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TR-IIS-19-001

A Prehistoric Calculator

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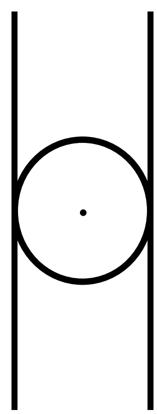
May 29, 2019 || Technical Report No. TR-IIS-19-001
http://www.iis.sinica.edu.tw/page/library/TechReport/tr2019/tr19.html

A PREHISTORIC CALCULATOR

There is a symbol of prehistoric origin, which is key to a powerful method of computation. It does not require the use of numbers, but can practically be used to add or subtract distances, compare areas and even to find the length of the hypotenuse of a right triangle. Thus its use would be of particular use to those of a primitive society which had to construct buildings, lay out camp sites and allot land for ownership. Later on, when that society understood the meaning and use of numbers, it could be used to perform virtually all arithmetic and even to solve quadratic equations. Whether or not it was ever used for these things is unknown, but we shall show here how it could have been.

The symbol consists of a circle, its center point and two parallel lines tangent to that circle. It is shown in `Figure 1` below:



`Figure 1`

The symbol has been found in various cultures of antiquity, and in particular on early Egyptian monuments, thus predating the building of King Solomon's temple in 974 BC. The society of Free and Accepted Masons, of which the ritual and traditions derive in great part from the building of Solomon's temple, has sometimes described itself as a "*system of morality veiled in allegory and illustrated by symbols*". In such as organization, it is sometimes the case that some of its symbols may have both an exoteric meaning, which is exposed to all its members and occasionally to the general public: as well as an esoteric meaning which is given only to a select few.

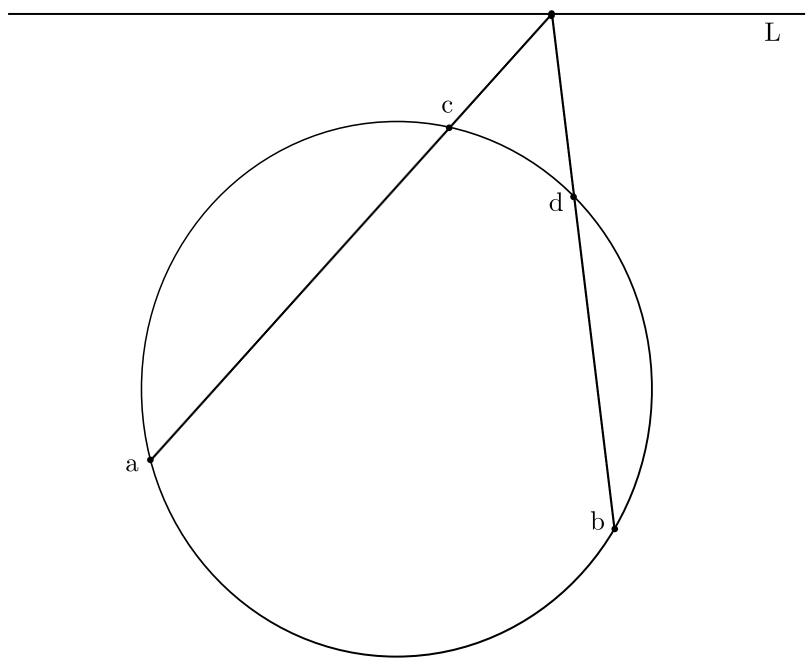
The symbol in `Figure 1` is one of the Masons' symbols, and is presented to its new initiates as one of its most important. When the Masons adopted it, they added an image of a holy book above the circle, a man representing Saint John the Baptist to the left of the left vertical line and a man representing Saint John the Evangelist to the right of the right vertical line.[1] The exoteric meaning given is that the point in the circle represents a Mason and the circle represents a boundary beyond which a Mason's behavior should not transgress. The circle may also represent the positions of the sun at meridian height during the course of the year, where the tangent point to the left signifies the lowest position at the winter solstice and the tangent point of the right the highest point at summer solstice. The placement of the Saints John to the left and the right are explained by the fact the birthday of John the Baptist was near the winter solstice and the birthday of John the Evangelist near the summer solstice.

Masons have sought an esoteric explanation from the occurrences and uses of the symbol in ancient times in Egypt and India. Most of these found seem to have been related to ideas and gods connected with procreation. But the name "Masonry", and the supposed origins of the society, are connected with the construction of buildings, and one might reasonably suspect that its most important symbols would

[1] Some depictions substitute a letter B for John the Baptist and an E for John the Evangelist. One of the fraternity's traditions is that the first Masonic Lodge was located in Jerusalem and was dedicated to these two men.

somehow be connected with the building craft. There is a meaning for the symbol in `Figure 1` which would be useful to that craft, which may extend back before recorded history, and may have been lost to the ages. The sequel is an exposition of this meaning.

