however for elements of Ω for which the images are not quadrupletons. Therefore we enlarge slightly the definition for equivalence for elements of Ω . We define, for all $\{f,g\}$ within Ω ,

[7] Sometimes called the **Klein** 4-**group**.

$$\mathtt{f} \approx \mathtt{g} \qquad \Longleftrightarrow \qquad (\exists\, \phi \in \mathcal{P} \text{ and } \kappa \in K) \qquad \phi \circ \mathtt{f} \circ \kappa = \mathtt{g}\,. \tag{7}$$

For any subset {w,x,y,z} of cardinality at least 3 we have

$$\boxed{\begin{smallmatrix} w&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \left(\boxed{\begin{smallmatrix} w&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \heartsuit&\spadesuit\\ \diamondsuit&\clubsuit \end{smallmatrix}}\right) = \boxed{\begin{smallmatrix} z&y&x&w\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,, \tag{8}$$

$$\boxed{\begin{smallmatrix} w&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \left(\boxed{\begin{smallmatrix} w&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \spadesuit&\clubsuit\\ \diamondsuit&\heartsuit \end{smallmatrix}}\right) = \boxed{\begin{smallmatrix} y&z&w&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,, \tag{9}$$

and

$$\boxed{\begin{smallmatrix} w&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \left(\boxed{\begin{smallmatrix} w&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \clubsuit&\spadesuit\\ \diamondsuit&\heartsuit \end{smallmatrix}}\right) = \boxed{\begin{smallmatrix} x&w&z&y\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,. \tag{10}$$

We shall denote the family of $\approx$-equivalence classes by

$$\mathfrak{W}_{\mathrm{P}}\,. \tag{11}$$

Von Staudt used the terminology "Wurf" for an element of $\mathfrak{W}_{\mathrm{P}}$: we shall use here the English translation **throw**. For any element $\mathtt{t}$ of Ω, we shall denote the $\approx$-equivalence class or throw containing it by

$$\boxed{\mathtt{t}}\,. \tag{12}$$

For a throw with member $\boxed{\begin{smallmatrix} w&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}$, we shall at times use the abbreviation

$$\boxed{\mathrm{w\ \ x\ \ y\ \ z}} \quad \text{for} \quad \boxed{\boxed{\begin{smallmatrix} w&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}}\,. \tag{13}$$

It follows from (8), (9) and (10) that

$$\boxed{\mathrm{w\ \ x\ \ y\ \ z}} = \boxed{\mathrm{z\ \ y\ \ x\ \ w}} = \boxed{\mathrm{y\ \ z\ \ w\ \ x}} = \boxed{\mathrm{x\ \ w\ \ z\ \ y}}\,. \tag{14}$$

The presence of K is only felt when the cardinality of {w,x,y,z} is 3: if $\approx$ denotes the equivalence relation, then, for any tripleton {x,y,z},

$$\boxed{\begin{smallmatrix} x&y&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} x&y&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \heartsuit&\spadesuit\\ \diamondsuit&\clubsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} x&z&y&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,,\ \boxed{\begin{smallmatrix} x&y&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} x&y&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \spadesuit&\clubsuit\\ \diamondsuit&\heartsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} z&x&x&y\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,,\ \boxed{\begin{smallmatrix} x&y&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} x&y&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \clubsuit&\spadesuit\\ \diamondsuit&\heartsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} y&x&x&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,;$$

$$\boxed{\begin{smallmatrix} y&x&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} y&x&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \heartsuit&\spadesuit\\ \diamondsuit&\clubsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} x&z&x&y\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,,\ \boxed{\begin{smallmatrix} y&x&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} y&x&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \spadesuit&\clubsuit\\ \diamondsuit&\heartsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} z&x&y&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,,\ \boxed{\begin{smallmatrix} y&x&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} y&x&z&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \clubsuit&\spadesuit\\ \diamondsuit&\heartsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} x&y&x&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,;$$

$$\boxed{\begin{smallmatrix} y&z&x&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} y&z&x&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \heartsuit&\spadesuit\\ \diamondsuit&\clubsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} x&x&z&y\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,,\ \boxed{\begin{smallmatrix} y&z&x&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} y&z&x&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \spadesuit&\clubsuit\\ \diamondsuit&\heartsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} x&x&y&z\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,,\ \boxed{\begin{smallmatrix} y&z&x&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \approx \boxed{\begin{smallmatrix} y&z&x&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \clubsuit&\spadesuit\\ \diamondsuit&\heartsuit \end{smallmatrix}} = \boxed{\begin{smallmatrix} z&y&x&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}\,.$$

Thus, members of Ω with ranges of cardinality 3 all fall inside the three distinct equivalence classes:

$$\bowtie\ \equiv \left\{\boxed{\begin{smallmatrix} a&b&c&a\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} : \{a,b,c\} \text{ is a coaligned tripleton}\right\} \cup \left\{\boxed{\begin{smallmatrix} a&b&b&c\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} : \{a,b,c\} \text{ is a coaligned tripleton}\right\}, \tag{15}$$

$$\mathtt{o} \equiv \left\{\boxed{\begin{smallmatrix} a&b&c&b\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} : \{a,b,c\} \text{ is a coaligned tripleton}\right\} \cup \left\{\boxed{\begin{smallmatrix} a&b&a&c\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} : \{a,b,c\} \text{ is a coaligned tripleton}\right\} \tag{16}$$

and

$$\mathtt{1} \equiv \left\{\boxed{\begin{smallmatrix} a&b&c&c\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} : \{a,b,c\} \text{ is a coaligned tripleton}\right\} \cup \left\{\boxed{\begin{smallmatrix} a&a&b&c\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}} : \{a,b,c\} \text{ is a coaligned tripleton}\right\}. \tag{17}$$

(3.2) A Meridian Isomorphism Let M be a line in a projective space P. We shall pick any element of **Ш**, say ♦, and any injection of the complement of {♦} in **Ш** into M, say {[♥,a],[♠,b],[♣,c]}. This provides us a function

$$\mathrm{M} \ni \mathrm{x} \hookrightarrow \boxed{\boxed{\begin{smallmatrix} a&b&c&x\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit \end{smallmatrix}}} \in \mathfrak{W}_{\mathrm{P}}\,. \tag{1}$$

It is of course dependent on the choice of ♦, [♥,a], [♠,b] and [♣,c]. However, in view of the definition of the equivalence relation $\approx$, if we had chosen something different than ♦, we could have altered the other three choices to get the same function.

It is evident that the function in (1) sends a to $\bowtie$, b to o and c to $\imath$. That suggests that we denote it by

$$\boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ a & b & c\end{smallmatrix}}\,. \tag{2}$$

It follows from (3.1.14) that

$$(\forall\, x\in M)\qquad \boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ a & b & c\end{smallmatrix}}(x) = \boxed{a\ \ b\ \ c\ \ x} = \boxed{b\ \ a\ \ x\ \ c} = \boxed{c\ \ x\ \ a\ \ b} = \boxed{x\ \ c\ \ b\ \ a}\,. \tag{3}$$

We shall denote the inverse function of $\boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ a & b & c\end{smallmatrix}}$ by

$$\boxed{\begin{smallmatrix}a & b & c\\ \bowtie & \mathsf{o} & \imath\end{smallmatrix}}\,. \tag{4}$$

