Natural Deduction for Propositional Logic

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1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
In our examples, we (informally) infer new sentences.
In natural deduction, we have a collection of proof rules.
  - These proof rules allow us to infer new sentences logically followed from existing ones.
Supose we have a set of sentences: $\phi_1, \phi_2, \ldots, \phi_n$ (called premises), and another sentence $\psi$ (called a conclusion).
The notation
  $$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$
is called a sequent.
A sequent is valid if a proof (built by the proof rules) can be found.
We will try to build a proof for our examples. Namely,
  $$p \land \neg q \implies r, \neg r, p \vdash q.$$
Suppose we want to prove a conclusion $\phi \land \psi$. What do we do?
- Of course, we need to prove both $\phi$ and $\psi$ so that we can conclude $\phi \land \psi$.

Hence the proof rule for conjunction is

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad \land i
\]

- Note that premises are shown above the line and the conclusion is below. Also, $\land i$ is the name of the proof rule.
- This proof rule is called “conjunction-introduction” since we introduce a conjunction ($\land$) in the conclusion.
For each connective, we have introduction proof rule(s) and also elimination proof rule(s).

Suppose we want to prove a conclusion $\phi$ from the premise $\phi \land \psi$. What do we do?
- We don’t do anything since we know $\phi$ already!

Here are the elimination proof rules:

$$
\frac{\phi \land \psi}{\phi} \quad ^{\land e_1} \\
\frac{\phi \land \psi}{\psi} \quad ^{\land e_2}
$$

The rule $^{\land e_1}$ says: if you have a proof for $\phi \land \psi$, then you have a proof for $\phi$ by applying this proof rule.

Why do we need two rules?
- Because we want to manipulate syntax only.
Example

Prove $p \land q, r \vdash q \land r$.

Proof.

We are looking for a proof of the form:

$$
\begin{array}{c}
p \land q \\
\vdots \\
q \land r
\end{array}
$$
Example

Prove $p \land q, r \vdash q \land r$.

Proof.

We are looking for a proof of the form:

\[
\begin{array}{c}
p \land q \\
q \land e_2 \\
\hline
r \\
\hline
q \land r \land i
\end{array}
\]

We will write proofs in lines:

\begin{align*}
1 & \quad p \land q \quad \text{premise} \\
2 & \quad r \quad \text{premise} \\
3 & \quad q \quad \land e_2 \; 1 \\
4 & \quad q \land r \quad \land i \; 3, 2
\end{align*}
Suppose we want to prove $\phi$ from a proof for $\neg\neg\phi$. What do we do?

- There is no difference between $\phi$ and $\neg\neg\phi$. The same proof suffices!

Hence we have the following proof rules:

- $\frac{\phi}{\neg\neg\phi}$ $\neg\neg i$
- $\frac{\neg\neg\phi}{\phi}$ $\neg\neg e$
Example

Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$.

Proof.

We are looking for a proof like:

```
p  \neg\neg(q \land r)
    ...
  \neg\neg p \land r
```
Example

Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$.

Proof.

We are looking for a proof like:

$$
\begin{array}{c}
p \\
\hline
\neg\neg p \\
\hline
\neg\neg q \land r \\
\hline
q \land r \\
\hline
\neg\neg r \\
\hline
\neg\neg p \land r \\
\hline
\end{array}
$$

$\neg\neg e$
Example

Prove $p, \neg \neg (q \land r) \vdash \neg \neg p \land r$.

Proof.

We are looking for a proof like:

1. $p$  \hspace{1cm} \text{premise}
2. $\neg \neg (q \land r)$  \hspace{1cm} \text{premise}
3. $\neg \neg p$  \hspace{1cm} $\neg \neg i$ 1
4. $q \land r$  \hspace{1cm} $\neg \neg e$ 2
5. $r$  \hspace{1cm} $\land e_2$ 4
6. $\neg \neg p \land r$  \hspace{1cm} $\land i$ 3, 5
Suppose we want to prove $\psi$ from proofs for $\phi$ and $\phi \implies \psi$. What do we do?

- We just put the two proofs for $\phi$ and $\phi \implies \psi$ together.

Here is the proof rule:

\[
\frac{\phi \quad \phi \implies \psi}{\psi} \implies e
\]

This proof rule is also called \textit{modus ponens}.

Here is another proof rule related to implication:

\[
\frac{\phi \implies \psi \quad \neg \psi}{\neg \phi} \quad MT
\]

This proof rule is called \textit{modus tollens}.
Example

Prove $p \implies (q \implies r), p, \neg r \vdash \neg q$.

Proof.

1. $p \implies (q \implies r)$ premise
2. $p$ premise
3. $\neg r$ premise
4. $q \implies r$ $\implies e \ 2, \ 1$
5. $\neg q$ $MT \ 4, \ 3$
Suppose we want to prove $\phi \implies \psi$. What do we do?

- We assume $\phi$ to prove $\psi$. If succeed, we conclude $\phi \implies \psi$ without any assumption.
- Note that $\phi$ is added as an assumption and then removed so that $\phi \implies \psi$ does not depend on $\phi$.

We use “box” to simulate this strategy.

Here is the proof rule:

$$\frac{\phi \quad \vdots \quad \psi}{\phi \implies \psi} \implies i$$

At any point in a box, you can only use a sentence $\phi$ before that point. Moreover, no box enclosing the occurrence of $\phi$ has been closed.
Example

Prove $\neg q \implies \neg p \vdash p \implies \neg \neg q$.

