Disjoint Pattern Database Heuristics
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Abstract

- Introducing a new form of heuristic called pattern databases.
  - Compute the cost of solving individual subgoals independently.
  - If the subgoals are disjoint, then we can use the sum of costs of the subgoals as a new and better admissible cost function.
    - A way to get a new and better heuristic function by composing known heuristic functions.
  - Make use of the fact that computers can memorize lots of patterns.
  - Solutions to pre-stored patterns can be pre-computed.
  - Speed up factor of over 2000 compared to previous results in 1985.
Definitions

- $n^2 - 1$ puzzle problem:
  - The numbers 1 through $n^2 - 1$ are arranged in a $n \times n$ square with one empty cell.
    - Let $N = n^2 - 1$.
  - Slide the tiles to a given goal position.

- 15 puzzle:
  - May be invented in 1874 and was popular in 1880.
  - It looks like one can rearrange an arbitrary state into a given goal state.
  - Publicized and published by Sam Loyd in January 1896.
    - A prize of US$ 1000 was offered to solve one “impossible”, but seems to be feasible case.
    - Note: average wage per hour for a worker is US$0.3.

- Generalizations:
  - $n \times m - 1$ puzzle.
  - Puzzles of different shapes.
15 puzzle

- **Rules:**
  - 15 tiles in a 4*4 square with numbers from 1 to 15.
  - One empty cell.
  - A tile can be slid horizontally or vertically into an empty cell.
  - From an initial position, slide the tiles into a goal position.

- **Examples:**

  - **Initial position:**
    
    | 10 | 8  |     |
    |----|----|-----|
    | 3  | 7  | 6   |
    | 1  | 14 | 4   |
    | 15 | 13 | 9   |
    |     |    | 11  |
    | 15 | 13 | 9   |

  - **Goal position:**
    
    | 1  | 2  | 3  | 4  |
    |----|----|----|----|
    | 5  | 6  | 7  | 8  |
    | 9  | 10 | 11 | 12 |
    | 13 | 14 | 15 |    |
State space is divided into subsets of even and odd permutations [Johnson & Story 1879].

- Treat a board into a permutation by appending the rows from left to right and from top to bottom.
- $f_1$ is number of inversions in a permutation $\pi_1\pi_2\cdots\pi_N$ where an inversion is a distinct pair $\pi_i > \pi_j$ such that $i < j$.
  - Let $inv(i, j) = 1$ if $\pi_i > \pi_j$ and $i < j$; otherwise, it is 0.
  - $f_1 = \sum_{i,j} inv(i,j)$.
  - Example: the permutation $10,8,12,3,7,6,2,1,14,4,11,15,13,9,5$ has $9+7+9+2+5+4+1+0+5+0+2+3+2+1$ inversions.
- $f_2$ is the row number, i.e., 1, 2, 3, or 4, of the empty cell.
- $f = f_1 + f_2$.
- Even parity: one whose $f$ value is even.
- Odd parity: one whose $f$ value is odd.
- The parity of a board is either even or odd.
15 Puzzle — Properties

- There is a board with an even parity.
  
  - The goal position:
    
    \[
    \begin{array}{cccc}
    1 & 2 & 3 & 4 \\
    5 & 6 & 7 & 8 \\
    9 & 10 & 11 & 12 \\
    13 & 14 & 15 & \\
    \end{array}
    \]
  
  - \( f_1 = 0 \) and \( f_2 = 4 \).

- There is a board with an odd parity.
  
  - \( f_1 = 1 \) and \( f_2 = 4 \).

- It is possible to prove mathematically that we can slide tiles in between two boards with the same parity.

- Slide a tile never change the parity of a 15-puzzle board.
  
  - This may not be true for other values of \( n \) and for other shapes.
  
  - A proof sketch is given in the next slide.
Proof: Sketch

- Slide a tile horizontally does not change the parity.
- Slide a tile vertically:
  - Change the parity of $f_2$, i.e., row number of the empty cell.
  - Change the value of $f_1$, i.e., the number of inversions by
    - $+3$
    - $+1$
    - $-1$
    - $-3$
  - Example: when “a” is slided down
    - only the relative order of “a”, “b”, “c” and “d” are changed
    - analyze the 4 cases according to the rank of “a” in “a”, “b”, “c” and “d”.

\[
\begin{array}{|c|c|c|}
\hline
* & a & b \\
\hline
* & * & c \\
\hline
d & * & * \\
\hline
* & * & * \\
\hline
\end{array}
\]
Core of past algorithms