For any element $\boxed{\begin{smallmatrix}w & x & y & z\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{smallmatrix}}$ of Ω, we have

$$\boxed{\begin{smallmatrix}a & b & c\\ \bowtie & \mathsf{o} & \imath\end{smallmatrix}}\left(\boxed{w\ \ x\ \ y\ \ z}\right) = \begin{cases} a & \text{if } z=w \text{ or } x=y\,,\\ b & \text{if } z=x \text{ or } w=y\,,\\ c & \text{if } z=y \text{ or } w=x\,, \text{ and}\\ \boxed{\begin{smallmatrix}a & b & c\\ w & x & y\end{smallmatrix}}(z) & \text{otherwise}\end{cases}\cdot \tag{5}$$

because

$$\boxed{\begin{smallmatrix}a & b & c\\ \bowtie & \mathsf{o} & \imath\end{smallmatrix}}^{-1}\left(\boxed{\begin{smallmatrix}a & b & c\\ w & x & y\end{smallmatrix}}(z)\right) \overset{\text{by (4)}}{=\!=} \boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ a & b & c\end{smallmatrix}}\left(\boxed{\begin{smallmatrix}a & b & c\\ w & x & y\end{smallmatrix}}(z)\right) \overset{\text{by (1)}}{=\!=} \boxed{a\ \ b\ \ c\ \ \boxed{\begin{smallmatrix}a & b & c\\ w & x & y\end{smallmatrix}}(z)} \overset{\text{by (3.1.7)}}{=\!=}$$

$$\boxed{\boxed{\begin{smallmatrix}w & x & y\\ a & b & c\end{smallmatrix}}(a)\ \ \boxed{\begin{smallmatrix}w & x & y\\ a & b & c\end{smallmatrix}}(b)\ \ \boxed{\begin{smallmatrix}w & x & y\\ a & b & c\end{smallmatrix}}(c)\ \ \boxed{\begin{smallmatrix}w & x & y\\ a & b & c\end{smallmatrix}}\left(\boxed{\begin{smallmatrix}a & b & c\\ w & x & y\end{smallmatrix}}(z)\right)} = \boxed{w\ \ x\ \ y\ \ z}\,.$$

In particular, for the case $\boxed{w\ \ x\ \ y\ \ z} = \boxed{a\ \ b\ \ c\ \ r}$, we obtain

$$(\forall\, r\in M \quad\text{and}\quad \mathfrak{r}\in\mathfrak{W}_P)\qquad \boxed{\begin{smallmatrix}a & b & c\\ \bowtie & \mathsf{o} & \imath\end{smallmatrix}}(\mathfrak{r}) = r \iff \boxed{\begin{smallmatrix}a & b & c & r\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{smallmatrix}}\in\mathfrak{r}\,. \tag{6}$$

(3.3) The Throw Meridian Group Let P, M, $\{a,b,c\}$, Ω, K, $\mathfrak{W}_P$, $\boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ a & b & c\end{smallmatrix}}$ and $\boxed{\begin{smallmatrix}a & b & c\\ \bowtie & \mathsf{o} & \imath\end{smallmatrix}}$ be as in (3.1) and (3.2). Let N be any other line meridian in P and let $\{r,s,t\}$ any tripleton in N. Following (3.2.1) and (3.2.1) we shall adopt the notation $\boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ r & s & t\end{smallmatrix}}$ for the function

$$N\ni x\hookrightarrow \boxed{\boxed{\begin{smallmatrix}r & s & t & x\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{smallmatrix}}}\in\mathfrak{W}_P\,. \tag{1}$$

For any x in N, we have

$$\boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ a & b & c\end{smallmatrix}}\left(\boxed{\begin{smallmatrix}a & b & c\\ r & s & t\end{smallmatrix}}(x)\right) \overset{\text{by (3.2.1) and (3.2.2)}}{=\!=\!=} \boxed{a\ \ b\ \ c\ \ \boxed{\begin{smallmatrix}a & b & c\\ r & s & t\end{smallmatrix}}(x)} \overset{\text{by (3.1.7)}}{=\!=}$$

$$\boxed{\boxed{\begin{smallmatrix}r & s & t\\ a & b & c\end{smallmatrix}}(a)\ \ \boxed{\begin{smallmatrix}r & s & t\\ a & b & c\end{smallmatrix}}(b)\ \ \boxed{\begin{smallmatrix}r & s & t\\ a & b & c\end{smallmatrix}}(c)\ \ \boxed{\begin{smallmatrix}r & s & t\\ a & b & c\end{smallmatrix}}\left(\boxed{\begin{smallmatrix}a & b & c\\ r & s & t\end{smallmatrix}}(x)\right)} = \boxed{r\ \ s\ \ t\ \ x} \overset{\text{by (1)}}{=\!=} \boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ r & s & t\end{smallmatrix}}(x)\,,$$

or, more succinctly,

$$\boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ r & s & t\end{smallmatrix}} = \boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ a & b & c\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}a & b & c\\ r & s & t\end{smallmatrix}}\,. \tag{2}$$

It follows from (2) that, if we define $\boxed{\begin{smallmatrix}r & s & t\\ \bowtie & \mathsf{o} & \imath\end{smallmatrix}}$ to be the inverse of the function $\boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ r & s & t\end{smallmatrix}}$, then the family

$$\widehat{\mathfrak{W}} \equiv \left\{\boxed{\begin{smallmatrix}\bowtie & \mathsf{o} & \imath\\ r & s & t\end{smallmatrix}}\circ\boxed{\begin{smallmatrix}a & b & c\\ \bowtie & \mathsf{o} & \imath\end{smallmatrix}} :\ \{r,s,t\} \text{ and } \{a,b,c\} \text{ are tripletons within } M\right\} \tag{3}$$

does not depend on the choice of meridian line M adopted for the definition. We shall say that $\widehat{\mathfrak{W}}$ is the **throw meridian group**.

It is evident that $\mathfrak{W}_P$ is a meridian relative to the group $\widehat{\mathfrak{W}}$ of permutations and that the function

$$\boxed{\begin{smallmatrix}\bowtie & o & 1\\ a & b & c\end{smallmatrix}} \quad \text{is a meridian isomorphism of M onto } \mathfrak{W}_P\,. \tag{4}$$

Suppose that $\{\infty, o, l\}$ is a tripleton of the meridian line M and that F is the field induced by that tripleton. As in (2.16.1), for any tripleton $\{w,x,y\}$ within M, we have a linear fractional transformation homography

$$M \ni z \hookrightarrow \frac{(y-w)\cdot(z-x)}{(y-x)\cdot(z-w)} \in M\,.$$

Evidently this homography is precisely $\boxed{\begin{smallmatrix}\infty & o & l\\ w & x & y\end{smallmatrix}}$. It follows from (3.2.5) that

$$\boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}}\left(\boxed{w\ \ x\ \ y\ \ z}\right) = \frac{(y-w)\cdot(z-x)}{(y-x)\cdot(z-w)}\,, \tag{5}$$

the **cross ratio** of w, x, y and z relative to the field F.

(3.4) Definition We shall say that the field of the meridian $\mathfrak{W}_P$ generated by the tripleton $\{\bowtie, o, 1\}$ is the **field of throws for the projective space**. We shall denote it by

$$\mathfrak{F} \tag{1}$$

and the associated field operations by $+$ and $\cdot$.

(3.5) Intuition The meridian of throws and the field of throws are rather abstract objects, so some intuition may not be amiss. Just as the terms applied for set and set element carry information about the context in which they are being applied (“families”, “functions”, “members”, “points”, “instances” *etc.*), the names assigned to the equivalence classes o, 1, and $\bowtie$ do as well. We shall use here the term **instance of a throw** to refer to an element of that throw, and adopt the term **standard instance** for an instance $\boxed{\begin{smallmatrix}w & x & y & z\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{smallmatrix}}$ for which $\{w,x,y\}$ is a quadrupleton.