Proof.

\[
\begin{array}{c}
\neg q \implies \neg p \\
\hline
\neg p \\
\hline
\neg \neg q \\
\hline
p \\
\implies
\end{array}
\]

\[
\begin{array}{c}
\hline
\neg \neg p \\
\hline
\neg \neg q \\
\hline
MT, 1, 3 \\
\hline
i, 2-4
\end{array}
\]

1. $\neg q \implies \neg p$ premise
2. $p$ assumption
3. $\neg \neg p$ $\neg \neg i$ 2
4. $\neg \neg q$ $MT$ 1, 3
5. $p \implies \neg \neg q$ $\implies i$ 2-4
Theorems

Example
Prove $\vdash p \implies p$.

Proof.

1. $p$ assumption
2. $p \implies p \implies i 1 - 1$

In the box, we have $\phi \equiv \psi \equiv p$.

Definition
A sentence $\phi$ such that $\vdash \phi$ is called a theorem.
Example

Prove \( p \land q \implies r \vdash p \implies (q \implies r) \).

Proof.

1. \( p \land q \implies r \) \hspace{1cm} \text{premise}
2. \( p \) \hspace{1cm} \text{assumption}
3. \( q \) \hspace{1cm} \text{assumption}
4. \( p \land q \) \hspace{1cm} \land \text{i} 2, 3
5. \( r \) \hspace{1cm} \implies \text{e} 4, 1
6. \( q \implies r \) \hspace{1cm} \implies \text{i} 3-5
7. \( p \implies (q \implies r) \) \hspace{1cm} \implies \text{i} 2-6
Suppose we want to prove $\phi \lor \psi$. What do we do?

- We can either prove $\phi$ or $\psi$.

Here are the proof rules:

\[
\begin{align*}
\frac{\phi}{\phi \lor \psi} \quad &\lor i_1 \\
\frac{\psi}{\phi \lor \psi} \quad &\lor i_2
\end{align*}
\]

- Note the symmetry with $\land e_1$ and $\land e_2$.

\[
\begin{align*}
\frac{\phi \land \psi}{\phi} \quad &\land e_1 \\
\frac{\phi \land \psi}{\psi} \quad &\land e_2
\end{align*}
\]

- Can we have a corresponding symmetric elimination rule for disjunction? Recall

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad \land i
\]
Suppose we want to prove $\chi$ from $\phi \lor \psi$. What do we do?

- We assume $\phi$ to prove $\chi$ and then assume $\psi$ to prove $\chi$.
- If both succeed, $\chi$ is proved from $\phi \lor \psi$ without assuming $\phi$ and $\psi$.

Here is the proof rule:

\[
\begin{array}{c}
\phi \lor \psi \\
\hline
\phi \\
\vdots \\
\chi \\
\psi \\
\vdots \\
\chi \\
\hline
\chi
\end{array}
\]

\[\lor e\]

In addition to nested boxes, we may have parallel boxes in our proofs.
Recall that our syntax does not admit commutativity.

Example

Prove $p \lor q \vdash q \lor p$.

Proof.

\[
\begin{array}{c}
p \lor q \\
\hline
q \lor p
\end{array}
\]

\[
\begin{array}{c}
p \\
q \lor p \quad \lor_i_2
\end{array} \\
\hline
q \lor p \quad \lor_i_1
\]

\[
\begin{array}{c}
p \lor q \\
\hline
q \lor p
\end{array}
\]

1. $p \lor q$ premise
2. $p$ assumption
3. $q \lor p$ $\lor_i_2$ 2
4. $q$ assumption
5. $q \lor p$ $\lor_i_1$ 4
6. $q \lor p$ $\lor_e$ 1, 2-3, 4-5
Example

Prove $q \implies r \vdash p \lor q \implies p \lor r$.

Proof.

1. $q \implies r$ premise
2. $p \lor q$ assumption
3. $p$ assumption
4. $p \lor r \lor i_{1} 3$
5. $q$ assumption
6. $r \implies e 5, 1$
7. $p \lor r \lor i_{2} 6$
8. $p \lor r \lor e 2, 3-4, 5-7$
9. $p \lor q \implies p \lor r \implies i 2-8$
Example

Prove $p \land (q \lor r) \vdash (p \land q) \lor (p \land r)$.

Proof.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p \land (q \lor r)$</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>$p$</td>
<td>$\land e_1 \ 1$</td>
</tr>
<tr>
<td>3</td>
<td>$q \lor r$</td>
<td>$\land e_2 \ 1$</td>
</tr>
<tr>
<td>4</td>
<td>$q$</td>
<td>assumption</td>
</tr>
<tr>
<td>5</td>
<td>$p \land q$</td>
<td>$\land i \ 2, 4$</td>
</tr>
<tr>
<td>6</td>
<td>$(p \land q) \lor (p \land r)$</td>
<td>$\lor i_1 \ 5$</td>
</tr>
<tr>
<td>7</td>
<td>$r$</td>
<td>assumption</td>
</tr>
<tr>
<td>8</td>
<td>$p \land r$</td>
<td>$\land i \ 2, 7$</td>
</tr>
<tr>
<td>9</td>
<td>$(p \land q) \lor (p \land r)$</td>
<td>$\lor i_2 \ 8$</td>
</tr>
<tr>
<td>10</td>
<td>$(p \land q) \lor (p \land r)$</td>
<td>$\lor e \ 3, 4-6, 7-9$</td>
</tr>
</tbody>
</table>
Example

Prove \((p \land q) \lor (p \land r) \vdash p \land (q \lor r)\).

