Two-Player Perfect Information Games: A Brief Survey

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Abstract

- Domain: two-player games.
- Which game characters are predominant when the solution of a game is the main target?
  - It is concluded that decision complexity is more important than state-space complexity.
  - There is a trade-off between knowledge-based methods and brute-force methods.
  - There is a clear correlation between the first-player’s initiative and the necessary effort to solve a game.
Definitions (1/4)

- **Domain:** 2-person zero-sum games with **perfect information**.
  - **Zero-sum** means one player’s loss is exactly the other player’s gain, and vice versa.
    - *There is no way for both players to win at the same time.*

- **Game-theoretic value of a game:** the outcome, i.e., win, loss or draw, when all participants play optimally.
  - Classification of games’ solutions according to L.V. Allis [Ph.D. thesis 1994] if they are considered solved.
    - **Ultra-weakly solved:** the game-theoretic value of the initial position has been determined.
    - **Weakly solved:** for the initial position a strategy has been determined to achieve the game-theoretic value against any opponent.
    - **Strongly solved:** a strategy has been determined for all legal positions.
  - The game-theoretical values of most games are **unknown** or are only known for some **legal positions**.
    - *A legal position is one that can be reached from the initial position.*
Definitions (2/4)

- **State-space** complexity of a game: the number of the legal positions in a game.

- **Game-tree (or decision)** complexity of a game: the number of the nodes in a solution search tree.
  - A solution search tree is a tree where the game-theoretic value of the root position can be decided.
  - Each node in the tree is a legal position. The children of a parent node $P$ are the positions that $P$ can reach in one step.
  - Some legal states may not be in a solution search tree. ▶ These are unreasonable positions.
  - Some children of a node may not be in a solution search tree.

- A **fair** game: the game-theoretic value is draw and both players have roughly an equal probability on making a mistake.
  ▶ *Paper-scissor-stone*
  ▶ *Roll a dice and compare who gets a larger number*

- **Initiative**: the right to move first.
Definitions (3/4)

- A **convergent** game: the size of the state space decreases as the game progresses.
  - Start with many pieces on the board and pieces are gradually removed during the course of the game.
    - Example: Checkers.
  - It means the number of possible configurations decreases as the game progresses.

- A **divergent** game: the size of the state space increases as the game progresses.
  - May start with an empty board, and pieces are gradually added during the course of the game.
    - Example: Connect-5 before the board is almost filled.
  - It means the number of possible configurations increases as the game progresses.
A game may be convergent at one stage and then divergent at other stage.

- Most games are dynamic.
- For the game of Tic-Tac-Toe, assume you have played $x$ plys with $x$ being even.
  
  Then you have a possible of

$$\binom{9}{x/2} \cdot \binom{9 - x/2}{x/2}$$

different configurations.

- This number is not monotone increasing or decreasing.
Predictions made in 1990

- Predictions were made in 1990 [Allis et al 1991] for the year 2000 concerning the expected playing strength of computer programs.

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- **Over champion means definitely over the best human player.**
- **World champion means equaling to the best human player.**
- **Grand master means beating most human players.**
A double dichotomy of the game space

<table>
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<th>Category 4</th>
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<tr>
<td>If solvable at all, then</td>
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$$\log \log(\text{game-tree complexity}) \rightarrow$$
Questions to be researched

- Can perfect knowledge obtained from solved games be translated into rules and strategies which human beings can assimilate?
- Are such rules generic, or do they constitute a multitude of ad hoc recipes?
- Can methods be transferred between games?
  - More specifically, are there generic methods for all category-$n$ games, or is each game in a specific category a law unto itself?
Convergent games

- Since most games are dynamic, here we consider games whose ending phases are convergent.
  - Can be solved by the method of endgame databases if we can enumerate and store all possible positions at a certain stage.

- Problems solved:
    - The game theoretic value is draw.
  - Mancala games
    - Kalah: in the year 2000 upto, but not equal, Kalah(6,6)
  - Checkers
    - By combining endgame databases, middle-game databases and verification of opening-game analysis.
    - Solved the so called 100-year position in 1994.
    - The game is proved to be a draw in 2007.
  - Chess endgames
  - Chinese chess endgames
Since most games are dynamic, here we consider games whose INITIAL phases are divergent.