Given a standard instance $\boxed{\begin{smallmatrix}w & x & y & z\\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit\end{smallmatrix}}$, one may imagine a line segment in P beginning at a point x and ending at a point w an infinite distance away; and suppose that y and z are two points on the line $\overleftrightarrow{x,w}$. The throw $\boxed{w\ \ x\ \ y\ \ z}$ may be thought of as the “ratio” of the line segment beginning at x and ending at z to the line segment beginning at x and ending at y. Thus, if $z=w$, then $\boxed{w\ \ x\ \ y\ \ z}$ is just $\bowtie$, and the corresponding ratio may be thought of as “infinite”. If $z=x$, then $\boxed{w\ \ x\ \ y\ \ z}$ is o, and the ratio may be thought of as 0. If $z=y$, then $\boxed{w\ \ x\ \ y\ \ z}$ is 1, the line segments are identical, and the ratio may be thought of as 1 or unity.

We could have used any permutation of the special elements $\bowtie$, o and 1 to generate the field of throws. Our choice was made so that the usual convention for the cross ratio would appear in (3.3.5).

(3.6) Theorem *Let* M *be a line in* P, *let* $\{\infty, o, l\}$ *be a tripleton in* M *and let* F *be the field induced by that tripleton. Then the restriction to* $\mathfrak{F}$ *of the function*

$$\boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}} \quad \textit{is an isomorphism of fields.} \tag{1}$$

In particular, for $\{r,s,t,u,v,w\}$ *within* $(M \triangle \{o\})$ *such that* $u\cdot v \neq -w$, *we have*

$$\frac{\boxed{\infty\ \ o\ \ l\ \ r}\cdot\boxed{\infty\ \ o\ \ l\ \ s}+\boxed{\infty\ \ o\ \ l\ \ t}}{\boxed{\infty\ \ o\ \ l\ \ u}\cdot\boxed{\infty\ \ o\ \ l\ \ v}+\boxed{\infty\ \ o\ \ l\ \ w}} = \boxed{\infty\ \ o\ \ l\ \ \frac{r\cdot s+t}{u\cdot v+w}} \tag{2}$$

and

$$\boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}} \left(\frac{\boxed{\infty\ o\ l\ r}\cdot\boxed{\infty\ o\ l\ s}+\boxed{\infty\ o\ l\ t}}{\boxed{\infty\ o\ l\ u}\cdot\boxed{\infty\ o\ l\ v}+\boxed{\infty\ o\ l\ w}}\right) = \frac{r\cdot s+t}{u\cdot v+w}\,. \tag{3}$$

Proof: Let x and y be elements of the field $F = M_{[\infty,\infty]}$. That

$$\boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ \infty & y\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}}$$

is an involution and that it leaves $\bowtie$ fixed is evident. Furthermore

$$\boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ \infty & y\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}} \left(\boxed{\infty\ o\ l\ y}\right) \overset{\text{by (3.2.3)}}{=\!=\!=} \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ \infty & y\end{smallmatrix}} (y) = \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} (x) \overset{\text{by (3.2.3)}}{=\!=\!=} \boxed{\infty\ o\ l\ x}\,.$$

It follows that

$$\boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ \infty & y\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}} = \boxed{\begin{smallmatrix}\bowtie & \boxed{\infty\ o\ l\ x}\\ \bowtie & \boxed{\infty\ o\ l\ y}\end{smallmatrix}}\,. \tag{4}$$

Consequently

$$\begin{aligned} &\boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} (x+y) \overset{\text{by (2.12.1)}}{=\!=\!=} \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \left(\boxed{\begin{smallmatrix}\infty & x\\ \infty & y\end{smallmatrix}} (o)\right) = \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ \infty & y\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}} (o) \overset{\text{by (4)}}{=\!=\!=} \\ &\boxed{\begin{smallmatrix}\bowtie & \boxed{\infty\ o\ l\ x}\\ \bowtie & \boxed{\infty\ o\ l\ y}\end{smallmatrix}} (o) \overset{\text{by (2.12.1)}}{=\!=\!=} \boxed{\infty\ o\ l\ x} + \boxed{\infty\ o\ l\ y} \overset{\text{by (3.2.3)}}{=\!=\!=} \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} (x) + \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} (y)\,. \end{aligned} \tag{5}$$

Let x and y be elements of the field $F = M_{[\infty,o]}$. That

$$\boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}}$$

is an involution and that it interchanges o and $\bowtie$ is evident. Furthermore

$$\boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}} \left(\boxed{\infty\ o\ l\ y}\right) \overset{\text{by (3.2.3)}}{=\!=\!=} \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}} (y) = \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} (x) \overset{\text{by (3.2.3)}}{=\!=\!=} \boxed{\infty\ o\ l\ x}\,.$$

It follows that

$$\boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}} = \boxed{\begin{smallmatrix}\bowtie & \boxed{\infty\ o\ l\ x}\\ o & \boxed{\infty\ o\ l\ y}\end{smallmatrix}}\,. \tag{6}$$

Consequently

$$\begin{aligned} &\boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} (x\cdot y) \overset{\text{by (2.12.2)}}{=\!=\!=} \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \left(\boxed{\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}} (l)\right) = \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & x\\ o & y\end{smallmatrix}} \circ \boxed{\begin{smallmatrix}\infty & o & l\\ \bowtie & o & 1\end{smallmatrix}} (1) \overset{\text{by (6)}}{=\!=\!=} \\ &\boxed{\begin{smallmatrix}\bowtie & \boxed{\infty\ o\ l\ x}\\ o & \boxed{\infty\ o\ l\ y}\end{smallmatrix}} (1) \overset{\text{by (2.12.2)}}{=\!=\!=} \boxed{\infty\ o\ l\ x} \cdot \boxed{\infty\ o\ l\ y} \overset{\text{by (3.2.3)}}{=\!=\!=} \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} (x) \cdot \boxed{\begin{smallmatrix}\bowtie & o & 1\\ \infty & o & l\end{smallmatrix}} (y)\,. \end{aligned} \tag{7}$$

That (1) holds follows from (5) and (7). That (2) and (3) hold is a consequence of (1). Q.E.D.

(3.7) Lemma *Let* $\{\{a,c\},\{b,d\}\}$ *be a harmonic pair within a meridian line of a projective space* P. *Let* ϕ *be a homography of* P. *Then*

$$\{\{\phi(a),\phi(c)\},\{\phi(b),\phi(d)\}\} \text{ is a harmonic pair of pairs.} \tag{1}$$

Proof: We know from (2.4.1) that there exists a homography θ of P such that

$$\theta(a) = b\,,\quad \theta(b) = c\,,\quad \theta(c) = d \quad\text{and}\quad \theta(d) = a\,.$$

Let $\psi \equiv \phi \circ \theta \circ \phi^{-1}$. Evidently

$$\psi(\phi(a)) = \phi(b)\,,\quad \psi(\phi(b)) = \phi(c)\,,\quad \psi(\phi(b)) = \phi(c) \quad\text{and}\quad \psi(\phi(d)) = \phi(a)\,.$$

It follows from (2.4.1) that (1) holds. Q.E.D.

(3.8) Harmonic Throws Let P be a non-trivial projective space and let Ω , $\mathfrak{W}_P$, $\widehat{\mathfrak{W}}$, $\bowtie$, $\mathbf{o}$, $\mathbf{1}$, $\mathfrak{F}$, $+$ and $\cdot$ be as in (3.1) and (3.4).

We shall say that **a coaligned quadrupleton within P is harmonic** if its elements can be arranged as a harmonic pair. We shall say that **a throw is harmonic** if it contains a harmonic quadrupleton. It follows from Lemma (3.7.1) that, if a throw is harmonic, all of its elements are harmonic quadrupletons.