Proof.

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((p \land q) \lor (p \land r))</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>(p \land q)</td>
<td>assumption</td>
</tr>
<tr>
<td>3</td>
<td>(p)</td>
<td>(\land e_1) 2</td>
</tr>
<tr>
<td>4</td>
<td>(q)</td>
<td>(\land e_2) 2</td>
</tr>
<tr>
<td>5</td>
<td>(q \lor r)</td>
<td>(\lor i_1) 4</td>
</tr>
<tr>
<td>6</td>
<td>(p \land (q \lor r))</td>
<td>(\land i) 3, 5</td>
</tr>
<tr>
<td>7</td>
<td>(p \land r)</td>
<td>assumption</td>
</tr>
<tr>
<td>8</td>
<td>(p)</td>
<td>(\land e_1) 7</td>
</tr>
<tr>
<td>9</td>
<td>(r)</td>
<td>(\land e_2) 7</td>
</tr>
<tr>
<td>10</td>
<td>(q \lor r)</td>
<td>(\lor i_2) 9</td>
</tr>
<tr>
<td>11</td>
<td>(p \land (q \lor r))</td>
<td>(\land i) 8, 10</td>
</tr>
<tr>
<td>12</td>
<td>(p \land (q \lor r))</td>
<td>(\lor e) 1, 2-6, 7-11</td>
</tr>
</tbody>
</table>
Contradiction

Definition

Contradictions are sentences of the form \( \phi \land \neg \phi \) or \( \neg \phi \land \phi \).

- Examples:
  - \( p \land \neg p, \neg(p \lor q \implies r) \land (p \lor q \implies r) \).
- Logically, any sentence can be proved from a contradiction.
  - If \( 0 = 1 \), then \( 100 \neq 100 \).
- Particularly, if \( \phi \) and \( \psi \) are contradictions, we have \( \phi \vdash \psi \).
  - \( \phi \vdash \psi \) means \( \phi \vdash \psi \) and \( \psi \vdash \phi \) (called provably equivalent).
- Since all contradictions are equivalent, we will use the symbol \( \bot \) (called “bottom”) for them.
- We are now ready to discuss proof rules for negation.
Since any sentence can be proved from a contradiction, we have

$$
\frac{\bot}{\phi} \quad \bot e
$$

When both $\phi$ and $\neg \phi$ are proved, we have a contradiction.

$$
\frac{\phi \quad \neg \phi}{\bot} \quad \neg e
$$

The proof rule could be called $\bot i$. We use $\neg e$ because it eliminates a negation.
Example

Prove $\neg p \lor q \vdash p \implies q$.

**Proof.**

| 1 | $\neg p \lor q$ | premise |
| 2 | $\neg p$ | assumption |
| 3 | $p$ | assumption |
| 4 | $\bot$ | $\neg e$ 3, 2 |
| 5 | $q$ | $\bot e$ 4 |
| 6 | $p \implies q$ | $\implies i$ 3-5 |
| 7 | $q$ | assumption |
| 8 | $p$ | assumption |
| 9 | $q$ | copy 7 |
| 10 | $p \implies q$ | $\implies i$ 8-9 |
| 11 | $p \implies q$ | $\lor e$ 1, 2-6, 7-10 |
Suppose we want to prove $\neg \phi$. What do we do?

- We assume $\phi$ and try to prove a contradiction. If succeed, we prove $\neg \phi$.

Here is the proof rule:

\[
\begin{array}{c}
\phi \\
\hline \\
\bot \\
\hline \\
\neg \phi \\
\end{array}
\]

$\neg i$
Example

Prove $p \rightarrow q$, $p \rightarrow \neg q \vdash \neg p$.

Proof.

1. $p \rightarrow q$ premise
2. $p \rightarrow \neg q$ premise
3. $p$ assumption
4. $q \rightarrow e\ 3,\ 1$
5. $\neg q \rightarrow e\ 3,\ 2$
6. $\bot \rightarrow \neg e\ 4,\ 5$
7. $\neg p \rightarrow \neg i\ 3-6$
**Example**

Prove \( p \land \neg q \rightarrow r, \neg r, p \vdash q \).

**Proof.**

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( p \land \neg q \rightarrow r )</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>( \neg r )</td>
<td>premise</td>
</tr>
<tr>
<td>3</td>
<td>( p )</td>
<td>premise</td>
</tr>
<tr>
<td>4</td>
<td>( \neg q )</td>
<td>assumption</td>
</tr>
<tr>
<td>5</td>
<td>( p \land \neg q )</td>
<td>( \land ) ( i ) 3, 4</td>
</tr>
<tr>
<td>6</td>
<td>( r )</td>
<td>( \rightarrow ) e 5, 1</td>
</tr>
<tr>
<td>7</td>
<td>( \bot )</td>
<td>( \neg ) e 6, 2</td>
</tr>
<tr>
<td>8</td>
<td>( \neg \neg q )</td>
<td>( \neg ) i 4-7</td>
</tr>
<tr>
<td>9</td>
<td>( q )</td>
<td>( \neg \neg ) e 8</td>
</tr>
</tbody>
</table>
Derived Rules

- Some rules can actually be derived from others.

**Examples**

Prove $p \implies q, \neg q \vdash \neg p$ (modus tollens).