- Using DEC 2060 a 1-MIPS machine: solves several random instances of the 15 puzzle problem within 30 CPU minutes in 1985.
- Using Iterative-deepening A*.
- Using the Manhattan distance heuristic as an estimation of the remaining cost.
  - Suppose a tile is currently at $(i, j)$ and its goal is at $(i', j')$, then
    \[
    \text{the Manhattan distance for this tile is } |i - i'| + |j - j'|.
    \]
  - The Manhattan distance between a board and a goal board is the sum of the Manhattan distance of all the tiles.
- Manhattan distance is a lower bound on the number of slides needed to reach the goal position.
  - It is admissible.
Non-additive pattern databases

- Intuition: do not measure the distance of one tile at a time.
  - Pattern database: measure the collective distance of a pattern, i.e., a group of tiles, at a time.

- Complications.
  - The tiles get in each other’s way.
  - Sliding a tile to reach its goal destination may make the other tiles that are already in their destinations to move away.
  - A form of interaction is called linear conflict:
    - To flip two adjacent tiles needs more than 2 moves.
    - In addition, sliding tiles other than the two adjacent tiles to be flipped is also needed in order to flip them.
Example: Linear conflict

- The sum of Manhattan distance for the following position is 4.

```
3 2 1 4
5 6 7 8
9 10 11 12
13 14 15
```

- However it takes much more than 4 slides to reach the goal.

```
3 2 1 4
5 6 7 8
9 10 11 12
13 14 15
```

```
1 2 3 4
5 6 7 8
9 10 11 12
13 14 15
```
A fringe is the arrangement of a subset of tiles, and may include the empty cell, by treating tiles not selected don’t-care.

- Don’t-cared tiles are indistinguishable within themselves.
- The subset of tiles selected is called a pattern.

<table>
<thead>
<tr>
<th>*</th>
<th>*</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>8</td>
<td>*</td>
</tr>
<tr>
<td>*</td>
<td>13</td>
<td>*</td>
</tr>
<tr>
<td>*</td>
<td>14</td>
<td>*</td>
</tr>
</tbody>
</table>

- Example:

Notations for specifying a pattern.

- “*” means don’t-care.
- We need to know the whereabout of the empty cell no matter it is selected or not.

- An empty space means a selected empty cell.
- “♥” means an unselected empty cell.
Example:

\[
\begin{array}{ccc}
\ast & \ast & 4 \\
\ast & 8 & \ast \\
\ast & 13 & \ast \\
\ast & \ast & 14 \\
\end{array}
\]

In this example, there are 7 selected tiles, including the empty cell.

- There are \(\frac{16!}{9!} = 57,657,600\) possible fringe arrangements which is called the pattern size.

The goal fringe arrangement for the selected subset of tiles:

\[
\begin{array}{ccc}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
13 & 14 & 15 \\
\end{array}
\]
Solving a fringe arrangement

For each fringe arrangement, pre-compute the minimum number of moves needed to make it into the goal fringe arrangement.

● This is called the fringe number for the given fringe arrangement.
● There are many possible ways to solve this problem since the pattern size is small enough to fit into the main memory.

▷ Sample solution 1: Using the original Manhattan distance heuristic to solve this smaller problem.
▷ Sample solution 2: BFS.
Comments on pattern size

- **Pro’s.**
  - Pattern with a larger size is better in terms of having a larger fringe number.
  - A larger fringe number usually means better estimation, i.e., closer to the goal fringe arrangement.

- **Con’s.**
  - Pattern with a larger size means consuming lots of memory to memorize these arrangements.
  - Pattern with a larger size also means consuming lots of time in constructing these arrangements.

▷ *Depends on your resource, pick the right pattern size.*
Usage of fringe numbers (1/2)

- **Divide and conquer.**
  - Reduce a 15-puzzle problem into a 8-puzzle one.
  - Solution =
    - First reach a goal fringe arrangement consisted of the first row and column.
    - Then solve the 8-puzzle problem without using the fringe tiles.
    - Finally combining these two partial solutions to form a solution for the 15-puzzle problem.
  - May not be optimal.