This "calculator symbol" of `Figure 1` works by tying two separate concepts together. One of these concepts is that of "balanced pairs of a circle relative to a line". Imagine a circle C in a plane, together with a line L in the same plane. We shall say that two pairs {a,c} and {b,d} of points on C are **balanced relative to L** if the line determined by a and c intersects the line determined by b and d at a point on L:[2]



`Figure 2: Balanced Pairs {a,c} and {b,d} Relative to a Line L`

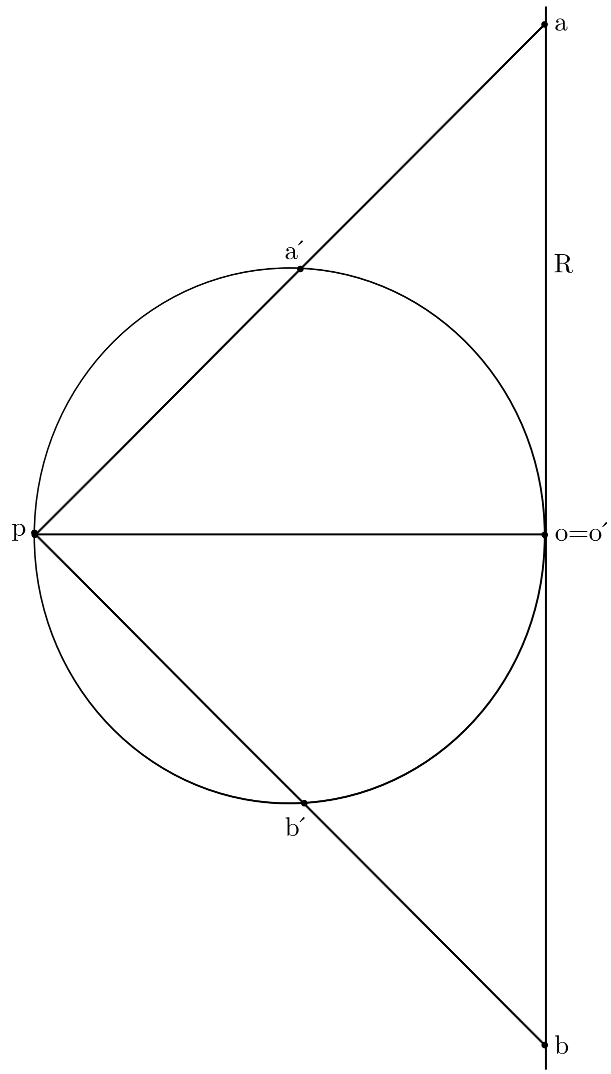
The line L in `Figure 2` does not intersect the circle C: two other cases, wherein L is tangent to C, and wherein L intersects C at two points, are also possible.

The other concept implicit within the calculator symbol is the projection of an infinite line R onto a circle C, where R is tangent to the circle and the focus of the projection is the point p on C directly across from the tangent point o. This projection is sometimes called a **stereographic projection**.[3]

It suits our purposes here to place the line R in a vertical manner just to the right of the circle C. Thus the focus p is the left-most point of C, and the tangent point o is the right-most point of C. If r is any point of R, we shall denote the projection of r onto C by r´. It should be noted that each point of C is a projection of exactly one point of R, the single exception being p itself. To eliminate this exception, it is common to introduce a new point to R, which one can imagine as a point **infinity**: we therefor shall denote it by the symbol ∞.

[2] If the elements of a pair {a,b} are identical, by the line though a and b we shall mean the line through a tangent to the circle C.

[3] The term "stereographic projection" properly applies to a similar notion wherein a plane is projected onto a sphere. Mathematics students commonly are first exposed to it in a course of complex analysis.