**Proof.**

1. $p \implies q$ premise
2. $\neg q$ premise
3. $p$ assumption
4. $q \implies e \ 3, \ 1$
5. $\bot \neg e \ 4, \ 2$
6. $\neg p \neg i \ 3-5$
Examples

Prove $p \vdash \neg
\neg p$ ($\neg
\neg i$)

Proof.

1. $p$ premise
2. $\neg p$ assumption
3. $\bot$ $\neg e$ 1, 2
4. $\neg
\neg p$ $\neg i$ 2-3

- These rules can be replaced by their proofs and are not necessary.
  - They are just macros to help us write shorter proofs.
Example

Prove $\neg p \implies \bot 
\vdash \neg p$ (RAA).

Proof.

1. $\neg p \implies \bot$  
   premise
2. $\neg p$  
   assumption
3. $\bot$  
   $\implies e \ 2, \ 1$
4. $\neg \neg p$  
   $\neg i \ 2-3$
5. $p$  
   $\neg \neg e \ 4$
Example

Prove $\vdash p \lor \neg p$.

Proof.

1. $\neg (p \lor \neg p)$ assumption
2. $p$ assumption
3. $p \lor \neg p \lor i_1 \ 2$
4. $\bot \neg e \ 3, 1 \lor i_2$
5. $\neg p \neg i \ 2-4$
6. $p \lor \neg p \lor i_2 \ 5$
7. $\bot \neg e \ 6, 1 \lor i_2$
8. $\neg \neg (p \lor \neg p) \neg i \ 1-7$
9. $p \lor \neg p \neg \neg e \ 8$
Proof Rules for Natural Deduction (Summary)

Conjunction ($\land$)

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad \land i
\]

Disjunction ($\lor$)

\[
\frac{\phi}{\phi \lor \psi} \quad \lor i_1 \quad \frac{\psi}{\phi \lor \psi} \quad \lor i_2
\]

Implication ($\implies$)

\[
\frac{\phi}{\phi \implies \psi} \implies i
\]

\[
\frac{\phi \quad \phi \implies \psi}{\psi} \implies e
\]
Proof Rules for Natural Deduction (Summary)

Negation ($\neg$)

\[
\begin{array}{c}
\phi \\
\vdots \\
\bot \\
\hline
\neg \phi \quad \neg i
\end{array}
\]

Contradiction ($\bot$)

(no introduction rule)

\[
\begin{array}{c}
\bot \\
\hline
\phi \quad \bot e
\end{array}
\]

Double negation ($\neg\neg$)

(no introduction rule)

\[
\begin{array}{c}
\neg\neg \phi \\
\hline
\phi \quad \neg\neg e
\end{array}
\]
Useful Derived Proof Rules

\[ \phi \rightarrow \psi \quad \neg \psi \]

\[ \neg \phi \]

\[ \phi \rightarrow \psi \quad \neg \psi \]

\[ \neg \phi \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \neg \phi \quad \neg \neg \]

\[ \neg \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]

\[ \vdots \]

\[ \bot \]

\[ \phi \]

\[ \neg \phi \quad \neg \neg \]

\[ \phi \rightarrow \neg \neg \phi \]

\[ \phi \neg \neg \phi \]

\[ \neg \]
Provable Equivalence

- Recall $p \vdash q$ means $p \vdash q$ and $q \vdash p$.
- Here are some provably equivalent sentences:

\[
\begin{align*}
\neg (p \land q) & \iff \neg q \lor \neg p \\
\neg (p \lor q) & \iff \neg q \land \neg p \\
p \implies q & \iff \neg q \implies \neg p \\
p \implies q & \iff \neg p \lor q \\
p \land q & \implies p \iff r \lor \neg r \\
p \land q & \implies r \iff p \implies (q \implies r)
\end{align*}
\]

- Try to prove them.
Proof by Contradiction

- Although it is very useful, the proof rule RAA is a bit puzzling.

\[
\begin{array}{c}
\neg \phi \\
\vdots \\
\bot \\
\hline \\
\phi \\
\end{array}
\]

RAA

- Instead of proving \( \phi \) directly, the proof rule allows indirect proofs.
  - If \( \neg \phi \) leads to a contradiction, then \( \phi \) must hold.
- Note that indirect proofs are not “constructive.”
  - We do not show why \( \phi \) holds; we only know \( \neg \phi \) is impossible.
- In early 20th century, some logicians and mathematicians chose not to prove indirectly. They are intuitionistic logicians or mathematicians.
- For the same reason, intuitionists also reject

\[
\begin{align*}
\phi \lor \neg \phi & \quad \text{LEM} \\
\neg \neg \phi & \quad \neg \neg \text{e}
\end{align*}
\]
Proof by Contradiction

**Theorem**

There are \( a, b \in \mathbb{R} \setminus \mathbb{Q} \) such that \( a^b \in \mathbb{Q} \).

**Proof.**

Let \( b = \sqrt{2} \). There are two cases:

1. If \( b^b \in \mathbb{Q} \), we are done since \( \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \).
2. If \( b^b \notin \mathbb{Q} \), choose \( a = b^b = \sqrt{2}^{\sqrt{2}} \). Then \( a^b = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2 \).

Since \( \sqrt{2}^{\sqrt{2}}, \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \), we are done.

An intuitionist would criticize the proof since it does not tell us what \( a, b \) give \( a^b \in \mathbb{Q} \).

