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ANO(min[m.n,n^2loglogn])MAXIMUM WEIGHT CLIQUE ALGORITHM FOR CIRCULAR-ARC GRAPHS+

AN O(min[$m \cdot n, n^2 loglogn$]) MAXIMUM WEIGHT CLIQUE ALGORITHM FOR CIRCULAR–ARC GRAPHS†

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ABSTRACT

Let F be a family of arcs in a circle, where each arc is associated with a weight. The maximum weight clique problem is to find a subset of pairwise overlapping arcs in F such that their total weight is maximum. This problem was originally solved by Hsu in O($m \cdot n$) time. Recently, a O($n^2 loglogn$) algorithm was discovered by Apostolico and Hambrusch for the unweighted case, whereas the weighted case was posed as an open problem. We extend their approach to solve the weighted case in O(min[$m \cdot n, n^2 loglogn$]) time. Our algorithm is based on a characterization of maximum weight independent sets in bipartite graphs, which is not related to bipartite matching.

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An O(min[$m \cdot n, n^2 \log\log n$]) Maximum Weight Clique Algorithm for Circular–Arc Graphs

1. Introduction

A circular arc family F is a collection of arcs on a circle. A graph G is a ***circular–arc graph*** if there is a circular arc family F and a one–to–one mapping of the vertices of G and the arcs in F such that two vertices in G are adjacent if and only if their corresponding arcs in F overlap. Circular–arc graphs have applications in compiler design and traffic light sequencing [2]. Various characterization and optimization problems on circular–arc graphs have been studied [2]. In this paper, we consider the ***maximum weight clique*** (MWC) problem for circular–arc graphs. A ***clique*** in a graph is a set of pairwise adjacent vertices. Let n be the number of vertices in G and m be the number of edges in G. An earlier result of Hsu [3] provides an O($m \cdot n$) algorithm for this problem (erroneously, this was quoted as O($n^2 \log n + m \cdot n$) in [1]). Later, Apostolico and Hambrusch [1] gave an O($n^2 \log\log n$) algorithm for the unweighted case based on bipartite matching. They posed the weighted case as an open problem. In this paper, we extend their technique to produce an O(min[$m \cdot n, n^2 \log\log n$]) algorithm for the weighted case. Our algorithm is based on a characterization of maximum weight independent sets in bipartite graphs, which is not related to bipartite matching.

Consider an arc family F and its intersection graph G. Without loss of generality, assume all arc endpoints are distinct and no arc covers the entire circle. Now, label the n arcs arbitrarily from 1 through n. Label the endpoints from 1 to 2n according to their clockwise order. Denote an arc that begins at endpoint a and ends at endpoint b in the clockwise direction by (a,b). Define a to be the ***head*** (or ***counterclockwise endpoint***) of the arc and b to be the ***tail*** (or ***clockwise endpoint***). An example is shown in Figure 1. The continuous part of the circle which begins

with endpoint c and ends with d in the clockwise direction is referred to as *segment* (c,d) of the circle. We use "arc" to refer to a member of F and "segment" to refer to a part of the circle between two endpoints. By this definition, an arc (a,b) of F is also a segment (a,b).

An arc i is said to *contain* another arc j if both endpoints of j are contained in arc i. Define a *minimal arc of a clique* Q to be any arc of Q which does not contain another arc of Q. For each arc u of F, we determine a MWC Q^u among all cliques containing arc u as a minimal arc. Among these n cliques, a Q^u with the maximum weight is a MWC of G.

Hence, we shall now consider a fixed arc u = (a,b) and search for Q^u. Note that arcs which pass through both a and b will naturally be included in Q^u. Therefore, our selection is among those arcs which pass through exactly one endpoint of u. Let A be the set of those arcs passing through endpoint a and B the set of those passing through endpoint b. Arcs in A (respectively, B) will be referred to as A–arcs (respectively, B–arcs). Obviously, arcs in A overlap with each other. The same holds for arcs in B. Our algorithm is to optimally select a subset of A and one of B such that every two arcs in these subsets overlap with each other. Assume all arcs not in A ∪ B have already been deleted. Starting from b, one can order all endpoints in the interior of segment (b,a) along the clockwise direction into a linear list **L**. Starting from a, one can order all endpoints in the interior of segment (a,b) along the clockwise direction into a linear list **R**. List L contains heads of all A–arcs and tails of all B–arcs. List R contains tails of all A–arcs and heads of all B–arcs. An example is shown in Figure 1.