Connection games
- Connect-four (6 * 7)
- Qubic (4 * 4 * 4)
- Go-Moku (15 * 15)
- Renju
- \(k\)-in-a-row games
  - Hex (10 * 10 or 11 * 11)

Polynmino games: place pieces inside a board without overlapping and alternatively until one cannot place more.
- Pentominoes
- Domineering

Othello
- Chess
- Chinese chess
- Shogi
- Go
Connection games (1/2)

- **Connect-four (6 × 7)**
  - Also solved by L.V. Allis in 1988 using a knowledge-based approach by combining 9 strategic rules that identify potential threats of the opponent.
    - Threats are something like forced moves or moves you have little choices.
    - Threats are moves with predictable counter-moves.
  - It is first-player win.
  - Weakly solved on a SUN-4 workstation using 300+ hours.

- **Qubic (4 × 4 × 4)**
  - A three-dimensional version of Tic-Tac-Toe.
  - Connect-four played on a 4 × 4 × 4 game board.
  - Solved in 1980 by O. Patashnik by combining the usual depth-first search with expert knowledge for ordering the moves.
    - It is first-player win for the 2-player version.
Connection games (2/2)

- **Go-Moku** \((15 \times 15)\)
  - First-player win.

- **Renju**
  - Does not allow the first player to play certain moves.
  - An asymmetric game.
  - Weakly solved by Wágner and Viráag in 2000 by combining search and knowledge.
    - Took advantage of an iterative-deepening search based on threat sequences up to 17 plies.
    - It is still first-player win.

- **\(k\)-in-a-row games**
  - \(mnk\)-Game: a game playing on a board of \(m\) rows and \(n\) columns with the goal of obtaining a straight line of length \(k\).
  - Variations: first ply picks only one stone, the rest picks two stones in a ply.
    - Connect 6.
    - Try to balance the advantage of the initiative!
Hex \((10 \times 10 \text{ or } 11 \times 11)\)

- **Properties:**
  - It is a finite game.
  - It is not possible for both players to win at the same time.
  - Exactly one of the players can win.

Red won

Courtesy of ICGA web site
Proof on exactly one player win (1/2)

- It is easy to known there cannot be two winners.
  - When the first player wins, allow the second player to play one more time. If the second player also wins, then the game is tie.

- A topological argument.
  - A vertical chain can only be cut by a horizontal chain and vice versa because each cell is connected with 6 adjacent cells.
    - Note if a cell has 4 neighbors as in the case of Go, then it is possible to cut off a vertical chain by cells that are not horizontally connected and vice versa.

- Other arguments such as one using graph theory exist.
Proof on exactly one player win (2/2)

- We then prove there is at least one winner.
  - Assume there is no winner.

- W.l.o.g. let $R$ be the set of red cells that can be reached by chains originated from the rightmost column.
  - $R$ must contain a cell of the leftmost column; otherwise we have a contradiction.

- Let $N(R)$ be the blue cells that can be reached by $R$ originated from the rightmost column.
  - $N(R)$ must contain a cell in the top row.
    - Otherwise, $R$ contains all cells in the first row, which is a contradiction.
  - $N(R)$ must contain a cell in the bottom row.
    - Otherwise, $R$ contains all cells in the bottom row, which is a contradiction.
  - $N(R)$ must be connected.
    - Otherwise, $R$ can advance further.
  - Hence $N(R)$ is a blue winning chain.
Illustration of the ideas (1/3)
Illustration of the ideas (2/3)
Illustration of the ideas (3/3)
The unrestricted form of Hex is a first-player win game. using the “strategy-stealing” argument made by John Nash in 1949.

- If there is a winning strategy for the second player, the first player can still win by making an arbitrary first move and using the second-player strategy from then on.
  - The first player ignores the arbitrary first move by assuming that move does not exist.
  - Hence the second move made by the second player becomes the first move.
  - The third move made by the first player becomes the second move.