- We know \((a, b)\) is either \((\sqrt{2}, \sqrt{2})\) or \((\sqrt{2}^{\sqrt{2}}, \sqrt{2})\).
1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
A well-formed formula is constructed by applying the following rules finitely many times:

- **atom**: Every propositional atom $p, q, r, \ldots$ is a well-formed formula;
- **$\neg$**: If $\phi$ is a well-formed formula, so is $(\neg \phi)$;
- **$\land$**: If $\phi$ and $\psi$ are well-formed formulae, so is $(\phi \land \psi)$;
- **$\lor$**: If $\phi$ and $\psi$ are well-formed formulae, so is $(\phi \lor \psi)$;
- **$\rightarrow$**: If $\phi$ and $\psi$ are well-formed formulae, so is $(\phi \rightarrow \psi)$.

More compactly, well-formed formulae are defined by the following grammar in Backus Naur form (BNF):

$$
\phi ::= p \mid (\neg \phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow \phi)
$$
Inversion Principle

- How do we check if $(((\neg p) \land q) \implies (p \land (q \lor (\neg r))))$ is well-formed?

- Although a well-formed formula needs five grammar rules to construct, the construction process can always be inverted.
  - This is called inversion principle.

To show $(((\neg p) \land q) \implies (p \land (q \lor (\neg r))))$ is well-formed, we need to show both $((\neg p) \land q)$ and $(p \land (q \lor (\neg r)))$ are well-formed.

- To show $((\neg p) \land q)$ is well-formed, we need to show both $(\neg p)$ and $q$ are well-formed.
  - $q$ is well-formed since it is an atom.

- To show $(\neg p)$ is well-formed, we need to show $p$ is well-formed.
  - $p$ is well-formed since it is an atom.

- Similarly, we can show $(p \land (q \lor (\neg r)))$ is well-formed.
The easiest way to decide whether a formula is well-formed is perhaps by drawing its parse tree.
Subformulae

- Given a well-formed formula, its subformulae are the well-formed
  formulae corresponding to its parse tree.

- For instance, the subformulae of the well-formed formulae
  \(((\neg p) \land q) \implies (p \land (q \lor \neg r)))\) are

  \[
  p, q, r, \neg p, \neg r, ((\neg p) \land q), (q \lor \neg r), (p \land (q \lor \neg r)), (((\neg p) \land q) \implies (p \land (q \lor \neg r)))
  \]
Outline

1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
We have developed a calculus to determine whether \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) is valid.

- That is, from the premises \( \phi_1, \phi_2, \ldots, \phi_n \), we can conclude \( \psi \).
- Our calculus is syntactic. It depends on the syntactic structures of \( \phi_1, \phi_2, \ldots, \phi_n \), and \( \psi \).

We will introduce another relation between premises \( \phi_1, \phi_2, \ldots, \phi_n \) and a conclusion \( \psi \).

\[
\phi_1, \phi_2, \ldots, \phi_n \models \psi.
\]

- The new relation is defined by ‘truth values’ of atomic formulae and the semantics of logical connectives.
Definition
The set of truth values is \{F, T\} where F represents ‘false’ and T represents ‘true.’

Definition
A valuation or model of a formula \( \phi \) is an assignment from each proposition atom in \( \phi \) to a truth value.
## Truth Values of Formulae

### Definition

Given a valuation of a formula \( \phi \), the truth value of \( \phi \) is defined inductively by the following truth tables:

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \psi )</th>
<th>( \phi \land \psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \psi )</th>
<th>( \phi \lor \psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \psi )</th>
<th>( \phi \iff \psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \neg \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \top )</th>
<th>( \bot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>
Example

- \( \phi \land \psi \) is T when \( \phi \) and \( \psi \) are T.
- \( \phi \lor \psi \) is F when \( \phi \) or \( \psi \) is T.
- \( \bot \) is always F; \( \top \) is always T.
- \( \phi \implies \psi \) is T when \( \phi \) “implies” \( \psi \).

Example

Consider the valuation \( \{ q \mapsto T, p \mapsto F, r \mapsto F \} \) of \((q \land p) \implies r\). What is the truth value of \((q \land p) \implies r\)?

Proof.

Since the truth values of \( q \) and \( p \) are T and F respectively, the truth value of \( q \land p \) is F. Moreover, the truth value of \( r \) is F. The truth value of \((q \land p) \implies r\) is T.
Truth Tables for Formulae

Given a formula $\phi$ with propositional atoms $p_1, p_2, \ldots, p_n$, we can construct a truth table for $\phi$ by listing $2^n$ valuations of $\phi$.

Example

Find the truth table for $(p \implies \neg q) \implies (q \lor \neg p)$.

Proof.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$p \implies \neg q$</th>
<th>$q \lor \neg p$</th>
<th>$(p \implies \neg q) \implies (q \lor \neg p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
1. Natural Deduction

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Validity of Sequent Revisited

- Informally \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) is valid if we can derive \( \psi \) with assumptions \( \phi_1, \phi_2, \ldots, \phi_n \).
  - We have formalized “deriving \( \psi \) with assumptions \( \phi_1, \phi_2, \ldots, \phi_n \)” by “constructing a proof in a formal calculus.”

- We can give another interpretation by valuations and truth values.
- Consider a valuation \( \nu \) over all propositional atoms in \( \phi_1, \phi_2, \ldots, \phi_n, \psi \).
  - By “assumptions \( \phi_1, \phi_2, \ldots, \phi_n \),” we mean “\( \phi_1, \phi_2, \ldots, \phi_n \) are T under the valuation \( \nu \).”
  - By “deriving \( \psi \),” we mean \( \psi \) is also T under the valuation \( \nu \).