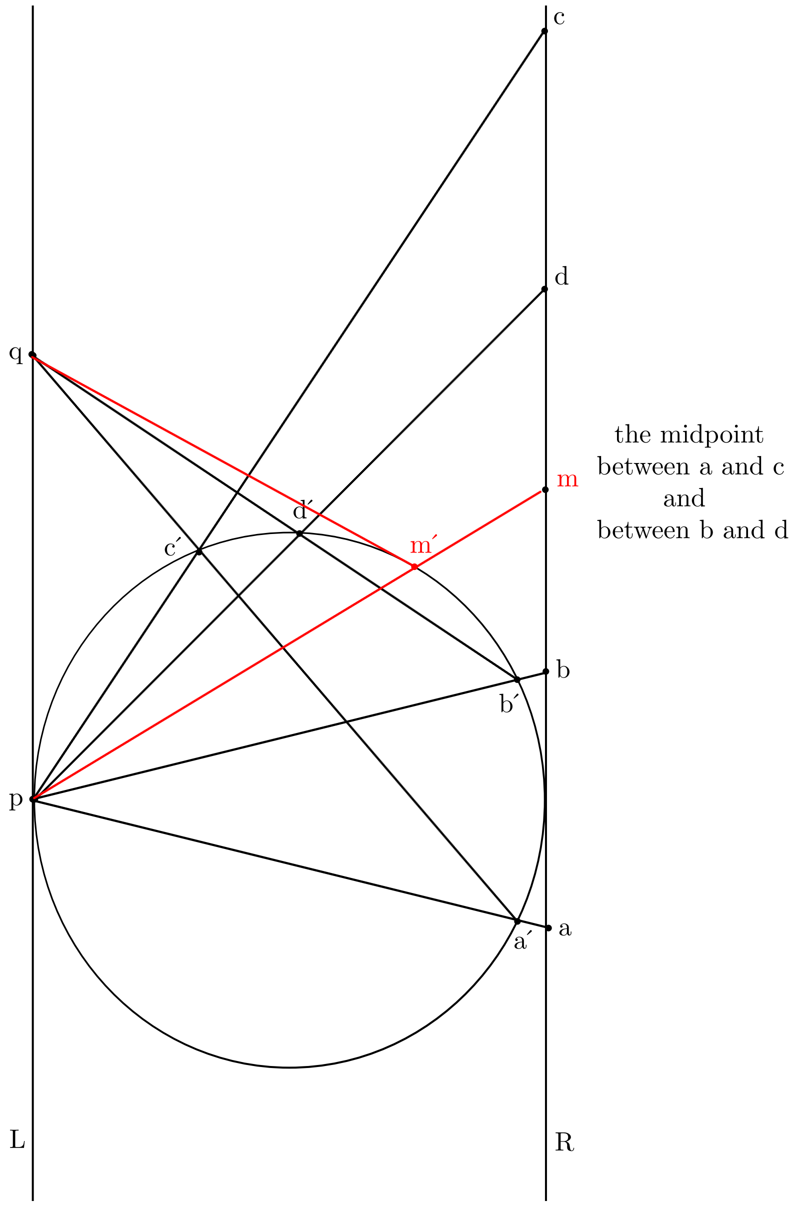
`Figure 3:  Projection of a Line upon a Circle`

The symbol of `Figure 1` suggests that we superimpose the circles in `Figure 2` and `Figure 3`, while taking L to be the vertical line tangent to the circle on its left.

Incorporating this in `Figure 4`, we observe a rather interesting fact[4]: if a, b, c and d are any four points on the line R, then

Fact 1: *the midpoint between* a *and* c *is the same as the midpoint between* b *and* d
if (and only if)
the pairs {a´,c´} *and* {b´,d´} *are balanced relative to* L.

[4] The reader may try other points as well to convince himself of this fact. Using methods of projective geometry, one can prove it, but we shall not do so here.



`Figure 4:  Midpoints and Balance`

`Figure 4` also shows how to find the midpoint m between two points a and b on the line R : if q is the point where the line between a´ and b´ crosses the line L , then m´ is the tangent point of the line through q tangent to the circle.

Any point x on R determines a line segment which extends from o to x (where o is as in `Figure 2`) . We shall denote that line segment and its length by

$$\vec{x} \quad \text{and} \quad |\vec{x}|\,.$$

If y is any other point on R and if m is the midpoint between x and y, then we have

$$|\vec{x}|+|\vec{y}|=|\vec{m}|+|\vec{m}|\,.$$

It follows from this that **Fact 1** may be restated as

Fact 2: *the sum* $|\vec{a}|+|\vec{c}|$ *equals the sum* $|\vec{b}|+|\vec{d}|$
if (and only if)
the pairs {a´,c´} *and* {b´,d´} *are balanced relative to* L.

Fact 2 tells us how to use the calculator symbol to add and subtract line segment lengths. For, if we let d be just o, then the line segment determined by d has zero length. It follows that $|\vec{a}|+|\vec{c}|=|\vec{b}|$ where b is the unique point such that {o,b´} and{a,c} are balanced relative to L.