- May not be working because often times you cannot combine two sub-solutions to form the final optimal solution easily.
  - In solving the second half, you may affect tiles that have reached the goal destinations in the first half.
  - The two partial solutions may not be disjoint.
Usage of fringe numbers (2/2)

- **New heuristic function** $h()$ for IDA*: using the fringe number as the new lower bound estimation.
  - The fringe number is a lower bound on the remaining cost.
    - *It is admissible.*

- **How to find better patterns for fringes?**
  - Large pattern require more space to store and more time to compute.
  - Can we combine smaller patterns to form bigger patterns?
    - *They are not disjoint.*
    - *May be overlapping physically.*
    - *May be overlapping in solutions.*
More than one patterns

- Can have many different patterns that may have some overlaps:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>3</th>
<th></th>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>*</td>
<td>*</td>
<td>7</td>
<td>*</td>
<td></td>
<td>5</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td></td>
<td>9</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>*</td>
<td>*</td>
<td>15</td>
<td>♠</td>
<td></td>
<td>13</td>
<td>*</td>
<td>*</td>
<td>♠</td>
</tr>
</tbody>
</table>

- Cannot use the divide and conquer approach anymore for some of the patterns.

- If you have many different pattern databases $P_1, P_2, P_3, \ldots$
  - The heuristics or patterns may not be disjoint.
    - Solving tiles in one pattern may help/hurt solving tiles in another pattern even if they have no common cells.
  - The heuristic function we can use is

\[
h(P_1, P_2, P_3, \ldots) = \max\{h(P_1), h(P_2), h(P_3), \ldots\}.
\]
Problems with multiple patterns (1/2)

- If you have many different pattern databases $P_1, P_2, P_3, \ldots$
  - It is better to have
    \[ h(P_1, P_2, P_3, \ldots) = h(P_1) + h(P_2) + h(P_3) + \cdots, \]
    instead of
    \[ h(P_1, P_2, P_3, \ldots) = \max\{h(P_1), h(P_2), h(P_3), \ldots\}. \]
  - A larger $h()$ means a better performance for $A^*$.
- Key problem: how to make sure $h()$ is admissible?
Problems with multiple patterns (2/2)

- Why not making the heuristics and the patterns disjoint?
  - Though patterns are disjoint, their costs are not disjoint.
    - Some moves are counted more than once.
  - If the patterns are not disjoint, then we cannot add them together.
    - Divide the board into several disjoint regions.

- Q: Why we add the Manhattan distance of all titles together to form a heuristic function?
  - We add 15 1-cell patterns together to form a better heuristic function.
  - What are the property of these patterns that can be added together?
Key observations (1/2)

- Partition the board into disjoint regions.
  - Using the tiles in a region of the goal arrangement as a pattern.

**Examples:**

```
<table>
<thead>
<tr>
<th>A</th>
<th>A</th>
<th>A</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>A</th>
<th>A</th>
<th>B</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>
```

- Can also divide the board into more than 2 disjoint patterns.

```
<table>
<thead>
<tr>
<th>A</th>
<th>A</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>C</td>
<td>B</td>
</tr>
</tbody>
</table>
```
For each region, solve the problem optimally and then count the moves **that are made only by tiles in this region**.
- Note: if the empty cell is selected, we do not count the moves of the empty cell.
- The “fringe” number for an arrangement is the minimum number of slides made on tiles in this region.
- It is now possible to add fringe numbers of all disjoint regions together to form a composite fringe number.

▷ **Q: How to prove this?**

For the Manhattan distance heuristic:
- Each pattern is a tile.
- They are disjoint.
  - They only count the number of slides made by each tile.
- Thus they can be added together to form a heuristic function.
Disjoint patterns

- A heuristic function $f()$ is **disjoint** with respect to two patterns $P_1$ and $P_2$ if
  - $P_1$ and $P_2$ have no common cells.
  - The solutions corresponding to $f(P_1)$ and $f(P_2)$ do not interfere each other.

- If they are disjoint, then $f(P_1) + f(P_2)$ is admissible if
  (1) $f()$ is disjoint with respect to $P_1$ and $P_2$ and
  (2) both $f(P_1)$ and $f(P_2)$ are admissible.
  - Q: How to prove this?
Revised fringe number

- Fringe number: for each fringe arrangement, the minimum number of moves needed to make it into the goal fringe arrangement.
  - Given a fringe arrangement $H$, let $f(H)$ be its fringe number.

- Revised fringe number: for each fringe arrangement $F$ during the course of making a sequence of moves to the goal fringe arrangement, the minimum number of fringe-only moves in the sequence of moves.
  - Given a fringe arrangement $H$, let $f'(H)$ be its revised fringe number.

