Chapter 7

RELATIONS

1. RELATIONS AND THEIR PROPERTIES

Definition 1. Let $A$ and $B$ be sets. A binary relation from $A$ to $B$ is a subset of $A \times B$.

Let $R$ be a binary relation. We sometimes write $aRb$ for $(a, b) \in R$. Examples of binary relations are $<, \in$, etc.

Functions belong to a special class of relations. Let $f : A \rightarrow B$. Define $F = \{(a, b) : b = f(a)\}$. $F$ is a binary relation from $A$ to $B$ such that $aFb$ and $aFb'$ implies $b = b'$.

Definition 2. Let $A$ be a set. A relation on $A$ is a relation from $A$ to $A$.

For instance, $<, =$ are relation on $\mathbb{R}$.

Definition 3. A relation $R$ on $A$ is called reflexive if $(a, a) \in R$ for all $a \in A$.

For instance, $\leq$ is reflexive.

Definition 4. A relation $R$ on $A$ is called symmetric if $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$.

A relation $R$ on $A$ is called antisymmetric if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.

For instance, $\neq$ is symmetric and $\geq$ is antisymmetric.

Definition 5. A relation $R$ on $A$ is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in R$.

For instance, $<$ is transitive.

Example. Do you still remember the “proof” that symmetry and transitivity imply reflexivity?

Definition 6. Let $R$ be a relation from $A$ to $B$ and $S$ a relation from $B$ to $C$. The composite of $R$ and $S$, $S \circ R$, is defined by

$$S \circ R = \{(a, c) : \exists b \in B. (a, b) \in R \land (b, c) \in S\}.$$  

Remark. When $R$ and $S$ happen to be functions, $S \circ R$ is equivalent to the function composition.

Definition 7. Let $R$ be a relation on $A$. Define

$$R^1 = R$$

and$$R^{n+1} = R^n \circ R.$$  

For convenience, we define $R^0 = I$ where $I$ is the identity relation $\{(a, a) : a \in A\}$.

Theorem 1. The relation $R$ on $A$ is transitive if and only if $R^n \subseteq R$ for $n \in \mathbb{Z}^+$.

Proof. $(\Rightarrow)$ Let $a, b, c \in A$ with $aRb$ and $bRc$. Then $aR^2c$ by definition. Hence $aRc$ for $R^2 \subseteq R$.

$(\Leftarrow)$ We prove by induction on $n$.

**Basis Step:** $n = 1$, $R \subseteq R$ is trivial.

**Inductive Step:** Assume $R^k \subseteq R$. We want to show $R^{k+1} \subseteq R$. Consider any $a, c$ such that $aR^{k+1}c$. There is a $b$ such that $aRb$ and $bR^kc$ because $R^{k+1} = R^k \circ R$. By inductive hypothesis, $bR^kc$ implies $bRc$. Hence $aRc$ follows from the transitivity of $R$. \qed

2. n-ARY RELATIONS AND THEIR APPLICATIONS

Definition 8. Let $A_0, A_1, \ldots, A_{n-1}$ be sets. An n-ary relation on these sets is a subset of $A_0 \times A_1 \times \cdots \times A_{n-1}$. $A_0, A_1, \ldots, A_{n-1}$ are called the domain of the relation and $n$ is its degree.

Definition 9. Let $R$ be an n-ary relation over $A_0, A_1, \ldots, A_{n-1}$ and $C : A_0 \times A_1 \times \cdots \times A_{n-1} \rightarrow \{\text{false, true}\}$. The selection operator $s_C : \varphi(A_0 \times A_1 \times \cdots \times A_{n-1}) \rightarrow \varphi(A_0 \times A_1 \times \cdots \times A_{n-1})$ is defined by

$$s_C(R) = \{(a_0, \ldots, a_{n-1}) \in R : C(a_0, \ldots, a_{n-1}) = \text{true}\}.$$  

Definition 10. The projection $P_{i_0, i_1, \ldots, i_{m-1}}$ maps the n-tuple $(a_0, a_1, \ldots, a_{n-1})$ to the m-tuple $(a_{i_0}, a_{i_1}, \ldots, a_{i_{m-1}})$, where $m \leq n$ and $0 \leq i_k < n$ for all $k$. 