- If using the second-player strategy requires playing the chosen first move or any move played before, then make another arbitrary move.
  - An arbitrary extra move can never be a disadvantage in Hex.

- We have obtained a contradiction, and thus the second player cannot win.

- Since we have proved there is no draw, and there is always a winner, and both players cannot win at the same time, the first player must have a winning strategy.
Assume the second player $P_2$ has a winning function $f(B)$ that tells the next ply towards winning when seeing the board $B$.

- Assume the initial board position is $B_0$.
- $f(B)$ has a value only when $B$ is a legal position for the second player.
- $rev(x)$: interchange colors of pieces in a board or ply $x$.

The steps taken by the first player $P_1$ to also win

- $P_1$ makes an arbitrary first ply $m_1$. Call it $m'$.
- $P_2$ uses $f(B_0 + m_1)$ to make the second ply $m_2$.
- $P_1$ makes the third ply $m_3 = rev(f(B_0 + rev(m_2)))$.
  - If $m_3 = m'$, then make another arbitrary ply and let it be the new $m'$.
- $P_2$ uses $f(B_0 + m_1 + m_2 + m_3)$ to make the forth ply $m_4$.
- $P_1$ makes the fifth ply $m_5 = rev(f(B_0 + rev(m_2) + m_3 + rev(m_4)))$.
  - If $m_5 = m'$ or any ply made before, then make another arbitrary ply and let it be the new $m'$.

- ...
Hence we know it is not possible for the second player to win.

We also know these.

- There is exactly one winner when the board is completely filled.
- The game is finite.
- Hence we can enumerate the whole solution search tree.
  
  *In this solution search tree, there is a way for one player to win all of the times no matter the opponent reacts.*

Since the second player cannot win, the first player must have a winning strategy.
Strategy-stealing argument (4/4)

- This is not a constructive proof.
- The strategy-stealing argument cannot be used for every game.
  - An arbitrary extra move can never be a disadvantage in Hex.
  - This may not be true for other games.
- The argument works for any game when
  - it is symmetric,
  - it is history independent,
  - it always has exactly one winner, and
  - namely, it cannot have a draw by having no winners or 2 winners,
  - an arbitrary extra move can never be a disadvantage.
    - Note: it requires that a player is always possible to place an arbitrary move which may not be true for some games.
Properties of Hex

- **Variations of Hex**
  - The **one-move-equalization** rule: one player plays an opening move and the other player then has to decide which color to play for the remainder of the game.
    - The revised version is a second-player win game (ultra-weakly).

- **Hex exhibits considerable mathematical structure.**
  - Hex in its general form has been proved to be PSPACE-complete by Even and Tarjan in 1976 by converting it to a Shannon switching game.
  - The state-space and decision complexities are comparable to those of Go on an equally-sized board.

- **Solutions**
  - (Weakly or strongly) solved on a 6 * 6 board in 1994.
  - Maybe possible to solve the 7 * 7 case.
    - The 7 * 7 case was solved in 2001. [Yang et. al. 2001]
  - Not likely to solve the 8 * 8 version without fundamental breakthroughs.
    - The 8 * 8 case was solved in 2009. [Henderson et. al. 2009]
More divergent games (1/3)

- **Polynmino games**: placing 2-D pieces of a connected subset of a square grid to construct a special form.
  - Pentominoes
  - Domineering
  - Games on smaller boards have been solved.

- **Othello**
  - M. Buro’s LOGISTELLO beat the resigning World Champion by 6-0 in 1997.
  - Weakly solved on a $6 \times 6$ board by J. Feinstein in 1993.
    - Second player win.

- **Chess**
  - DEEP BLUE beat the human World Champion in 1997!
More divergent games (2/3)

- **Chinese chess**
  - Still in progress.
  - Professional 7-dan since 2007.