Suppose that $\{\infty,o,l,x\}$ is a harmonic quadrupleton and that $+$ and $\cdot$ are the binary field operations for the field F generated by the tripleton $\{\infty,o,l\}$.

The condition for $\{\{\infty,o\},\{l,x\}\}$ to be a harmonic pair is for $\boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}(l) = x$. From (2.16.1) we see

$$\boxed{\begin{smallmatrix}\infty & o\\ \infty & o\end{smallmatrix}}\Big| F \ni x \hookrightarrow -x \in F\,,$$

whence follows that $x = -l$. Consequently $\boxed{\infty \;\; o \;\; l \;\; x}$ must be in the throw $-\mathbf{1}$.

The condition for $\{\{\infty,l\},\{o,x\}\}$ to be a harmonic pair is for $\boxed{\begin{smallmatrix}\infty & l\\ \infty & l\end{smallmatrix}}(o) = x$. From (2.16.1) we see

$$\boxed{\begin{smallmatrix}\infty & l\\ \infty & l\end{smallmatrix}}\Big| F \ni x \hookrightarrow -x+2 \in F\,,$$

whence follows that $x = 2$. Consequently $\boxed{\infty \;\; o \;\; l \;\; x}$ must be in the throw $\mathbf{2}$.

The condition for $\{\{o,l\},\{\infty,x\}\}$ to be a harmonic pair is for $\boxed{\begin{smallmatrix}o & l\\ o & l\end{smallmatrix}}(\infty) = x$. From (2.16.1) we see

$$\boxed{\begin{smallmatrix}o & l\\ o & l\end{smallmatrix}}\Big| F \ni x \hookrightarrow \tfrac{x}{2\cdot x - 1} \in F\,,$$

whence follows that $x = \frac{1}{2}$. Consequently $\boxed{\infty \;\; o \;\; l \;\; x}$ must be in the throw $\frac{\mathbf{1}}{\mathbf{2}}$.

It follows that each harmonic quadrupleton is in one of the three throws

$$-\mathbf{1}\,,\quad \mathbf{2} \quad\text{and}\quad \frac{\mathbf{1}}{\mathbf{2}}\,. \tag{1}$$

The three throws in (1) are identical if the field $\mathfrak{F}$ has characteristic 3 . Otherwise they are pairwise distinct.

Section 4. Higher Dimensional Projective Spaces

(4.1) Definitions Let P be a finite dimensional projective space of dimension greater than one, and let $\mathcal{P}$ be its homography group of permutations. Elements of P are often called **points**. If S is a maximal proper subspace, we say that S is a **co-point**.[8] A subspace of dimension 2 is called a **plane**.

(4.2) Projections Let p be a point and B a co-point not containing p. It follows from (1.13) that, for any point x distinct from p, the meridian $\overleftrightarrow{x,p}$ intersects B at one point: we shall denote as $\pi(x)$. This function π will be said to be the **projection from p onto B**. If N is a meridian neither within B nor through p, the restriction of π to N sends N onto a meridian within B: this function is called a **perspectivity**.

(4.3) Theorem I *Each perspectivity is a projective meridian isomorphism.*

Proof: Let β be a perspectivity from a meridian L onto a meridian M defined by a projection from a point b. The meridian M is in the plane determined by b and L. From (1.13) we know that M and L intersect in a single point q. Let {c,d} be a doubleton within L not containing q. Let x be the intersection point of M with $\overleftrightarrow{b,c}$ and y be the intersection point of M with $\overleftrightarrow{b,d}$. The set {c,d,x,y} is fundamental and so by the *fundamental axiom* (1.3.2) (1.3.2) there exists an element θ of $\mathcal{P}$ such that

$$\theta(c) = x\,,\ \ \theta(x) = c\,,\ \ \theta(d) = y \quad \text{and} \quad \theta(y) = d\,. \tag{1}$$

Since θ interchanges c and x, it sends the meridian $\overleftrightarrow{b,x}$ onto itself; since θ interchanges d and y, its sends the meridian $\overleftrightarrow{b,y}$ onto itself. Hence the intersecting point of those two meridians must be fixed:

$$\theta(b) = b\,. \tag{2}$$

Also the meridian $\overleftrightarrow{c,d}$ is sent onto the meridian $\overleftrightarrow{x,y}$ and *vice versa*: that is, θ interchanges the meridians L and M. Thus

$$\theta(q) = q\,. \tag{3}$$

In view of (2) and the fact that the points c and x are interchanged on $\overleftrightarrow{c,x}$, it follows from the *fixed point property* (2.1.4) that there is precisely one other point o on $\overleftrightarrow{b,x}$ such that

$$\theta(o) = o\,. \tag{4}$$

It follows from (3) and (4) that the meridian $\overleftrightarrow{q,o}$ is sent onto itself by θ. Since the meridian $\overleftrightarrow{b,y}$ is also sent onto itself, it follows that the intersection point p of these two meridians is fixed by θ. It follows from (3), (4) and the *fundamental property* (2.1.3) that

$$(\forall\, t \in \overleftrightarrow{q,o})\quad \theta(t) = t\,. \tag{5}$$

Now let z be any element of L. The meridian $\overleftrightarrow{b,z}$ intersects $\overleftrightarrow{q,o}$ at one point, say w. In view of (2) and (5), the meridian $\overleftrightarrow{b,z}$ is sent onto itself. Since L is sent onto M, the intersection point z of $\overleftrightarrow{b,z}$ with L is sent to the intersection point of $\overleftrightarrow{b,z}$ with M: that is

$$(\forall\, z \in L)\quad \theta(z) = \beta(z)\,.$$

But by definition (3.1.1), θ is a projective meridian isomorphism. Q.E.D.

(4.4) Affine Spaces Let V be a co-point within P. We shall define a ternary operator on the set complement A of V in P and show that it is an abelian libra operator. We shall call A the **affine space complementary to the co-point** V.

For each meridian M which intersects A, shall write $\mathfrak{v}(M)$ for the intersection point of M with V. If $M = \overleftrightarrow{u,v}$ for {u,v} a doubleton within P which intersects A, we abbreviate $\mathfrak{v}(\overleftrightarrow{u,v})$ to $\mathfrak{v}(u,v)$. Thus, for

[8] Sometimes called **hyperplane**.

$u \neq v$,

$$\{\mathfrak{v}(u,v)\} = \overleftrightarrow{u,v} \cap V .$$

Let $\{a,b,c\}$ be within A . We define m to be a if $a = c$: otherwise we define

$$m \equiv \boxed{\begin{smallmatrix} a & c \\ a & c \end{smallmatrix}}(\mathfrak{v}(a,c)) . \tag{1}$$

It follows from (2.4.4) that, if $a \neq c$,

$$m \quad \text{is the unique element of } \overleftrightarrow{a,c} \text{ such that } \{\{a,c\},\{m,\mathfrak{v}(a,c)\}\} \text{ is a harmonic pair.} \tag{2}$$

If $m = b$, we define $\lfloor a,b,c \rfloor$ to be b : otherwise we define

$$\lfloor a,b,c \rfloor \equiv \boxed{\begin{smallmatrix} \mathfrak{v}(m,b) & m \\ \mathfrak{v}(m,b) & m \end{smallmatrix}}(b) . \tag{3}$$

Thus, if $m \neq b$,

$$\lfloor a,b,c \rfloor \quad \text{is the unique element of } \overleftrightarrow{b,m} \text{ such that } \{\{b,\lfloor a,b,c \rfloor\},\{m,\mathfrak{v}(m,b)\}\} \text{ is a harmonic pair.} \tag{4}$$