- Hence, “we can derive \( \psi \) with assumptions \( \phi_1, \phi_2, \ldots, \phi_n \)” actually means “if \( \phi_1, \phi_2, \ldots, \phi_n \) are T under a valuation, then \( \psi \) must be T under the same valuation.”
Semantic Entailment

Definition

We say

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

holds if for every valuations where $\phi_1, \phi_2, \ldots, \phi_n$ are T, $\psi$ is also T. In this case, we also say $\phi_1, \phi_2, \ldots, \phi_n$ semantically entail $\psi$.

Examples

- $p \land q \models p$. For every valuation where $p \land q$ is T, $p$ must be T. Hence $p \land q \models p$.
- $p \lor q \not\models q$. Consider the valuation $\{p \mapsto T, q \mapsto F\}$. We have $p \lor q$ is T but $q$ is F. Hence $p \lor q \not\models q$.
- $\neg p, p \lor q \models q$. Consider any valuation where $\neg p$ and $p \lor q$ are T. Since $\neg p$ is T, $p$ must be F under the valuation. Since $p$ is F and $p \lor q$ is T, $q$ must be T under the valuation. Hence $\neg p, p \lor q \models q$.

The validity of $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ is defined by syntactic calculus. $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ is defined by truth tables. Do these two relations coincide?
Soundness Theorem for Propositional Logic

Theorem (Soundness)

Let \( \phi_1, \phi_2, \ldots, \phi_n \) and \( \psi \) be propositional logic formulae. If \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) is valid, then \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \) holds.

Proof.

Consider the assertion \( M(k) \):

“For all sequents \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi (n \geq 0) \) that have a proof of length \( k \), then \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \) holds.”

\( k = 1 \). The only possible proof is of the form

\[
1 \quad \phi \quad \text{premise}
\]

This is the proof of \( \phi \vdash \phi \). For every valuation such that \( \phi \) is T, \( \phi \) must be T. That is, \( \phi \models \phi \).
Proof (cont’d).

Assume $M(i)$ for $i < k$. Consider a proof of the form

1. $\phi_1$ premise
2. $\phi_2$ premise
   
   $\vdots$
   
   n. $\phi_n$ premise
   
   $\vdots$
   
   k. $\psi$ justification

We have the following possible cases for justification:

i. $\land i$. Then $\psi$ is $\psi_1 \land \psi_2$. In order to apply $\land i$, $\psi_1$ and $\psi_2$ must appear in the proof. That is, we have $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_2$. By inductive hypothesis, $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_2$. Hence $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1 \land \psi_2$ (Why?).
Soundness Theorem for Propositional Logic

Proof (cont’d).

ii \( \lor e \). Recall the proof rule for \( \lor e \):

\[
\begin{array}{c}
\eta_1 \lor \eta_2 \\
\psi \\
\eta_1 \lor \eta_2 \\
\psi \\
\eta_1 \lor \eta_2 \\
\psi
\end{array}
\]

\( \psi \lor e \)

In order to apply \( \lor e \), \( \eta_1 \lor \eta_2 \) must appear in the proof. We have \( \phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \lor \eta_2 \). By turning “assumptions” \( \eta_1 \) and \( \eta_2 \) to “premises,” we obtain proofs for \( \phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi \) and \( \phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi \). By inductive hypothesis, \( \phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \lor \eta_2 \), \( \phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi \), and \( \phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi \). Consider any valuation such that \( \phi_1, \phi_2, \ldots, \phi_n \) evaluates to \( T \). \( \eta_1 \lor \eta_2 \) must be \( T \). If \( \eta_1 \) is \( T \) under the valuation, \( \psi \) is also \( T \) (Why?). Similarly for \( \eta_2 \) is \( T \). Thus \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \).
Other cases are similar. Prove the case of $\equiv e$ to see if you understand the proof.

- The soundness theorem shows that our calculus does not go wrong.
- If there is a proof of a sequent, then the conclusion must be true for all valuations where all premises are true.
- The theorem also allows us to show the non-existence of proofs.

Given a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, how do we prove there is no proof for the sequent?
  - Try to find a valuation where $\phi_1, \phi_2, \ldots, \phi_n$ are T but $\psi$ is F.
Outline

1. Natural Deduction

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Completeness Theorem for Propositional Logic

- “φ₁, φ₂, ..., φₙ ⊢ ψ is valid” and “φ₁, φ₂, ..., φₙ ⊨ ψ holds” are very different.
  - “φ₁, φ₂, ..., φₙ ⊢ ψ is valid” requires proof search (syntax);
  - “φ₁, φ₂, ..., φₙ ⊨ ψ holds” requires a truth table (semantics).
- If “φ₁, φ₂, ..., φₙ ⊨ ψ holds” implies “φ₁, φ₂, ..., φₙ ⊢ ψ is valid,”
  then our natural deduction proof system is complete.
- The natural deduction proof system is both sound and complete.
  That is
  φ₁, φ₂, ..., φₙ ⊢ ψ is valid iff φ₁, φ₂, ..., φₙ ⊨ ψ holds.
Completeness Theorem for Propositional Logic

- We will show the natural deduction proof system is complete.
- That is, if $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds, then there is a natural deduction proof for the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.
- Assume $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$. We proceed in three steps:
  1. $\vdash \phi_1 \implies (\phi_2 \implies (\ldots (\phi_n \implies \psi)))$ holds;
  2. $\vdash \phi_1 \implies (\phi_2 \implies (\ldots (\phi_n \implies \psi)))$ is valid;
  3. $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.
Completeness Theorem for Propositional Logic (Step 1)

Lemma

If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds, then $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots))$ holds.