- Given two patterns $P_1$ and $P_2$ without overlapping cells, then
  - $f(P_1)$ and $f'(P_1)$ are both admissible.
  - $f(P_2)$ and $f'(P_2)$ are both admissible.
  - $f(P_1) + f(P_2)$ is not admissible.
  - $f'(P_1) + f'(P_2)$ is admissible.

- Note: the Manhattan distance of a 1-cell pattern is a lower bound of its revised fringe number.
A special form of divide and conquer with additional properties.  
Spaces required by patterns must be within the main memory.  
Each pattern must be able to be solved optimally by “primitive” methods.  
It is better to put near-by tiles together to better deal with the conflicting problem.  
It is now possible to design a better admissible heuristic function $f$ by composing two simple admissible heuristic functions $f_1$ and $f_2$.  
  
- Let $f'_1$ be the function that does not count moves of tiles not in its region when computing $f_1$. 
  \[ f'_1(x) \leq f_1(x) \]  
- Let $f'_2$ be the function that does not count moves of tiles not in its region when computing $f_2$. 
  \[ f'_2(x) \leq f_2(x) \]  
- Let $f = f'_1 + f'_2$. 
  \[ \text{Hopefully, } f(x) > f_1(x) \text{ and } f(x) > f_2(x). \]
Performance

- Running on a 440-MHZ Sun Ultra 10 workstation.
  - SPECint = 1.0 (1 MIPS) in 1985.
  - SPECint = 17.9 in 2002.

- Solves the 15 puzzle problem that is more than 2,000 times faster than the previous result by using the Manhattan distance heuristic.
  - 2,000 * 17.9 times faster in wall time time.

- Solves the 24-puzzle problem
  - An average of two days per problem instance.
  - Generates 2,110,000 nodes per second.
  - The average solution length was 100.78 moves.
  - The maximum solution length was 114 moves.
  - Prediction: using the Manhattan distance heuristic, it would take an average of about 50,000 years to solve a problem instance.
    - The average Manhattan distance is 76.078 moves.
    - The average value for the disjoint database heuristic is 81.607 moves, which gives a tighter bound.
Other heuristics (1/2)

- The main drawback of disjoint heuristics is that they do not capture interactions between tiles in different regions.

- 2-tile pattern database:
  - For each pair of tiles, and for each pair of possible locations, compute the optimal solution, i.e., minimum number of all moves made by these 2 tiles, for this pair of tiles to both move to their destinations.
    - This is called pairwise distance.
    - For an $n^2 - 1$ puzzle, we have $O(n^4)$ different combinations.
    - For $n = 4$, $n^4 = 256$.
    - For $n = 5$, $n^4 = 625$.

- It is usually the case that the pairwise distance of 2 tiles $x$ and $y$ is larger than the sum of the Manhattan distances of $x$ and $y$.
  - The pairwise distance is at least the sum of the Manhattan distances.
For a given board, partition the board into a collection of 2-tiles so that the sum of cost is maximized.

- This can be done using a maximum weighted perfect matching.
- Build a complete graph with the tiles being the vertices.
- The edge cost is the pairwise distance between these two tiles.
- Try to find a perfect matching with the sum of edge costs being the largest possible.
- Algorithm runs in $O(n(m + n \log n))$ is known where $n$ is the number of vertices and $m$ is the number of edges.
The Manhattan distance is a partition into 1-tile patterns.

For 2-tile patterns:
- Faster approximation algorithms for finding maximum perfect matchings on complete graphs are known.
- The cost for exhaustive enumeration is
  $$\binom{16}{2} \binom{14}{2} \cdots \binom{4}{2} \binom{2}{2} / 8!$$
  $$= 16!/(2^8 \cdot 8!) = 2,027,025$$

Can also build 3-tile databases, but the corresponding 3-D matching problem for partitioning is NP-C.

Requires much less memory than that of the fringe method.

Some kinds of bootstrapping: solving smaller problems using primitive methods, and then using these results to solve larger problems.
What else can be done?

- Looks like some kinds of two-stage search.
  - First stage searching means building pre-computed results, e.g., patterns.
  - Second stage searching meets the pre-computed results if found.
- Better way of partitioning.
- Is it possible to generalize this result to other problem domains?
- How to decide the amount of time used in searching and the amount of time used in retrieving pre-computed knowledge?
  - Memorize vs Compute
References and further readings