(4.5) Lemma I *Let* V *be a co-point of a projective space* P *and let* A *be the affine space complementary to* P . *Let* p *be an element of* V , *let* S *be a co-point of* P *not containing* p *and let* π *be the projection onto* S *from* p . *Let* $\{a,b,c\}$ *be a tripleton in* A . *Then*

$$\pi(\lfloor a,b,c \rfloor) = \lfloor \pi(a),\pi(b),\pi(c) \rfloor . \tag{1}$$

Proof: Let m be as in (4.4.1). We know from Theorem I (4.3) that the restriction of π to $\overleftrightarrow{a,c}$ is a meridian isomorphism μ of $\overleftrightarrow{a,c}$ onto $\overleftrightarrow{\pi(a),\pi(c)}$. Since p is in V, it follows that $\pi(\mathfrak{v}(a,c))$ is in V as well: that is, $\mathfrak{v}(\pi(a),\pi(c)) = \pi(\mathfrak{v}(a,c))$. Since μ preserves harmonic pairs by (3.7),

$$\pi(m) \quad \text{is the element of } \overleftrightarrow{\pi(a),\pi(c)} \text{ such that } \{\{\pi(a),\pi(c)\},\{\pi(m),\mathfrak{v}(\pi(a),\pi(c))\}\} \text{ is a harmonic pair.} \tag{2}$$

The perspectivity from p of $\overleftrightarrow{m,b}$ onto $\overleftrightarrow{\pi(m),\pi(b)}$ is also a meridian isomorphism ψ . Since p is in V, it follows that $\pi(\mathfrak{v}(b,m))$ is in V as well: that is, $\mathfrak{v}(\pi(b),\pi(m)) = \pi(\mathfrak{v}(b,m))$. Since ψ preserves harmonic pairs, it follows that $\pi(\lfloor a,b,c \rfloor)$ is the unique element of $\overleftrightarrow{\pi(b),\pi(m)}$ such that

$$\{\{\pi(b),\pi(\lfloor a,b,c \rfloor)\},\{\pi(m),\mathfrak{v}(\pi(b),\pi(m))\}\} \text{ is a harmonic pair:}$$

but by definition (4.4.4) and by (2), that element is just $\lfloor \pi(a),\pi(b),\pi(c) \rfloor$. Q.E.D.

(4.6) Theorem II *Let* V *be a co-point of a projective space* P *and let* A *be the affine space complementary to* P . *Then* A , *relative to the ternary operator defined in* (4.4), *is an abelian libra.*

Proof: It is evident from the definition of the libra operator that $\lfloor a,b,c \rfloor = \lfloor c,b,a \rfloor$: that the operator is abelian.

Suppose that $a = b$ or that $c = b$. Then we are in the 1-dimensional case and that (2.6.4) holds follows from (2.8) .

Let n be a positive integer and suppose that we have proven that (2.6.4) holds for all subspaces of dimension less than or equal to n . In view of (2.8), we may presume that n is 1 or more. Let the set $\{v,w,x,y,z\}$ be within $A \cap S$ where S is a subspace of P of dimension $n{+}1$ and assume that

$$\lfloor \lfloor v,w,x \rfloor ,y,z \rfloor \neq \lfloor v,w,\lfloor x,y,z \rfloor \rfloor .$$

Let T be a maximal proper subspace of S containing the elements $\lfloor \lfloor v,w,x \rfloor ,y,z \rfloor$ and $\lfloor v,w,\lfloor x,y,z \rfloor \rfloor$. Let p be any element of $(V \cap S) \triangle (V \cap T)$. Let π be the projection of S from p onto T . Evidently

$$\pi(\lfloor\lfloor v,w,x\rfloor,y,z\rfloor) \neq \pi(\lfloor v,w,\lfloor x,y,z\rfloor\rfloor)\ . \tag{1}$$

In view of Lemma I (4.5), we have both

$$\pi(\lfloor\lfloor v,w,x\rfloor,y,z\rfloor) = \lfloor\pi(\lfloor v,w,x\rfloor),\pi(y),\pi(z)\rfloor = \lfloor\lfloor\pi(v),\pi(w),\pi(x)\rfloor,\pi(y),\pi(z)\rfloor \tag{2}$$

and

$$\pi(\lfloor v,w,\lfloor x,y,z\rfloor\rfloor) = \lfloor\pi(v),\pi(w),\pi(\lfloor x,y,z\rfloor)\rfloor = \lfloor\pi(v),\pi(w),\lfloor\pi(x),\pi(y),\pi(z)\rfloor\rfloor\ . \tag{3}$$

Since T is of dimension n , we know that

$$\lfloor\lfloor\pi(v),\pi(w),\pi(x)\rfloor,\pi(y),\pi(z)\rfloor = \lfloor\pi(v),\pi(w),\lfloor\pi(x),\pi(y),\pi(z)\rfloor\rfloor$$

whence from (2) and (3) follows that

$$\pi(\lfloor\lfloor v,w,x\rfloor,y,z\rfloor) = \pi(\lfloor v,w,\lfloor x,y,z\rfloor\rfloor)\ .$$

But this is inconsistent with (1). Thus (2.6.4) holds for all subspaces of dimension n+1. By induction, it follows that A is a libra. Q.E.D.

(4.7) Lemma II *Let M be a meridian and let* {a,b,c,d,e} *be within* M *with* a *and* b *distinct. Then*

$$\boxed{\substack{a\,b\,d\\a\,b\,c}}\circ\boxed{\substack{a\,b\,c\\a\,b\,e}} = \boxed{\substack{a\,b\,c\\a\,b\,e}}\circ\boxed{\substack{a\,b\,d\\a\,b\,c}}\ . \tag{1}$$

Furthermore,

$$\boxed{\substack{a\,b\,d\\a\,b\,c}}(e) = \boxed{\substack{a\,b\,e\\a\,b\,c}}(d)\ . \tag{2}$$

Proof: We have $\boxed{\substack{a\,b\,d\\a\,b\,c}} = \boxed{\substack{a\,c\\b\,d}}\circ\boxed{\substack{a\,c\\b\,c}}$ and $\boxed{\substack{a\,b\,c\\a\,b\,e}} = \boxed{\substack{a\,c\\b\,e}}\circ\boxed{\substack{a\,e\\b\,e}}$. Since by (2.8) $\mathcal{M}_{[a,b]}$ is an abelian libra, we have

$$\boxed{\substack{a\,b\,d\\a\,b\,c}}\circ\boxed{\substack{a\,b\,c\\a\,b\,e}} = \boxed{\substack{a\,c\\b\,d}}\circ\boxed{\substack{a\,c\\b\,c}}\circ\boxed{\substack{a\,c\\b\,e}}\circ\boxed{\substack{a\,e\\b\,e}} = \llbracket\boxed{\substack{a\,c\\b\,d}},\boxed{\substack{a\,c\\b\,c}},\boxed{\substack{a\,c\\b\,e}}\rrbracket\circ\boxed{\substack{a\,e\\b\,e}} = \llbracket\boxed{\substack{a\,c\\b\,e}},\boxed{\substack{a\,c\\b\,c}},\boxed{\substack{a\,c\\b\,d}}\rrbracket\circ\boxed{\substack{a\,e\\b\,e}} = \boxed{\substack{a\,c\\b\,e}}\circ\boxed{\substack{a\,c\\b\,c}}\circ\boxed{\substack{a\,c\\b\,d}}\circ\boxed{\substack{a\,e\\b\,e}} =$$

$$\boxed{\substack{a\,c\\b\,e}}\circ\llbracket\boxed{\substack{a\,c\\b\,c}},\boxed{\substack{a\,c\\b\,d}},\boxed{\substack{a\,e\\b\,e}}\rrbracket = \boxed{\substack{a\,c\\b\,e}}\circ\llbracket\boxed{\substack{a\,e\\b\,e}}\cdot\boxed{\substack{a\,c\\b\,d}},\boxed{\substack{a\,c\\b\,c}}\rrbracket = \boxed{\substack{a\,c\\b\,e}}\circ\boxed{\substack{a\,e\\b\,e}}\circ\boxed{\substack{a\,c\\b\,d}}\circ\boxed{\substack{a\,c\\b\,c}} = \boxed{\substack{a\,b\,c\\a\,b\,e}}\circ\boxed{\substack{a\,b\,d\\a\,b\,c}}\ ,$$

which proves (1).