Fig. 1. The two lists L and R

Define the *complement* $\overline{G}$ of G to be the graph whose vertex set is the same as G and whose edges are exactly those missing in G. Since both A and B induce complete subgraphs of G, the subgraph K of $\overline{G}$ induced on vertices corresponding to

arcs in A ∪ B is a bipartite graph. Hence, finding a MWC in A ∪ B in F is equivalent to finding a maximum weight independent set (MWIS) of K. In the unweighted case, there is a close relationship between a maximum matching and a maximum independent set in a bipartite graph. However, such relationship does not exist in the weighted case. Instead, we resort to the following characterization of MWISs in bipartite graphs.

Theorem 1. Let $K = (X \cup Y, E)$ be a bipartite graph with weights assigned to its vertices. Then an independent set $P \subseteq X \cup Y$ is a MWIS in K if and only if both of the following two conditions hold:

(1.1) For each subset X' of $X \backslash P$, the set Y' of vertices in $Y \cap P$ adjacent to some vertices in X' satisfies $w(Y') \geq w(X')$.

(1.2) For each subset Y'' of $B \backslash P$, the set X'' of vertices in $X \cap P$ adjacent to some vertices in Y'' satisfies $w(X'') \geq w(Y'')$.

Proof: The "only if" part: Let P be a MWIS in K. If there exists a subset X' in $X \backslash P$ such that its corresponding Y' satisfies $w(Y') < w(X')$, then $X' \cup (P \backslash Y')$ is an independent set of K with weight $w(X') + w(P) - w(Y') > w(P)$, a contradiction. Hence, (1.1) holds. Similarly, one can show that (1.2) holds.

The "if" part: Let P be an independent set of K satisfying conditions (1.1) and (1.2). Let P^* be a MWIS of K. Let X' be $X \cap (P^* \backslash P)$ and Y' be its corresponding adjacent set in $Y \cap P$. Let Y'' be $Y \cap (P^* \backslash P)$ and X'' be its corresponding adjacent set in $X \cap P$. Then we have $w(Y') \geq w(X')$, $w(X'') \geq w(Y'')$, $P \cap (X' \cup Y'') = \emptyset$ and $P^* \cap (X'' \cup Y') = \emptyset$. Since P^* is the disjoint union of sets $(P \cap P^*)$, X' and Y'' and P contains the following three disjoint sets $(P \cap P^*)$, Y' and X'', we have

$$w(P^*) = w(P \cap P^*) + w(X') + w(Y'') \leq w(P \cap P^*) + w(Y') + w(X'') \leq w(P)$$

Hence, P is also a MWIS of K. ∎

2. The MWC Algorithm

We use u to denote an arc in F, j to denote an arc in A, i to denote an arc in B and Q to denote a MWC among all arcs scanned. The idea of the algorithm can be described as follows. We scan the list L from left to right. Every time the tail of a B–arc i is encountered, arc i is included in Q. Every time the head of an A–arc j is encountered, we test whether there is a clique of larger weight containing j and update Q. When all endpoints in L are scanned, the current Q is a MWC in A ∪ B. The testing involves exchanging certain arcs in Q with some arcs not in Q, which is the main part of our algorithm.

Denote the original weight of each arc u by w(u). Its modified weight at each iteration is denoted by w'(u). We use a graph H to keep track of the relationships of all arcs scanned so far and modify H by adding vertices and edges (but never delete vertices or edges) along the algorithm. Each vertex of H corresponds to an arc already scanned. Each edge of H connects a vertex representing an A–arc to one representing a B–arc and is assigned a weight. The edges of H are added so that the entire graph becomes a forest. Each component of H is a tree which is associated with a root. If this root corresponds to an A–arc, then the tree is called an ***A–tree***; otherwise, it is called a ***B–tree***. The union of all A–trees is called the ***A–forest*** of H. The union of all B–trees is called the ***B–forest***. Denote the vertex of H corresponding to an arc i in F by v_i. Denote the arc of F corresponding to a vertex v in H by i_v. A priority queue AVAIL is used to store the roots of all the B–trees. The algorithm maintains the following conditions for H:

(2.1) Let v be a vertex in a tree other than the root. Then $w(i_v)$ is equal to the total weight of edges incident to it.