1
Definition 11. Let $R$ be a relation of degree $m$ and $S$ a relation of degree $n$. The \textit{join} $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ such that

$$(a_0, a_1, \ldots, a_{m-p-1}, c_0, c_1, \ldots, c_{p-1}, b_p, b_{p+1}, \ldots, b_{n-1}) \in J_p(R, S)$$

if and only if

$$(a_0, a_1, \ldots, a_{m-p-1}, c_0, c_1, \ldots, c_{p-1}) \in R$$

and

$$(c_0, c_1, \ldots, c_{p-1}, b_p, b_{p+1}, \ldots, b_{n-1}) \in S.$$  

We can think databases as $n$-ary relations. The database query language SQL (Structured Query Language) has the operations we defined here.

Example 1. Try to interpret the following SQL query:

```
SELECT Grade
FROM Transcripts
WHERE Department='Information Management'
```

Solution. The command \texttt{SELECT} corresponds to projection. The clause \texttt{WHERE} specifies the condition of the selection operator. Finally, \texttt{FROM} denotes the $n$-ary relation we’re interested in.

Although database queries in SQL seem to be simple from mathematical point of view, they require intensive computation to implement. Consider the example and ask yourself: how many students in the university are there? Note that we have only one transcript database for all these years. It is by no mean “simple” to collect information from the database.

3. Representing Relations

We can represent relations by matrices or graphs.

3.1. Matrix Representation.

Example 2. Consider any binary relation $R$ on $\{a_0, a_1, \ldots, a_{n-1}\}$. Define $M_R = [m_{ij}]_{n \times n}$ where

$$m_{ij} = \begin{cases} 
1 & \text{if } (a_i, a_j) \in R \\
0 & \text{if } (a_i, a_j) \notin R 
\end{cases}$$

If $R$ is symmetric, $M_R$ is symmetric. If $R$ is reflexive and $M_R = [m_{ij}]_{n \times n}$, $m_{ii} = 1$ for $0 \leq i < n$.

We introduce the following matrix operations:

Definition 12. Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ and $a_{ij}, b_{ij} \in \{0, 1\}$ for $0 \leq i < m$, $0 \leq j < n$. Define $A \land B = [c_{ij}]_{m \times n}$ where

$$c_{ij} = \begin{cases} 
1 & \text{if } a_{ij} = 1 \land b_{ij} = 1 \\
0 & \text{otherwise}
\end{cases}$$

Definition 13. Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ and $a_{ij}, b_{ij} \in \{0, 1\}$ for $0 \leq i < m$, $0 \leq j < n$. Define $A \lor B = [c_{ij}]_{m \times n}$ where

$$c_{ij} = \begin{cases} 
1 & \text{if } a_{ij} = 1 \lor b_{ij} = 1 \\
0 & \text{otherwise}
\end{cases}$$

Definition 14. Let $A = [a_{ik}]_{l \times m}$, $B = [b_{kj}]_{l \times n}$ and $a_{ik}, b_{kj} \in \{0, 1\}$ for $0 \leq i < m$, $0 \leq k < l$, $0 \leq j < n$. Define $A \otimes B = [c_{ij}]_{m \times n}$ where

$$c_{ij} = \begin{cases} 
1 & \text{if } \bigvee_{k=0}^{l-1} a_{ik} = 1 \land b_{kj} = 1 \\
0 & \text{otherwise}
\end{cases}$$

When $m = l$, define $A^{[0]} = I_n$ and $A^{[r+1]} = A \land A^{[r]}$.

Remark. If we think 0 as \texttt{false} and 1 as \texttt{true}, $A \land B$ and $A \lor B$ are simply defined by replacing $\land$ for $\cdot$ and $\lor$ for $+$ in $A + B$ and $A \times B$ respectively.

Let $R_0$ and $R_1$ be relations on $\{0, \ldots, n-1\}$. It is straightforward to see $M_{R_0 \cup R_1} = M_{R_0} \lor M_{R_1}$, $M_{R_0 \cap R_1} = M_{R_0} \land M_{R_1}$ and $M_{R_1 \circ R_0} = M_{R_0} \otimes M_{R_1}$. Note that the order of $R_0$ and $R_1$ is reversed in the composite.
3.2. Graph Representation.

Definition 15. A directed graph (or digraph), $G = (V, E)$, consists of the set $V$ of vertices and $E \subseteq V \times V$ the set of edges. For the edge $(a, b)$, the vertex $a$ is its initial vertex and $b$ its terminal vertex. The edge $(a, a)$ is called a loop.