Noting that $\boxed{\substack{a\,b\,e\\a\,b\,c}}^{-1} = \boxed{\substack{a\,b\,c\\a\,b\,e}}$, we have

$$\boxed{\substack{a\,b\,d\\a\,b\,c}}(e) = \boxed{\substack{a\,b\,e\\a\,b\,c}}\circ\boxed{\substack{a\,b\,e\\a\,b\,c}}^{-1}\circ\boxed{\substack{a\,b\,d\\a\,b\,c}}(e) = \boxed{\substack{a\,b\,e\\a\,b\,c}}\circ\Big(\boxed{\substack{a\,b\,c\\a\,b\,e}}\circ\boxed{\substack{a\,b\,d\\a\,b\,c}}\Big)(e) \overset{\text{by (1)}}{=\!=\!=} \boxed{\substack{a\,b\,e\\a\,b\,c}}\circ\Big(\boxed{\substack{a\,b\,d\\a\,b\,c}}\circ\boxed{\substack{a\,b\,c\\a\,b\,e}}\Big)(e) = \boxed{\substack{a\,b\,e\\a\,b\,c}}(d)$$

which proves (2). Q.E.D.

(4.8) Scalar Multiplication Let P be a finite dimensional projective space of dimension greater than one, and let $\mathcal{P}$ be its homography group of permutations. Let V be a co-point in P and A the affine space complementary to V. Let o be any element of A . The group associated with the libra A , with identity o , has the operation (see (2.6.7))

$$(\forall\ \{a,b\}\subset A)\quad a+b \equiv \lfloor a,o,b\rfloor\ . \tag{1}$$

For {s,t} within A , recall that $\mathfrak{v}(s,t)$ denotes the intersection point of V with the meridian $\overleftrightarrow{s,t}$.

Let the field of throws $\mathfrak{F}$, along with its binary operators + and · , be as in (3.4) . For each element x of A△{o} and each $\mathfrak{r}$ in $\mathfrak{F}$ we define

$$\mathfrak{r}\bullet x \equiv \boxed{\substack{\mathfrak{v}(o,x)\ \ o\ x\\ \bowtie\ \ \ \mathfrak{o}\ \mathfrak{1}}}(\mathfrak{r})\ . \tag{2}$$

It follows from (3.2.6) that,

$$(\forall\, r \in \overleftrightarrow{o,x}) \qquad \mathfrak{r} \bullet x = r \iff \boxed{\begin{smallmatrix} \mathfrak{v}(o,x) & o & x & r \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}} \in \mathfrak{r}\,. \tag{3}$$

For x in $V \cup \{o\}$ we define

$$\mathfrak{r} \bullet x \equiv x\,. \tag{4}$$

(4.9) Intuition We chose the letter V in (4.8) because in the context of an affine space we think of the elements of V as being at "infinity" and therefor "virtual" rather than real. The libra operator applied to a triple {x,y,z} within A can be thought of as producing a fourth point p so that the "line segments" determined by {y,x} , {y,z} , {x,p} , and set {z,p} form a "parallelogram".

By choosing an element o of A to form a group libra associated with A , we are creating a linear space with origin o . Thus we may identify each point x of A with the line segment beginning at o and ending at x . For x in A distinct from o , and $\mathfrak{r}$ an element of $\mathfrak{F}$, the product $\mathfrak{r} \bullet x$ may be thought of as the element of A on the meridian $\overleftrightarrow{o,x}$ of which the associated line segment has "ratio" $\mathfrak{r}$ to the line segment associated with the point x .

(4.10) Theorem III *Let P be a finite dimensional projective space with projective group $\mathcal{P}$, V a co-point of P with affine complement A , and o an element of A . Then, for each $\mathfrak{r}$ in $\mathfrak{W}_P \triangle \{\bowtie,\mathfrak{o}\}$, the function*

$$P \ni x \hookrightarrow \mathfrak{r} \bullet x \in P \quad \textit{is an element of } \mathcal{P}. \tag{1}$$

Proof: Let M be any meridian in P containing o , let ∞ be the intersection point of M with V and let l be any element of M not in $\{o,\infty\}$. Let r be the element of M such that $\boxed{\infty \;\; o \;\; l \;\; r} = \mathfrak{r}$. Then, for each x in M ,

$$\mathfrak{r} \bullet x \overset{\text{by } (4.8.2)}{=\!=\!=} \boxed{\begin{smallmatrix} \infty & o & x \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{r}) = \boxed{\begin{smallmatrix} \infty & o & x \\ \infty & o & l \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \infty & o & l \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{r}) \overset{\text{by } (4.8.2)}{=\!=\!=} \boxed{\begin{smallmatrix} \infty & o & x \\ \infty & o & l \end{smallmatrix}}(\mathfrak{r} \bullet l) \overset{\text{by } (4.8.3)}{=\!=\!=} \boxed{\begin{smallmatrix} \infty & o & x \\ \infty & o & l \end{smallmatrix}}(r) \overset{\text{by } (4.7.2)}{=\!=\!=} \boxed{\begin{smallmatrix} \infty & o & r \\ \infty & o & l \end{smallmatrix}}(x)\,. \tag{2}$$

It follows from (2) that (1) is true if P is 1-dimensional. We shall presume then that the dimension of P is larger than 1 . Letting B be a maximal independent subset of V, we know that B has at least two elements. It follows from (1.8) that $B \cup \{o\}$ is a maximal independent subset of P . Let l be any element of $(B \cup \{o\})^{\square}$: the set $B \cup \{o,l\}$ is a fundamental subset of P . Evidently $B \cup \{o,\mathfrak{r} \bullet l\}$ is a fundamental subset of P as well. It follows from the *fundamental axiom* (1.3.2) that there exists ϕ in $\mathcal{P}$ such that

$$\phi(l) = \mathfrak{r} \bullet l \;\text{ and }\; (\forall\, x \in (B \cup \{o\})) \quad \phi(x) = x\,. \tag{3}$$

We wish to show that

$$(\forall\, x \in A) \qquad \phi(x) = \mathfrak{r} \bullet x\,. \tag{4}$$

Let r and ∞ be as in (2): that is, we are letting M be $\overleftrightarrow{o,l}$. Since ϕ agrees with $\boxed{\begin{smallmatrix} \infty & o & r \\ \infty & o & l \end{smallmatrix}}$ at o , at l and at ∞ , it follows from the *fundamental property* (2.1.3) that they are identical on $\overleftrightarrow{o,l}$. It follows from (2) that (4) holds for x in $\overleftrightarrow{o,l}$.