- **Shogi**
  - Still in progress.
  - Claimed to be professional 2-dan in 2007.
  - Defeat a Lady professional player in 2010.
Go

- Recent success and breakthrough using Monte Carlo UCT based methods between 2004 and 2012.
- Lack major theoretical or practical breakthrough since 2012.
  - Beat a professional 8-dan by having an 8-stone advantage.
  - Beaten by a professional 9-dan by giving a 7-stone advantage.
- Amateur 1 dan in 2010.
- Amateur 3 dan in 2011.
- The program Zen beat a 9-dan professional master at March 17, 2012.
  - First game: Five stone handicap and won by 11 points.
  - Second game: four stones handicap and won by 20 points.
## Table of complexity

<table>
<thead>
<tr>
<th>Game</th>
<th>$\log_{10}$ (state-space)</th>
<th>$\log_{10}$ (game-tree size)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nine Men’s Morris</td>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>Pentominoes</td>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>Awari</td>
<td>12</td>
<td>32</td>
</tr>
<tr>
<td>Kalak (6,4)</td>
<td>13</td>
<td>18</td>
</tr>
<tr>
<td>Connect-four</td>
<td>14</td>
<td>21</td>
</tr>
<tr>
<td>Domineering (8 * 8)</td>
<td>15</td>
<td>27</td>
</tr>
<tr>
<td>Dakon-6</td>
<td>15</td>
<td>33</td>
</tr>
<tr>
<td>Checkers</td>
<td>21</td>
<td>31</td>
</tr>
<tr>
<td>Othello</td>
<td>28</td>
<td>58</td>
</tr>
<tr>
<td>Qubic</td>
<td>30</td>
<td>34</td>
</tr>
<tr>
<td>Draughts</td>
<td>30</td>
<td>54</td>
</tr>
<tr>
<td>Chess</td>
<td>46</td>
<td>123</td>
</tr>
<tr>
<td>Chinese chess</td>
<td>48</td>
<td>150</td>
</tr>
<tr>
<td>Hex (11 * 11)</td>
<td>57</td>
<td>98</td>
</tr>
<tr>
<td>Shogi</td>
<td>71</td>
<td>226</td>
</tr>
<tr>
<td>Renju (15 * 15)</td>
<td>105</td>
<td>70</td>
</tr>
<tr>
<td>Go-Moku (15 * 15)</td>
<td>105</td>
<td>70</td>
</tr>
<tr>
<td>Go (19 * 19)</td>
<td>172</td>
<td>360</td>
</tr>
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State-space versus game-tree size

- In 1994, the boundary of solvability by complete enumeration was set at $10^{11}$.
  - The current estimation is about $10^{13}$ (since the year 2007).
- It is often possible to use heuristics in searching a game tree to cut the number of nodes visited tremendously when the structure of the game is well studied.
  - Example: Connect-Four.
Methods developed for solving games

- **Brute-force methods**
  - Retrograde analysis
  - Enhanced transposition-table methods

- **Knowledge-based methods**
  - Threat-space search and $\lambda$-search
  - Proof-number search
  - Depth-first proof-number search
  - Pattern search
    - To search for threat patterns, which are collections of cells in a position.
    - A threat pattern can be thought of as representing the relevant area on the board, an area that human players commonly identify when analyzing a position.

- **Recent advancements:**
  - Monte Carlo UCT based game tree simulation.
    - Monte Carlo method has a root from statistic.
    - Biased sampling.
    - Using methods from machine learning.
    - Combining domain knowledge with statistics.
  - A majority vote algorithm.
Brute-force versus knowledge-based methods

Games with both a relative low state-space complexity and a low game-tree complexity have been solved by both methods.
  
  - **Category 1**
    - Connect-four and Qubic

Games with a relative low state-space complexity have mainly been solved with brute-force methods.
  
  - **Category 2**
    - Namely by constructing endgame databases
    - Nine Men’s Morris

Games with a relative low game-tree-complexities have mainly been solved with knowledge-based methods.
  