Example 3. Draw a digraph to represent $<$ on $\{0, 1, 2, 3\}$.

Solution. Figure 1 shows the solution.

\[ \square \]

4. Closures of Relations

Definition 16. Let $R$ be a relation on $A$. The smallest transitive relation that contains $R$ is called the transitive closure of $R$. Similarly, the smallest reflexive relation that contains $R$ is called the reflexive closure of $R$. And the smallest symmetric relation that contains $R$ is called the symmetric closure of $R$.

The relation $\Delta_A = \{(a, a) : a \in A\}$ is the diagonal relation on $A$. Note that when $|A| = n$, $M_{\Delta_A} = I_n$.

Example 4. What is the reflexive closure of $<$ on $\mathbb{Z}$?

Solution. Consider $\leq$. It is easy to see $\leq$ is the reflexive closure of $<$.

Remark. What is the symmetric closure of $<$?

Definition 17. Let $G = (V, E)$ be a digraph. A path from $v_0$ to $v_n$ is a sequence of edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ in $G$. We usually write $v_0, v_1, \ldots, v_n$ to denote the path and say the length of path is $n$. For any $v \in V$, we view the empty sequence as a path (of length 0). If $n > 0$ and $v_0 = v_n$, we say the path is a circuit or cycle.

Just like a relation can be represented by a digraph. A digraph corresponds to a relation $E$ on $V$.

Theorem 2. Let $R$ be a relation on $A$. There is a path of length $n > 0$ from $a$ to $b$ if and only if $(a, b) \in R^n$.

Proof. We prove by induction.

BASIS STEP: $n = 1$. Obvious.

INDUCTIVE STEP: Assume there is a path of length $k$ from $c$ to $b$ if and only if $(c, b) \in R^k$. Consider any path of length $k + 1$ from $a$ to $b$. The path consists a path of length 1 from $a$ to $c$ and a path of length $k$ from $c$ to $b$. Hence $(a, c) \in R$ and $(c, b) \in R^k$. We have $(a, b) \in R^{k+1}$.

On the other hand, if $(a, b) \in R^{k+1}$, there exists a $c$ such that $(a, c) \in R$ and $(c, b) \in R^k$. The result follows from the inductive hypothesis as well.

Definition 18. Let $R$ be a relation on $A$. The connectivity relation $R^*$ consists of $(a, b)$ such that there is a path from $a$ to $b$ in $R$.

By Theorem 2, we have

$$ R^* = \bigcup_{n=1}^{\infty} R^n. $$
The transitive closure of Theorem 3.

Proof. Note that \( R \subseteq R^* \) by definition. Furthermore, \((a, b), (b, c) \in R^*\) implies \((a, b) \in R^i\) and \((b, c) \in R^j\) for some \(i, j\). Hence \((a, c) \in R^{i+j} \subseteq R^*\). \( R^* \) is transitive.

It remains to show that any transitive relation containing \( R \) must contain \( R^* \). Let \( S \) be any transitive relation containing \( R \). Since \( S \) is transitive, \( S^n \subseteq S \) by Theorem 1 for \( n \in \mathbb{Z}^+ \). Therefore

\[
R^* = \bigcup_{n=1}^{\infty} R^n \subseteq \bigcup_{n=1}^{\infty} S^n \subseteq S.
\]

\( \square \)

**Lemma 1.** Let \( A \) be a set with \( |A| = n \) and \( R \) a relation on \( A \). If there is a path of length \( > 0 \) in \( R \) from \( a \) to \( b \), then there is a path of length \( \leq n \). When \( a \neq b \), if there is a path of length \( > 0 \) in \( R \), then there is a non-empty path of length \( < n \).

Proof. Consider the shortest path \( v_0 = a, v_1, v_2, \ldots, v_m = b \) from \( a \) to \( b \) of length \( m \). If \( v_0 = v_m \) and \( m > n \), there must be some \( i, j \) with \( i < j \) such that \( v_i = v_j \) by the pigeonhole principle. Then \( v_0, v_1, \ldots, v_i, v_{j-1}, v_j, \ldots, v_m \) is a shorter path from \( a \) to \( b \). A contradiction.