Suppose then that x is not in $\overleftrightarrow{o,l}$. Let S be any co-point such that $\overleftrightarrow{o,x}$ is within S but l is not in S . Let v be the intersection point of the meridian $\overleftrightarrow{l,x}$ with the co-point V , and let π be the projection from v onto S . It follows from (4.3) that the restriction of π to $\overleftrightarrow{o,l}$ is a meridian isomorphism from $\overleftrightarrow{o,l}$ onto the meridian $\overleftrightarrow{o,x}$. In fact, if w denotes the intersection point of $\overleftrightarrow{o,x}$ with V, we have

$$\pi\big|_{\overleftrightarrow{o,l}} = \boxed{\begin{smallmatrix} w & o & x \\ \infty & o & l \end{smallmatrix}}\,. \tag{5}$$

It follows from (3) that ϕ fixes both w and o , whence follows that ϕ sends the meridian $\overleftrightarrow{o,x}$ onto itself. Since ϕ sends l to r and fixes v , it sends the meridian $\overleftrightarrow{v,l}$ onto the meridian $\overleftrightarrow{v,r}$. Thus x , which is the intersection point of the meridians $\overleftrightarrow{v,l}$ and $\overleftrightarrow{o,w}$, is sent to the intersection point of the meridians $\overleftrightarrow{v,r}$ and $\overleftrightarrow{o,w}$. This intersection point is precisely the projection π from v of r : that is,

$$\phi(\mathrm{x}) = \pi(\mathrm{r}) \overset{\text{by (5)}}{=\!=\!=} \boxed{\begin{smallmatrix} \mathrm{w} & \mathrm{o} & \mathrm{x} \\ \infty & \mathrm{o} & 1 \end{smallmatrix}}(\mathrm{r}) \,. \tag{6}$$

Since $\boxed{\begin{smallmatrix} \infty & \mathrm{o} & 1 & \mathrm{r} \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}}$ is in $\mathfrak{r}$, it follows that $\boxed{\begin{smallmatrix} \mathrm{w} & \mathrm{o} & \mathrm{x} \\ \infty & \mathrm{o} & 1 \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \infty & \mathrm{o} & 1 & \mathrm{r} \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}}$ is in $\mathfrak{r}$ as well. By (6) this last is just $\boxed{\begin{smallmatrix} \mathrm{w} & \mathrm{o} & \mathrm{x} & \phi(\mathrm{x}) \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}}$. It follows from (4.8.3) that (4) holds for this x. Q.E.D.

(4.11) Theorem IV *Let* P *be a finite dimensional projective space,* V *a co-point of* P *with affine complement* A*. Let* ϕ *be any element of* $\mathcal{P}$ *which sends each element of* V *into* V*. Then*

$$(\forall\, \{\mathrm{x,y,z}\} \subset \mathrm{A}) \qquad \phi(\lfloor \mathrm{x,y,z} \rfloor) = \lfloor \phi(\mathrm{x}), \phi(\mathrm{y}), \phi(\mathrm{z}) \rfloor \,. \tag{1}$$

Furthermore, if o *is an element of* A*,* + *is defined as in* (4.8.1)*, and* $\phi(\mathrm{o}) = \mathrm{o}$*, then*

$$(\forall\, \{\mathrm{x,y}\} \subset \mathrm{A}) \qquad \phi(\mathrm{x+y}) = \phi(\mathrm{x}) + \phi(\mathrm{y}) \,. \tag{2}$$

Proof: That (1) holds is a consequence of (4.5) and the definition of the libra operator for A. For (2) we have

$$\phi(\mathrm{x+y}) \overset{\text{by (4.8.1)}}{=\!=\!=} \phi(\lfloor \mathrm{x,o,y} \rfloor) \overset{\text{by (1)}}{=\!=\!=} \lfloor \phi(\mathrm{x}), \phi(\mathrm{o}), \phi(\mathrm{y}) \rfloor \overset{\text{by (4.8.1)}}{=\!=\!=} \phi(\mathrm{x}) + \phi(\mathrm{y}) \,.$$

Q.E.D.

(4.12) Theorem V *Let* P *be a finite dimensional projective space,* V *a co-point of* P *with affine complement* A *and* o *an element of* A*. Let* + *and* $\bullet$ *be as defined in* (4.8)*. Let* + *and* $\cdot$ *be defined as in* (3.4)*. Then* A*, relative to these operations, is a vector space over the throw field* $\mathfrak{F}$ *of* P*.*

Furthermore, if ϕ *is an element of* $\mathcal{P}$ *which fixes* o *and sends elements of* V *into* V*, then*

$$(\forall\, \mathrm{r} \in \mathfrak{W}_{\mathrm{P}} \text{ and } \mathrm{x} \in \mathrm{A}) \qquad \mathfrak{r} \cdot (\phi(\mathrm{x})) = \phi(\mathfrak{r} \cdot \mathrm{x}) \,. \tag{1}$$

Proof: It follows from (4.6) that A is an abelian group with identity o relative to the binary operator +. We need to prove the interactions between $\mathfrak{F}$ and A. Let x and y be elements of $\mathrm{A} \triangle \{\mathrm{o}\}$ and let $\mathfrak{r}$ and $\mathfrak{s}$ be elements of $\mathfrak{F} \triangle \{\mathfrak{o}\}$.

(1): $\mathfrak{r} \bullet (\mathfrak{s} \bullet \mathrm{x}) = (\mathfrak{r} \cdot \mathfrak{s}) \bullet \mathrm{x}$. Let F be the field $\overleftrightarrow{\mathrm{o,x}} \triangle \{\mathfrak{v}(\mathrm{o,x})\}$ with additive identity o and multiplicative identity x. Let r and s be the elements of $\overleftrightarrow{\mathrm{o,x}}$ such that

$$\boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} & \mathrm{r} \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}} \in \mathfrak{r} \quad \text{and} \quad \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} & \mathrm{s} \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}} \in \mathfrak{s} \,.$$

We have

$$(\mathfrak{r} \cdot \mathfrak{s}) \bullet \mathrm{x} \overset{\text{by (4.8.2)}}{=\!=\!=} \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{r} \cdot \mathfrak{s}) \overset{\text{by (3.6)}}{=\!=\!=} \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{r}) \cdot \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{s}) \overset{\text{by (4.8.3)}}{=\!=\!=} \mathrm{r{\cdot}s} \overset{\text{by (2.16.1)}}{=\!=\!=}$$

$$\frac{\mathrm{r{\cdot}s}}{\mathrm{x}} = \frac{\mathrm{s}}{\mathrm{x}} \cdot \mathrm{r} \overset{\text{by (2.16.1)}}{=\!=\!=} \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{s} \\ \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \end{smallmatrix}}(\mathrm{r}) \overset{\text{by (4.8.3)}}{=\!=\!=} \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{s} \\ \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \end{smallmatrix}} \circ \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{r}) =$$

$$\boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{s} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{r}) \overset{\text{by (4.8.2)}}{=\!=\!=} \mathfrak{r} \bullet \mathrm{s} \overset{\text{by (4.8.3)}}{=\!=\!=} \mathfrak{r} \bullet \Big(\boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{s}) \Big) \overset{\text{by (4.8.2)}}{=\!=\!=} \mathfrak{r} \bullet (\mathfrak{s} \bullet \mathrm{x}) \,.$$

(2): $\mathfrak{1} \bullet \mathrm{x} = \mathrm{x}$. Evidently $\boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} & \mathrm{x} \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \end{smallmatrix}}$ is in $\mathfrak{1}$ so we have

$$\mathfrak{1} \bullet \mathrm{x} \overset{\text{by (4.8.2)}}{=\!=\!=} \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{1}) \overset{\text{by (4.8.3)}}{=\!=\!=} \mathrm{x} \,.$$

(3): $(\mathfrak{r} + \mathfrak{s}) \bullet \mathrm{x} = (\mathfrak{r} \bullet \mathrm{x}) + (\mathfrak{s} \bullet \mathrm{x})$. We have

$$(\mathfrak{r} + \mathfrak{s}) \bullet \mathrm{x} \overset{\text{by (4.8.2)}}{=\!=\!=} \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{r} + \mathfrak{s}) \overset{\text{by (3.6)}}{=\!=\!=} \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{r}) + \boxed{\begin{smallmatrix} \mathfrak{v}(\mathrm{o,x}) & \mathrm{o} & \mathrm{x} \\ \bowtie & \mathfrak{o} & \mathfrak{1} \end{smallmatrix}}(\mathfrak{s}) \overset{\text{by (4.8.2)}}{=\!=\!=} (\mathfrak{r} \bullet \mathrm{x}) + (\mathfrak{s} \bullet \mathrm{x}) \,.$$

(4): $\mathfrak{r} \cdot (\mathrm{x+y}) = (\mathfrak{r} \cdot \mathrm{x}) + (\mathfrak{r} \cdot \mathrm{y})$. This follows from (4.10) and (4.11.2).