(2.2) Let v be the root of a tree. Then $w(i_v)$ is no less than the total weight of edges incident to it. Furthermore, if i_v is a B–arc, then $w(i_v)$ is strictly greater than the total weight of edges incident to it.

(2.3) Every A–arc corresponding to a vertex in the A–forest overlaps with every B–arc corresponding to a vertex in the B–forest.

The clique Q constructed from arcs scanned so far consists of all A–arcs corresponding to vertices in the A–forest and all B–arcs corresponding to vertices in the B–forest. An A–tree, once formed, will never be modified in later iterations.

We now describe the modification of H at each iteration. If the tail of a B–arc i is scanned, then we add a new vertex v_i representing i to H as a (single vertex) B–tree. If the head of an A–arc j is scanned, then we add a new vertex v_j to H representing j and add new edges incident to v_j according to the two cases below. Let $\{v_{i_1}, ..., v_{i_r}\}$ be the roots of the B–forest whose corresponding arcs $i_1, ..., i_r$ do not overlap with j and let them be arranged according to the R–order.

Case 1. $w(j) \geq \sum_{t=1}^{r} w'(i_t)$. In this case, we can show that j is contained in some MWC among all arcs scanned. Hence, it is included in the current Q. Form an A–tree T with root v_j by adding edges (v_j,v_{i_t}), $t = 1,...,r$ with weight $w'(i_t)$. Set $w'(j) = w(j) - \sum_{t=1}^{r} w'(i_t)$.

Case 2. $w(j) < \sum_{t=1}^{r} w'(i_t)$. In this case, we can show that j is not contained in any MWC among all arcs scanned and our Q remains unchanged. Let i_s be the first arc in the list $i_1, ..., i_r$ such that $w(j) < \sum_{t=1}^{s} w'(i_t)$. Form a B–tree T with root v_{i_s} by adding edges (v_j,v_{i_t}), $t = 1,...,s-1$ with weight $w'(i_t)$ and edge (v_j,v_{i_s}) with weight $w(j) - \sum_{t=1}^{s-1} w'(i_t)$. Reduce the current weight of i_s to $\sum_{t=1}^{s} w'(i_t) - w(j)$.

We describe the above procedure more formally in Figure 2. Let g be $|A \cup B|$.

Procedure FIND–CLIQUE
INPUT: Two sets A, B of weighted arcs ordered according to the list L
OUTPUT: A MWC from A ∪ B

begin
$w'(u) \leftarrow w(u)$; AVAIL $\leftarrow \emptyset$; (* the priority queue storing all roots of B–trees *)
for k = 1 to g **do**