Proof.

Suppose $\not\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots))$ does not hold. Then there is valuation where $\phi_1, \phi_2, \ldots, \phi_n$ is T but $\psi$ is F. A contradiction to $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.

Definition

Let $\phi$ be a propositional logic formula. We say $\phi$ is a tautology if $\vdash \phi$.

- A tautology is a propositional logic formula that evaluates to T for all of its valuations.
Our goal is to show the following theorem:

**Theorem**

If $\models \eta$ holds, then $\vdash \eta$ is valid.

Similar to tautologies, we introduce the following definition:

**Definition**

Let $\phi$ be a propositional logic formula. We say $\phi$ is a theorem if $\vdash \phi$.

Two types of theorems:
- If $\vdash \phi$, $\phi$ is a theorem proved by the natural deduction proof system.
- The soundness theorem for propositional logic is another type of theorem proved by mathematical reasoning (less formally).
Completeness Theorem for Propositional Logic (Step 2)

Proposition

Let $\phi$ be a formula with propositional atoms $p_1, p_2, \ldots, p_n$. Let $l$ be a line in $\phi$'s truth table. For all $1 \leq i \leq n$, let $\hat{p}_i$ be $p_i$ if $p_i$ is T in $l$; otherwise $\hat{p}_i$ is $\neg p_i$. Then

1. $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi$ is valid if the entry for $\phi$ at $l$ is T;
2. $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi$ is valid if the entry for $\phi$ at $l$ is F.

Proof.

We prove by induction on the height of the parse tree of $\phi$.

- $\phi$ is a propositional atom $p$. Then $p \vdash p$ or $\neg p \vdash \neg p$ have one-line proof.
- $\phi$ is $\neg \phi_1$.
  - If $\phi$ is T at $l$. Then $\phi_1$ is F. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi_1(\equiv \phi)$.
  - If $\phi$ is F at $l$. Then $\phi_1$ is T. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$. Using $\neg \neg i$, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \neg \phi_1(\equiv \neg \phi)$. 
Completeness Theorem for Propositional Logic (Step 2)

**Proof (cont’d).**

- \( \phi \) is \( \phi_1 \implies \phi_2 \).
  - If \( \phi \) is F at \( l \), then \( \phi_1 \) is T and \( \phi_2 \) is F at \( l \). By IH, \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \) and \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi_2 \). Consider

\[
\begin{array}{c|c|c|c}
1 & \phi_1 & \implies & \phi_2 \\
\vdots & & & \\
 i & \phi_1 & & \text{IH} \\
 i + 1 & \phi_2 & & \implies \ e \ i, \ 1 \\
\vdots & & & \\
 j & \neg \phi_2 & & \text{IH} \\
j + 1 & \bot & & \neg \ e \ i+1, \ j \\
j + 2 & \neg (\phi_1 \implies \phi_2) & & \neg \ i \ 1-(j+1)
\end{array}
\]
Proof (cont’d).

- \( \phi \) is \( \phi_1 \implies \phi_2 \).
  - If \( \phi \) is T at \( l \), we have three subcases. Consider the case where \( \phi_1 \) and \( \phi_2 \) are F at \( l \). Then

\[
\begin{align*}
1 & \quad \phi_1 & \text{assumption} \\
\vdots & & \\
2 & \quad \neg \phi_1 & \text{IH} \\
3 & \quad \bot & \neg \text{e } 1, i \\
4 & \quad \phi_2 & \bot \text{ e } (i+1) \\
5 & \quad \phi_1 \implies \phi_2 & \implies i 1-(i+2)
\end{align*}
\]

The other two subcases are simple exercises.
Completeness Theorem for Propositional Logic (Step 2)

Proof (cont’d).

- \( \phi \) is \( \phi_1 \land \phi_2 \).
  - If \( \phi \) is T at \( l \), then \( \phi_1 \) and \( \phi_2 \) are T at \( l \). By IH, we have \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \) and \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_2 \). Using \( \land \ i \), we have \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \land \phi_2 \).
  - If \( \phi \) is F at \( l \), there are three subcases. Consider the subcase where \( \phi_1 \) and \( \phi_2 \) are F at \( l \). Then

\[
\begin{array}{c}
1 & \phi_1 \land \phi_2 & \text{assumption} \\
2 & \phi_1 & \land \text{e} 1 1 \\
\vdots \\
i & \neg \phi_1 & \text{IH} \\
i + 1 & \bot & \neg \text{e} 2, i \\
i + 2 & \neg (\phi_1 \land \phi_2) & \neg \text{i} 1-(i+1)
\end{array}
\]

The other two subcases are simple exercises.
Completeness Theorem for Propositional Logic (Step 2)

Proof.