Now suppose \( a = v_0 \neq v_m = b \) and \( m \geq n \). There must be \( i, j \) with \( i < j \) such that \( v_i = v_j \). We can also construct a shorter path and lead to a contradiction. \( \square \)

**Theorem 4.** Let \( M_R = [m_{ij}]_{n \times n} \) be the matrix of the relation \( R \) on a set with \( n \) elements. Then the matrix \( M_{R^*} \) of \( R^* \) is

\[
M_{R^*} = M_R \lor M_R^2 \lor M_R^3 \lor \cdots \lor M_R^n.
\]

Proof. \((a, b) \in R^*\), then there is a path from \( a \) to \( b \) by definition. By Lemma 1, it suffices to consider paths of length at most \( n \). By Theorem 2,

\[
R^* = R \lor R^2 \lor \cdots \lor R^n.
\]

The result follows from the matrix representation of relations. \( \square \)

Figure 2 shows an algorithm for computing transitive closure of \( R \). Let’s analyze its time complexity. First observe that computing the entry \( c_{ij} \) in \( A \lor B \)

\[
c_{ij} = \bigvee_{k=0}^{n-1} a_{ik} \land b_{kj}
\]

requires \( O(n) \) steps. Since there are \( n^2 \) entries, \( A \lor M_R \) takes \( O(n^3) \) steps. Moreover, \( B \lor A \) takes \( O(n^2) \) steps. Thus an iteration of the loop (line (5) and (6)) takes \( O(n^3) + O(n^2) = O(n^3) \) steps. There are \( n \) iterations, we have the time complexity \( O(n^4) \) for Line (4) to (7). Line (2) and (3) take \( O(n^2) \) steps respectively. The time complexity of the algorithm is \( O(n^4) \).

We can actually do better than \( O(n^4) \). Figure 3 shows the algorithm developed by Warshall. If we let \( w_{ij}^{[k]} \) denote the value of \( w_{ij} \) at the end of the \( k \)-th iteration of outmost loop, then \( w_{ij}^{[k]} \) is 1 if and only if there is a path from \( i \) to \( j \) via vertices \( \{v_0, v_1, \ldots, v_k\} \). Line (6) can be rewritten as

\[
w_{ij}^{[k]} = w_{ij}^{[k-1]} \lor (w_{ik}^{[k-1]} \land w_{kj}^{[k-1]}).
\]

We can rephrase it as the equivalence of the following two statements:
Example 5. Proof.
For any \( v \) a relation \( \text{Definition 19.} \) Clearly, these two statements are equivalent. Hence line (6) basically tells us whether there is a path from

\( \text{Definition 20.} \) Let \( \text{Theorem 5.} \) The equivalence classes of the relation congruence modulo \( n \) are called congruence classes modulo \( n \).

Let \( A \) be a set. A partition of \( A \) is a collection of disjoint nonempty subsets of \( A \). The equivalence classes of \( R \) on \( A \) form a partition of \( A \). More precisely,

\begin{enumerate}
  \item \textbf{procedure} Warshall (\( M_R : n \times n \) zero-one matrix)
  \item \( W := M_R \)
  \item \text{for} \( k := 0 \) \text{to} \( n - 1 \) \text{do}
  \item \text{for} \( i := 0 \) \text{to} \( n - 1 \) \text{do}
  \item \text{for} \( j := 0 \) \text{to} \( n - 1 \) \text{do}
  \item \( w_{ij} := w_{ij} \lor (w_{ik} \land w_{kj}) \)
  \item \text{od}
  \item \text{od}
  \item \{ \( W = [w_{ij}] \) is \( M_R \) \}
  \item \text{end}
\end{enumerate}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{warshall.png}
\caption{Warshall’s Algorithm}
\end{figure}

- There is a path from \( v_i \) to \( v_j \) via \( \{v_0, v_1, \ldots, v_k\} \);
- There is a path from \( v_i \) to \( v_j \) via \( \{v_0, v_1, \ldots, v_{k-1}\} \) or a path from \( v_i \) to \( v_k \) and \( v_k \) to \( v_j \) via \( \{v_0, v_1, \ldots, v_{k-1}\} \).

Clearly, these two statements are equivalent. Hence line (6) basically tells us whether there is a path from \( v_i \) to \( v_j \) via \( \{v_0, v_1, \ldots, v_k\} \). Since \( w_{ij}^{[n-1]} \) denotes that there is a path from \( v_i \) to \( v_j \) via all vertices, the result follows.

Since line (6) takes \( O(1) \) steps and it repeats \( n^3 \) times, we conclude the Warshall algorithm has time complexity \( O(n^3) \).