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    if the k–th arc is a B–arc i then do
        add v_i as a single vertex B–tree to H;
        insert arc i into AVAIL according to its R–order;
    else do (* the k–th arc is an A–arc denoted by j *)
        exchange ←. "true"; add v_j as a new vertex to H;
        while exchange is true & there is a B–arc in AVAIL not overlapping
        with j do
            i_s ← the B–arc of the smallest R–order in AVAIL not overlapping
                  with j;
            connect v_j to v_{i_s} by an edge and call the new tree T;
            if w'(j) < w'(i_s) then do (* do not exchange *)
                w'(i_s) ← w'(i_s) – w'(j); w(j,i_s) ← w'(j);
                exchange ←. "false"; make v_{i_s} the root of T;
            else do (* continue the process *)
                w'(j) ← w'(j) – w'(i_s); w(j,i_s) ← w'(i_s);
                delete i_s from AVAIL; make v_i the root of T;
        endwhile;
    endif;
endfor;
Q ← all A–arcs corresponding to vertices in the A–forest and all B–arcs
    corresponding to vertices in the B–forest;
end;
```

Figure 2. The clique construction procedure

3. Correctness of the Algorithm

To facilitate our discussion, denote the graph H formed at iteration k by H_k and the clique by Q_k. Let H_k' be the subgraph of $\overline{G}$ induced on vertices of H_k, k = 1,...,g. It is easy to verify that H_k is a spanning forest of H_k'. Denote the complement of H_k' by $\overline{H_k'}$

Theorem 2. For k = 1,...,g, *the set Q_k constructed is a clique of maximum weight among all cliques in $\overline{H_k'}$.*

Proof. We prove this by induction. Assume it is true for Q_1, ..., Q_{k-1} and consider Q_k.

First of all, it is easy to verify that the subgraph H_k satisfies properties (2.1) and (2.2). We show that Q_k is a clique in G and, as an independent set in H_k',

satisfies (1.1) and (1.2). We do this by proving the following three claims. Denote the root of a tree T in a subgraph by r(T). Denote the R–order of the arc i corresponding to vertex v_i of H_k by $R(v_i)$.

Claim 1. Let i and i' be two B–arcs so that v_i and v_i' are contained in two different B–trees T_s, T_s' of H_s, respectively with $R(r(T_s)) < R(r(T_s'))$. If v_i and v_i' are also contained in two different B–trees T_p, T_p' of H_p, respectively, with $p > s$, then $R(r(T_p)) < R(r(T_p'))$.

Proof. We show this by induction on the subgraph index. Assume this is true for all subgraphs H_ℓ with $1 \le \ell < p$ and consider H_p. Let the two trees containing v_i and v_i' at iteration p–1 be T_{p-1} and T_{p-1}', respectively. By induction, $R(r(T_p)) < R(r(T_{p-1}'))$. If the p–th arc is a B–arc, then $T_p = T_{p-1}$, $T_p' = T_{p-1}'$ and the claim is true by induction. Hence, assume the p–th arc j^* is an A–arc. Let $S = \{i_1, ..., i_r\}$ be the set of arcs corresponding to roots of the B–forest in H_{p-1} which do not overlap with j^* and let them be arranged according to the R–order. Because i and i' are contained in different B–trees at iteration p, v_j^* cannot be adjacent to both $r(T_{p-1})$ and $r(T_{p-1}')$ in H_{p-1}. If v_j^* is adjacent to neither $r(T_{p-1})$ nor $r(T_{p-1}')$, then again, $T_p = T_{p-1}$, $T_p' = T_{p-1}'$ and we are done. Otherwise, consider two cases:

Case (a). v_j^* is adjacent to $r(T_{p-1})$ but not $r(T_{p-1}')$: then $r(T_p') = r(T_{p-1}')$ and the arc for $r(T_p')$ must be in S. By Case 2 of the algorithm, v_i and v_j^* are contained in the same B–tree with root $r(T_p)$ and $R(r(T_p)) < R(r(T_p'))$.

Case (b). v_j^* is adjacent to $r(T_{p-1}')$ but not $r(T(_{p-1}))$: then $r(T_p) = r(T_{p-1})$ and $R(r(T_p')) \ge R(r(T_{p-1}')) > R(r(T_{p-1})) = R(r(T_p))$. ∎

Claim 2. Q_k is a clique in G, namely, H_k satisfies (2.3).

Proof. This is obvious if the k–th arc is a B–arc i because $Q_k = Q_{k-1} \cup \{i\}$ and i overlaps with all A–arcs in H_k. Hence, suppose the k–th arc is an A–arc j. Let i_1,

..., i_r be the arcs corresponding to roots of the B–forest in H_{k-1} which do not overlap with arc j and let them be arranged according to the R–order. Consider the two cases discussed in the construction procedure.

Case 1. $w(j) \geq \sum_{t=1}^{r} w'(i_t)$: In this case, the algorithm connects v_j to each of i_1, ..., i_r and make v_j the root of this new A–tree in H_k. It also adds to Q_k all A–arcs in T and remove all B–arcs in T from Q_{k-1}. Hence, we need to show that the A–arcs corresponding to vertices in T are overlap with all B–arcs corresponding to vertices in the B–forest of H_k. For each A–arc j' in T, let $S_j' = \{i_1', ..., i_w'\}$ be the set of B–arcs whose corresponding vertices appear in H_k which do not overlap with j'. All arcs in S_j' must have their tails scanned before j'. We show that S_j' is contained in the A–forest of H_k.

Suppose, to the contrary, there is an arc i_q' of S_j' with $v_{i_q'}$ contained in a B–tree of H_k. Let s be the iteration in which the head of j' is scanned. Let T_s be the B–tree containing v_j' and T_s' be the B–tree containing $v_{i_q'}$ at the s–th iteration. Then $R(r(T_s)) < R(r(T_s'))$. Since $v_{i_p'}$ is in the same B–tree with v_j' in H_{k-1}, $v_{i_p'}$ and $v_{i_q'}$ must be in different B–trees in H_{k-1}. Let T_{k-1} and T_{k-1}' be the subtrees in H_{k-1} containing $v_{i_p'}$ and $v_{i_q'}$, respectively. By Claim 1, $R(r(T_{k-1})) < R(r(T_{k-1}'))$. This implies that the arc for $r(T_{k-1}')$ is among i_1, ..., i_r and would have been made adjacent to v_j' at the k–th iteration, a contradiction to the assumption that i_q' is in a B–tree in H_k.

Case 2. $w(j) < \sum_{t=1}^{r} w'(i_t)$: in this case $Q_k = Q_{k-1}$ and the claim is trivially true. ∎

Claim 3. Q_k satisfies (1.1) and (1.2) in H_k'.

Proof. Let A_k be the set of vertices in H_k that correspond to A–arcs and B_k be those correspond to B–arcs. We first show that (1.1) holds for the set V_k of vertices corresponding to arcs in Q_k. Let A' be any subset in $A_k \backslash V_k$. Because Q_k contains