Finally, we shall prove (1). We have, for x distinct from o,

$$\mathfrak{r}\cdot(\phi(\mathrm{x}))\ \underset{}{\overset{\text{by (4.8.2)}}{=\!=\!=}}\ \boxed{\begin{array}{ccc}\mathfrak{v}(\mathrm{o},\phi(\mathrm{x})) & \mathrm{o} & \phi(\mathrm{x})\\ \bowtie & 0 & 1\end{array}}(\mathfrak{r}) = \phi\circ\boxed{\begin{array}{ccc}\mathfrak{v}(\mathrm{o},\mathrm{x}) & \mathrm{o} & \mathrm{x}\\ \bowtie & 0 & 1\end{array}}(\mathfrak{r})\ \overset{\text{by (4.8.2)}}{=\!=\!=}\ \phi(\mathfrak{r}\cdot\mathrm{x})\,.$$

Q.E.D.

(4.13) Isomorphisms Each meridian M is a meridian and the meridian group is

$$\{\theta\big|_{\mathrm{M}}:\ \theta\in\mathcal{P}\ \text{ and }\ (\forall\,\mathrm{x}\in\mathrm{M})\ \ \theta(\mathrm{x})\in\mathrm{M}\}\,. \tag{1}$$

From the point of view of the Erlanger Programm, a **projective isomorphism** ω from one projective space P onto another Q is a bijection of P onto Q such that

$$\mathcal{P} = \{\omega^{-1}\circ\theta\circ\omega:\ \ \theta \text{ is a homography of Q}\}\,. \tag{2}$$

Of course, when P equals Q, we speak of a projective **automorphism**. Some automorphisms of P may not be homographies,[9] that is, not elements of $\mathcal{P}$. This begs the question what extra condition must be applied to an isomorphism to obviate this.

We shall say that a projective isomorphism $\omega\big|\mathrm{P}\hookrightarrow\mathrm{Q}$ is **homographic** provided that

$$(\forall\,\mathfrak{w}\in\mathfrak{W}_{\mathrm{Q}})\ \ \{\boxed{\begin{array}{cccc}\mathrm{w}&\mathrm{x}&\mathrm{y}&\mathrm{z}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{array}}:\ \omega\circ\boxed{\begin{array}{cccc}\mathrm{w}&\mathrm{x}&\mathrm{y}&\mathrm{z}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{array}}\in\mathfrak{w}\}\in\mathfrak{W}_{\mathrm{P}}\,. \tag{3}$$

That is, ω is homographic if it preserves throws.

(4.14) Theorem VI *A projective space automorphism of a finite dimensional projective space is homographic if, and only if, it is a homography.*

Proof: That a homography is homographic follows from the definition (3.1.7) of the equivalence relation $\approx$.

Let $\omega\big|\mathrm{P}\hookrightarrow\mathrm{P}$ be a homographic projective automorphism. Let F be a fundamental subset of P. By *fundamental axiom* (1.3.2) there exists a unique element ψ of $\mathcal{P}$ which agrees with ω on F. Let θ be the composition $\psi^{-1}\circ\omega$. Evidently θ is a homographic automorphism which agrees with the identity mapping on P everywhere on F. To complete the proof of (4.14), it will suffice to show that

$$\theta \text{ fixes each point in P}\,. \tag{1}$$

Let M be any meridian within P on which θ fixes each element of a tripleton {a,b,c}. For each x in M, we have

$$\boxed{\begin{array}{cccc}\mathrm{a}&\mathrm{b}&\mathrm{c}&\mathrm{x}\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{array}} \approx \boxed{\begin{array}{cccc}\theta(\mathrm{a})&\theta(\mathrm{b})&\theta(\mathrm{c})&\theta(\mathrm{x})\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{array}} = \boxed{\begin{array}{cccc}\mathrm{a}&\mathrm{b}&\mathrm{c}&\theta(\mathrm{x})\\ \heartsuit&\spadesuit&\clubsuit&\diamondsuit\end{array}}\,,$$

which implies that $\mathrm{x}=\theta(\mathrm{x})$. We have shown that

$$\theta \text{ agrees with the identity } \iota \text{ on each meridian if it fixes three or more points of that meridian.} \tag{2}$$

Let X be the set of all the points of P which are fixed by θ. Let n be the dimension of P: thus F has cardinality n+2. If p is an element of F, it follows from (1.9) that, for each $\mathrm{a}\in(\mathrm{F}\triangle\{\mathrm{p}\})$, the meridian $\overleftrightarrow{\mathrm{a,p}}$ intersects $(\mathrm{F}\triangle\{\mathrm{a,p}\})^{\square}$ at some point c. Since $(\mathrm{F}\triangle\{\mathrm{a,p}\})^{\square}$ is sent to itself by θ and $\overleftrightarrow{\mathrm{a,p}}$ is sent to itself by θ, it follows that c must be fixed by θ. We have just shown that

$$\mathrm{X}\cap\left(\mathrm{S}^{\square}\right)\neq\varnothing \ \text{ for any subset S of F of cardinality n}\,. \tag{3}$$

Taking each of the subsets S of (3) in turn, and applying the above argument to it, we can show that (3) holds for all subsets S of cardinality n−1. Continuing in this way we eventually obtain that

[9] In the case of the complex meridian, the complex plane with a point "at infinity" attached, complex conjugation is an automorphism but not a homography.

$$X \cap \left(S^{\square}\right) \neq \emptyset \text{ for any subset S of F of any cardinality greater than } 1\,.$$

This, for the case of cardinality 2 , together with (2), implies that

$$(\forall \{a,b\} \text{ a doubleton within F}) \quad \overleftrightarrow{a,b} \subset X\,.$$

Suppose that m is any positive integer between 1 and n+1 and that we have shown that,

$$(\forall\, S \subset F \text{ of cardinality m}) \quad \boxed{S} \subset X\,. \tag{4}$$

Let T be a subset of F of cardinality m+1 . It follows from (4) that the facade (see (1.6)) of T is a subset of X . Thus, if T were not within X , there would be a point x of $T^{\square}$ not in X . Let u and v be distinct elements of T . It would then follow from (1.11.1) that there would exist an element c of $(T \triangle \{u\})^{\square}$ such that x is in $\overleftrightarrow{u,c}$ and an element d of $(T \triangle \{v\})^{\square}$ such that x is in $\overleftrightarrow{u,d}$. It would follow from (4) that {u,v,c,d} would be within X . Consequently θ would send the meridian $\overleftrightarrow{u,c}$ onto itself and the meridian $\overleftrightarrow{v,d}$ onto itself. Thus the intersection point x of those two meridians would be fixed by θ : an absurdity. We have shown that (4) also holds for sets S of cardinality m+1 . By induction it will hold when m equals n+1 , and in this case $\boxed{S} = P$. Thus (1) must hold. Q.E.D.

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