\section{Equivalence Relations}

\textbf{Definition 19.} A relation \( R \) on \( A \) is called an equivalence relation if it is reflexive, symmetric, and transitive.

\textbf{Example 5.} Let \( n \in \mathbb{Z}^+ \). Show \( \{(a, b) : a \equiv b(n)\} \) is an equivalence relation.

\textit{Proof.} For any \( a, b, c \in \mathbb{N} \), we have

- \( a \equiv a(n) \);
- \( a \equiv b(n) \). Then \( n|a - b \). Thus \( n|b - a \). So \( b \equiv a(n) \);
- \( a \equiv b(n) \) and \( b \equiv c(n) \). Then \( nk = a - b \) and \( nk' = b - c \). So \( n(k + k') = (a - b) + (b - c) = a - c \). \( n|a - c \).

We have \( a \equiv c(n) \).

\textit{Definition 20.} Let \( R \) be an equivalence relation on \( A \). Let \( a \in A \). Define the equivalence class of \( a \), \( [a]_R \), to be

\[ [a]_R = \{ b : (a, b) \in R \} \]

Sometimes, we may write \( [a] \) if \( R \) is clear from context.

\textbf{Example 6.} Find all equivalence classes in Example 5.

\textit{Solution.}

\begin{align*}
[0] &= \{ \ldots, -2n, -n, 0, n, 2n, \ldots \} \\
[1] &= \{ \ldots, -2n + 1, -n + 1, 1, n + 1, 2n + 1, \ldots \} \\
& \ldots \\
[n - 1] &= \{ \ldots, -2n + (n - 1), -n + (n - 1), 1, n + (n - 1), 2n + (n - 1), \ldots \}
\end{align*}

The equivalence classes of the relation congruence modulo \( n \) are called congruence classes modulo \( n \).
Proof. ((i)⇒(ii)) Assume aRb. Consider any c ∈ [a]. We have aRc by definition. Then bRc by reflexivity and transitivity. So c ∈ [b]. [a] ⊆ [b], [b] ⊆ [a] follows by symmetric arguments. Hence [a] = [b].

((ii)⇒(iii)) Trivial.

((iii)⇒(i)) Let c ∈ [a] ∩ [b]. Then aRc and bRc. aRb follows by reflexivity and transitivity. Since \{0\}, \{1\}, \{2\}, ..., \{n − 1\} forms a partition of \( \mathbb{Z} \), any integer must belong to one and only one of the congruence classes.

Conversely, we can obtain an equivalence relation from a partition.

Theorem 6. Let \( A \) be a set and \( \{ A_i : i \in I \} \) a partition of \( A \). Define

\[ R = \{ (a, b) : \exists i. a \in A_i \land b \in A_i \}. \]

Then \( R \) is an equivalence relation with \( A_i, i \in I \) its equivalence classes.

Proof. We have

- \( aRa \). Since \( \{ A_i \} \) is a partition, \( a \in A_i \) for some \( i \).
- \( aRb \) implies \( bRa \). Since the definition of \( R \) is symmetric, the result follows.
- \( aRb \) and \( bRc \) implies \( aRc \). By definition, there are \( i, j \) such that \( a \in A_i \land b \in A_i \) and \( b \in A_j \land c \in A_j \).

But \( \{ A_i \} \) is a partition, \( i = j \). We have \( a \in A_i \land c \in A_i \), \( aRc \).

The equivalences classes of \( R \) follows by definition. □

6. Partial Orderings

Definition 21. Let \( R \) be a relation on \( A \). \( R \) is called a partial ordering or a partial order if it is reflexive, antisymmetric, and transitive. The set \( A \) is called a partially ordered set, or poset, and is denoted by \((A, R)\).

For instance, \((\mathbb{Z}, \geq)\) is a poset. And \((\mathcal{P}(S), \subseteq)\) is a poset for any set \( S \). Note that not all pairs of elements can be ordered; neither \( \{0\} \subseteq \{1\} \) nor \( \{1\} \subseteq \{0\} \).

Definition 22. Let \((S, \leq)\) be a poset and \( a, b \in S \). \( a \) and \( b \) are called comparable if either \( a \leq b \) or \( b \leq a \). Otherwise, they are called incomparable.

Notation. We will write \( a < b \) for \( a \leq b \) but \( a \neq b \).

When all pairs of elements are comparable, we call the relation a total ordering.