all A–arcs corresponding to vertices in the A–forest of H_k, the vertices in A' must be contained in the B–forest. Hence, A' does not contain any root. Let E_1 be the set of edges of H_k incident to some vertices in A'. Let B_1 be the subset of vertices in V_k which are adjacent to some vertices in A' through edges in E_1. Let B' be the subset of vertices in V_k which are adjacent to some vertices in A' through edges of H_k'. Then, $B_1 \subseteq B'$. By property (2.1), $w(A') = w(E_1) \leq w(B_1) \leq w(B')$. The fact that (1.2) holds for Q_k can be proved similarly. ∎

4. Conclusions

We claim that Q^u can be found in $O(\min[m,n^2\text{loglog}n])$ time for each u. There will be g iterations in the FIND–CLIQUE procedure. Since the graph H_g constructed has $O(g)$ edges, it is easy to see that the major work involved is to process the set AVAIL. Depending on which of m and $n\text{loglog}n$ is smaller, We can adopt one of the following two strategies:

The first strategy is used in Fig. 1. Following the idea of Van Emde Boas [4], we use a priority queue over the fixed list R to process the active arcs in AVAIL. Our algorithm uses operations *insert, successor, delete,* each of which can be implemented in $O(\text{loglog}g)$ time. The total number of such operations is bounded by $O(g)$ and the total time spent in finding Q^u is $O(g\text{loglog}g)$, which is bounded by $O(n\text{loglog}n)$.

The second strategy does not require any sophisticated data structure. It results in the same complexity as that in Hsu [2]. However, the approach used in this paper is simpler. For each arc (e,f), order its adjacent arcs according to their first endpoint encountered traversing from f along the clockwise direction. This can be done in $O(m)$ time assuming all endpoints are sorted in the clockwise direction. Initially, set AVAIL to be all arcs in A ∪ B, represented by a linked list ordered according to R. At iteration k, update AVAIL to be the set of all A–arcs and the

set of all B–arcs that are roots of the B–forest of H_k. That is, every time a root of a B–tree becomes a non–root, we delete it from AVAIL. If the k–th arc is a B–arc, then AVAIL is not changed. If the k–th arc j is an A–arc, then root arcs of the B–forest in H_{k-1} (whose weights are to be compared against w'(j)'one by one) are exactly those in AVAIL but not in the adjacency list of j. Since both lists are ordered according to R, we can find the first (and subsequent ones, if necessary) such B–root by a linear scan of AVAIL and the adjacency list of j. Thus, the total time spent in finding Q^u is proportional to $O(m)$.

In the current algorithm, we have not been able to pass the maximum clique information from one arc to another. This appears to be the only approach which could lead to a more efficient algorithm for the maximum clique problem on circular–arc graphs.

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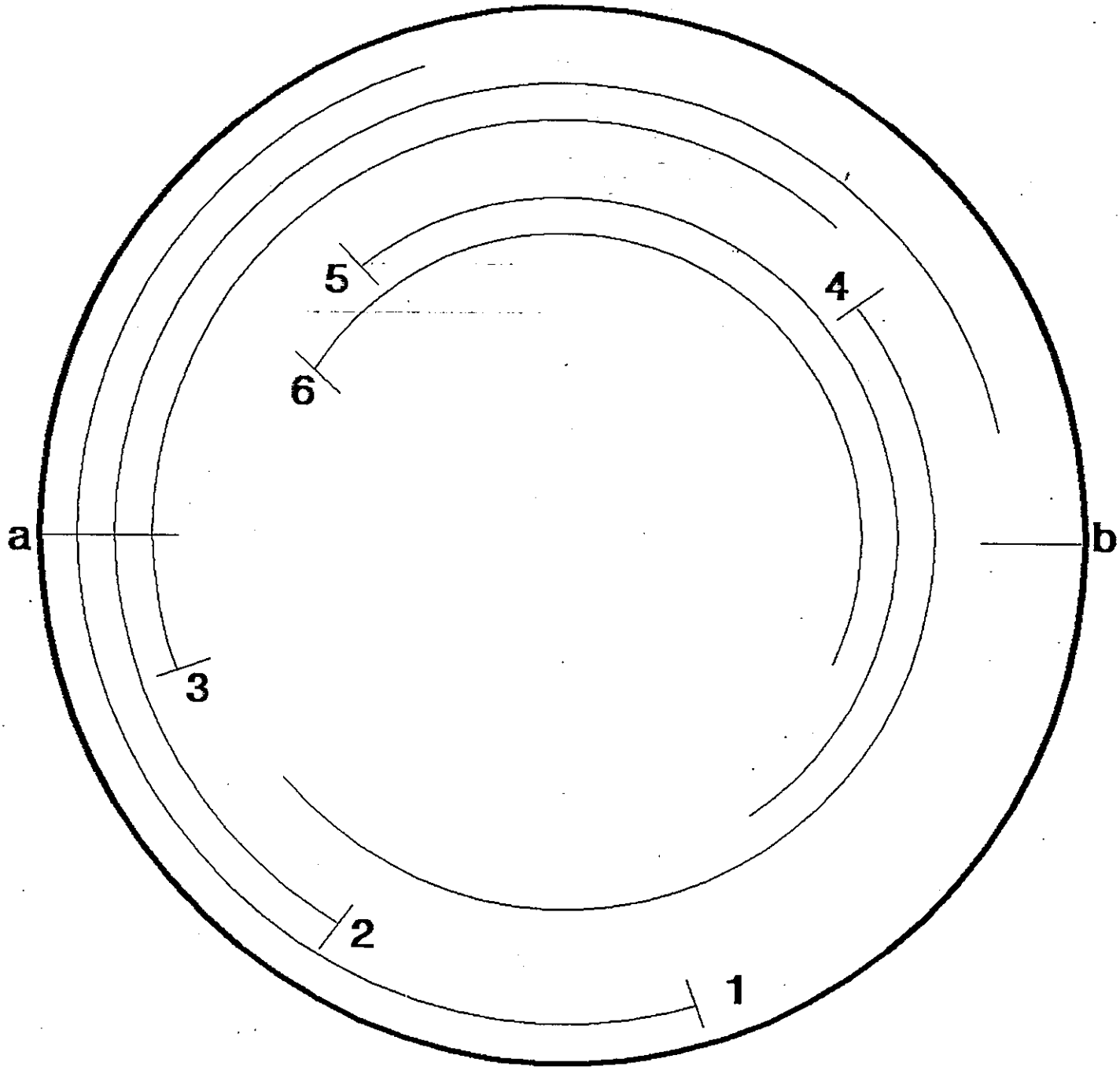

Arcs in A: 1, 2, 3

Arcs in B: 4, 5, 6

List L: 6, 5, 1, 2, 4, 3

List R: 6, 5, 1, 3, 4, 2

Figure 1. The two lists L and R