  - **Category 3**
    - Namely, by intelligent (heuristic) searching
    - Sometimes, with the helps of endgame databases
    - Go-Moku, Renju, and $k$-in-a-row games
Advantage of the initiative

- Theorem (or argument) made by Singmaster in 1981: The first player has advantages.
  - Two kinds of positions
    - \( P \)-positions: the previous player can force a win.
    - \( N \)-positions: the next player can force a win.
  - Arguments
    - For the first player to have a forced win, just one of the moves must lead to a \( P \)-position.
    - For the second player to have a forced win, all of the moves must lead to \( N \)-positions.
    - It is easier to the first player to have a forced win assuming all positions are randomly distributed.
    - Can be easily extended to games with draws.

- Remarks:
  - One small boards, the second player is able to draw or even to win for certain games.
  - Cannot be applied to the infinite board.
How to make use of the initiative

- **A potential universal strategy for winning a game:**
  - Try to obtain a small advantage by using the initiative.
    - *The opponent must react adequately on the moves played by the other player.*
  - To reinforce the initiative the player searches for threats, and even a sequence of threats using an evaluation function $E$.
  - Force the opponent to always play the moves you expected.

- **Threat-space search**
  - Search for threats only!
Offsetting the initiative

- An example of a game with a huge initiative:
  - A connection $mn_1$-game.
    - 一子棋 was mentioned in 張系國著名小說”棋王” (1978年出版).
  - A connection $mn_2$-game.
  - A connection $mn_3$-game.

- Need to offset the initiative.
  - The offsetting rule must be simple.
  - The revised game must be as fair as possible.
    - It is difficult to prove a game is fair.
    - Example: Paper-scissor-stone is fair.
  - The revised game needs be fun to play with.
  - The revised game cannot be too much different from the original game.

- Knowing how to properly offsetting the initiative may uncover some fundamental properties of the game such as its level of difficulty.
Examples (1/2)

- Enforce rules so that the first player cannot win by selective patterns.
  - Renju.
    - Still first-player win.
  - Go (19 * 19).
    - The first player must win by more than 7 stones.
    - Komi = 7.5 in 2011.
    - The value of Komi changes with the time and maybe different because of using different set of rules.

- The one-move-equalization rule: one player plays an opening move and the other player then has to decide which color to play for the reminder of the game.
  - Hex.
  - Second-player win.
Examples (2/2)

- The first move plays one stone, the rest plays two stones each.
  - Connect 6.
  - Intuitively, in each turn the initiative goes to different players alternatively.
  - Still not able to prove the game is fair (at 2015).

- The first player uses less resource.
  - For example: using less time.
    - Chinese chess.
  - A resource-auctioning scheme.

- Unclear about how to redesign a game to make it fair.
Conclusions

- The knowledge-based methods mostly inform us on the structure of the game, while exhaustive enumeration rarely does.
- Many ad-hoc recipes are produced currently.
  - The database can be used as a corrector or verifier of strategies formulated by human experts.
- It may be hopeful to use data mining techniques to obtain cross-game methods.
  - Currently not very successful.
Comments

- Can combine knowledge-based method with exhaustive enumeration.
  - For converging games, build endgame databases when the remaining state spaces is manageable.
    - Example: build endgames with at most 5 pieces in Chess and stop searching when the number of pieces on the board is less than 6.
  - For diverging games, pre-compute all possible opening moves and solve them one by one in sequence or in parallel.

- This is different from the usage of pattern databases in solving one-player games.
  - Patterns are used to guide the search in solving one-player games.
  - Endgame databases are used here to stop the search earlier. The idea has a flavor like that of bi-directional search.
1990’s Predictions — 2000’s Status

- Predictions were made in 1990 [Allis et al 1991] for the year 2000 concerning the expected playing strength of computer programs.

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- **color code**
  - **Green**: Performs much better than expected
  - **Red**: right on the target.
  - **Black**: have some progress towards the target.
  - **Blue**: not so.
Predictions for 2010

Predictions were made at the year 2000 for the year 2010 concerning the expected playing strength of computer programs.

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<td></td>
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Predictions for 2010 – Status

- My personal opinion about the status of Prediction-2010 at October, 2010, right after the Computer Olympiad held in Kanazawa, Japan.

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<tr>
<td></td>
<td>Lines of Action</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **color code**
  - **Red**: right on the target.
  - **Black**: have some progress towards the target.
  - **Blue**: not so.