- \( \phi \) is \( \phi_1 \lor \phi_2 \).
  - If \( \phi \) is F at \( l \), then \( \phi_1 \) and \( \phi_2 \) are F at \( l \). Then
    1. \( \phi_1 \lor \phi_2 \) assumption
    2. \( \phi_1 \) assumption
    3. \( \phi_2 \) assumption
    ... 
    i. \( \neg \phi_1 \) IH
    i + 1. \( \bot \) \( \neg e \ 2, \ i \)
    i + 2. \( \phi_2 \) assumption
    ... 
    j. \( \neg \phi_2 \) IH
    j + 1. \( \bot \) \( \neg e \ i+2, \ j \)
    j + 2. \( \bot \) \( \lor e \ 2-(i+1), \ (i+2)-(j+1) \)
    j + 3. \( \neg(\phi_1 \lor \phi_2) \) \( \neg i \ 1-(j+2) \)
  - If \( \phi \) is T at \( l \), there are three subcases. All of them are simple exercises.
Completeness Theorem for Propositional Logic (Step 2)

**Theorem**

If $\phi$ is a tautology, then $\phi$ is a theorem.

**Proof.**

Let $\phi$ have propositional atoms $p_1, p_2, \ldots, p_n$. Since $\phi$ is a tautology, each line in $\phi$’s truth table is T. By the above proposition, we have the following $2^n$ proofs for $\phi$:

\[ \neg p_1, \neg p_2, \ldots, \neg p_n \vdash \phi \]
\[ p_1, \neg p_2, \ldots, \neg p_n \vdash \phi \]
\[ \neg p_1, p_2, \ldots, \neg p_n \vdash \phi \]
\[ \vdots \]
\[ p_1, p_2, \ldots, p_n \vdash \phi \]

We apply the rule LEM and the $\lor$ rule to obtain a proof for $\vdash \phi$. (See the following example.)
Example

Observe that $\vdash p \implies (q \implies p)$. Prove $\vdash p \implies (q \implies p)$.

Proof.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$p \lor \neg p$</td>
<td>LEM</td>
<td></td>
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<tr>
<td>2</td>
<td>$p$</td>
<td>assumption</td>
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<tr>
<td>3</td>
<td>$q \lor \neg q$</td>
<td>LEM</td>
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</tr>
<tr>
<td>4</td>
<td>$q$</td>
<td>assumption</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$i$</td>
<td>$p \implies (q \implies p)$</td>
<td>$p, q \vdash p \implies (q \implies p)$</td>
<td></td>
</tr>
<tr>
<td>$i+1$</td>
<td>$\neg q$</td>
<td>assumption</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$j$</td>
<td>$p \implies (q \implies p)$</td>
<td>$p, \neg q \vdash p \implies (q \implies p)$</td>
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</tr>
<tr>
<td>$j+1$</td>
<td>$p \implies (q \implies p)$</td>
<td>$\lor e\ 3, 4-i, (i+1)-j$</td>
<td></td>
</tr>
<tr>
<td>$j+2$</td>
<td>$\neg p$</td>
<td>assumption</td>
<td></td>
</tr>
<tr>
<td>$j+3$</td>
<td>$q \lor \neg q$</td>
<td>LEM</td>
<td></td>
</tr>
<tr>
<td>$j+4$</td>
<td>$q$</td>
<td>assumption</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>$p \implies (q \implies p)$</td>
<td>$\neg p, q \vdash p \implies (q \implies p)$</td>
<td></td>
</tr>
<tr>
<td>$k+1$</td>
<td>$\neg q$</td>
<td>assumption</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$l$</td>
<td>$p \implies (q \implies p)$</td>
<td>$\neg p, \neg q \vdash p \implies (q \implies p)$</td>
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</tr>
<tr>
<td>$l+1$</td>
<td>$p \implies (q \implies p)$</td>
<td>$\lor e\ (j+3), (j+4)-k, (k+1)-l$</td>
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<tr>
<td>$l+2$</td>
<td>$p \implies (q \implies p)$</td>
<td>$\lor e\ 1, 2-(j+1), (j+2)-(l+1)$</td>
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</tbody>
</table>
Completeness Theorem for Propositional Logic (Step 3)

Lemma

If \( \phi_1 \implies (\phi_2 \implies (\ldots (\phi_n \implies \psi))) \) is a theorem, then \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) is valid.

Proof.

Consider

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>( \phi_1 )</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>( \phi_2 )</td>
<td>premise</td>
</tr>
<tr>
<td></td>
<td>\vdots</td>
<td></td>
</tr>
<tr>
<td>n</td>
<td>( \phi_n )</td>
<td>premise</td>
</tr>
<tr>
<td></td>
<td>\vdots</td>
<td></td>
</tr>
<tr>
<td>i</td>
<td>( \phi_1 \implies (\phi_2 \implies (\ldots (\phi_n \implies \psi))) )</td>
<td>theorem</td>
</tr>
<tr>
<td>i + 1</td>
<td>( \phi_2 \implies (\ldots (\phi_n \implies \psi)) )</td>
<td>( \implies ) e 1, i</td>
</tr>
<tr>
<td>i + 2</td>
<td>( \phi_3 \implies (\ldots (\phi_n \implies \psi)) )</td>
<td>( \implies ) e 2, (i+1)</td>
</tr>
<tr>
<td></td>
<td>\vdots</td>
<td></td>
</tr>
<tr>
<td>i + n - 1</td>
<td>( \phi_n \implies \psi )</td>
<td>( \implies ) e (n-1), (i+n-2)</td>
</tr>
<tr>
<td>i + n</td>
<td>( \psi )</td>
<td>( \implies ) e n, (i+n-1)</td>
</tr>
</tbody>
</table>