Definition 23. If \((S, \leq)\) is a poset and every two elements of \( S \) are comparable, \( S \) is called a totally ordered or linearly ordered set, and \( \leq \) is called a total order or a linear order. A totally ordered set is also called a chain.

For instance, \((\mathbb{Z}, \leq)\) is a chain.

Definition 24. Let \((S, \leq)\) be a poset. \((S, \leq)\) is a well-ordered set if \( \leq \) is a total ordering such that every nonempty subset of \( S \) has a least element (according to \( \leq \)).

For instance, \((\mathbb{Z}^+, \leq)\) is a well-ordered set but \((\mathbb{Z}, \leq)\) is not.

Example 7. Consider the relation \( \preceq \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \) such that \((a, b) \preceq (a', b')\) if \( a < a' \), or \( a = a' \land b \leq b' \). Then \((\mathbb{Z}^+ \times \mathbb{Z}^+, \preceq)\) is a well-ordered set.

Solution. Consider any subset \( A \) of \( \mathbb{Z}^+ \times \mathbb{Z}^+ \). Define \( A_0 \) to be the projection of \( A \) on the \( x \)-coordinate. \( A_0 \subseteq \mathbb{Z}^+ \), then \( A_0 \) has a least element, say \( a_0 \). Define \( A_1 \) to be the selection of \( A \) on the condition that the \( x \)-coordinate equals to \( a_0 \). Let \( A_2 \) be the projection of \( A_1 \) on the \( y \)-coordinate. Then \( A_2 \subseteq \mathbb{Z}^+ \). Hence it has a least element \( a_2 \). Then \((a_0, a_2)\) is a least element of \( A \). □

Theorem 7. (The Principle of Well-Ordered Induction) Let \((S, \leq)\) be a well-ordered set. Then \( P(x) \) is true for all \( x \in S \) if

- BASIS STEP: \( P(x_0) \) is true for the least element of \( S \), and
- INDUCTIVE STEP: For every \( y \in S \) if \( P(x) \) is true for all \( x < y \), then \( P(y) \) is true.

Proof. Consider the set \( A = \{ y : \neg P(y) \} \). \( A \subseteq S \), \( A \) has a least element \( y_0 \) for \( S \) is well-ordered. \( y_0 \) cannot be the least element of \( S \) because of basis step. Consider the set \( B = \{ x : x < y \} \). \( B \) is not empty and \( \forall x \in B. P(x) \) by the choice of \( y_0 \). Then \( P(y_0) \) holds by the inductive step. A contradiction. □

The following example represents a partially ordered relation by an undirected graph. The graph is called a Hasse diagram.
Definition 25. Let \((S, \preceq)\) be a poset and \(a \in S\). Then \(a\) is maximal in \((S, \preceq)\) if there is no \(b \in S\) such that \(a \prec b\). Similarly, \(a\) is minimal in \((S, \preceq)\) if there is no \(b \in S\) such that \(b \prec a\).

Example 10. What are the greatest and least elements of the poset \((\{2, 3, 4, 5, 10, 12, 15, 20, 24\}, \subseteq)\)?

Solution. The maximal elements are 15, 20, 24. And the minimal elements are 2, 3, 5.

Definition 26. Let \((S, \preceq)\) be a poset and \(a \in S\). Then \(a\) is the greatest element of \((S, \preceq)\) if \(b \preceq a\) for all \(b \in S\). Similarly, \(a\) is the least element of \((S, \preceq)\) if \(a \preceq b\) for all \(b \in S\).

Example 11. Is the poset \((\mathbb{Z}^+, |)\) a lattice?

Solution. Let \(a, b \in \mathbb{Z}^+\). The \(\text{lcm}(a, b)\) and \(\gcd(a, b)\) are the least upper bound and greatest lower bound of \(\{a, b\}\) respectively. \((\mathbb{Z}^+, |)\) is a lattice.

Example 12. Let \(S\) be a set. Is the poset \((\wp(S), \subseteq)\) a lattice?

Solution. Let \(A, B \subseteq S\). Then \(A \cup B\) and \(A \cap B\) are the least upper bound and greatest lower bound of \(\{A, B\}\) respectively. \((\wp(S), \subseteq)\) is a lattice.

An important application of partially ordered sets is the denotational semantics of programming languages. In denotational semantics, the meaning of a statement is defined as a monotonic function over a partially ordered set. Program constructs are interpreted by operations on such monotonic functions.