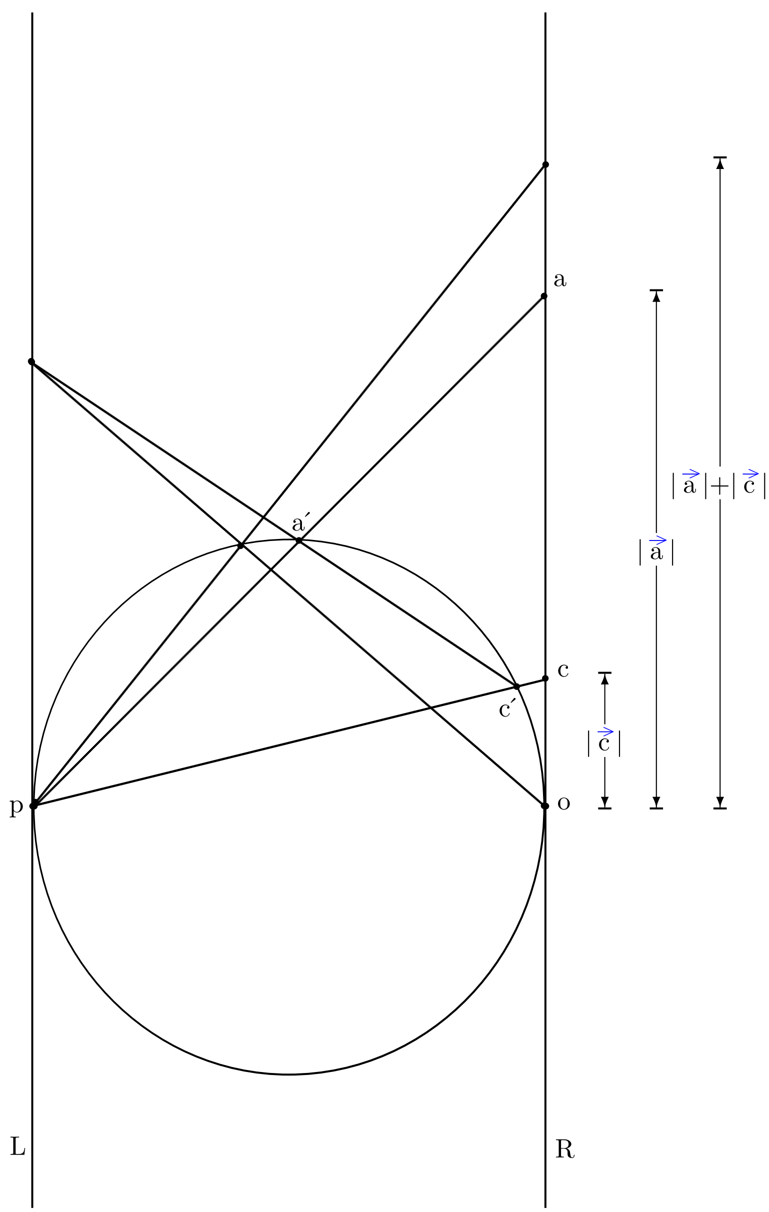
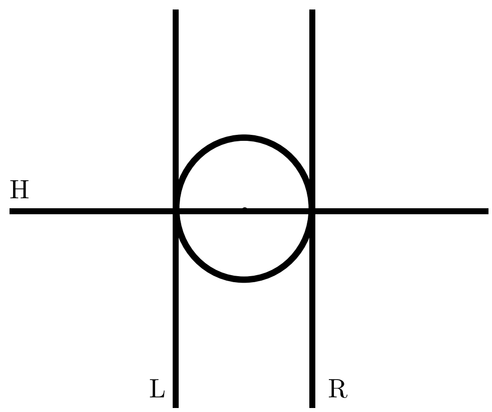


Figure 5: Finding the Sum of Two Line Segments

The symbol of `Figure 1` has a point in the center, which we have not yet used. The center is one of three special points in that figure, the other two being the two tangent points. All three of these points lie on a common line. This suggests that we draw that line: we shall call it H . This is a fruitful suggestion, because the balanced sets relative to H will enable us to do for multiplication which the balanced sets of L did for addition.[5]



`Figure 6:  the Augmented Calculator Symbol`

Recall that hitherto we have used no numbers. We can add the lengths of line segments without numbers: we simply "pick up" one segment and place its bottom end on the top end of the other. This forms a new line segment longer than either of its constituents and its length is the length of the sum.

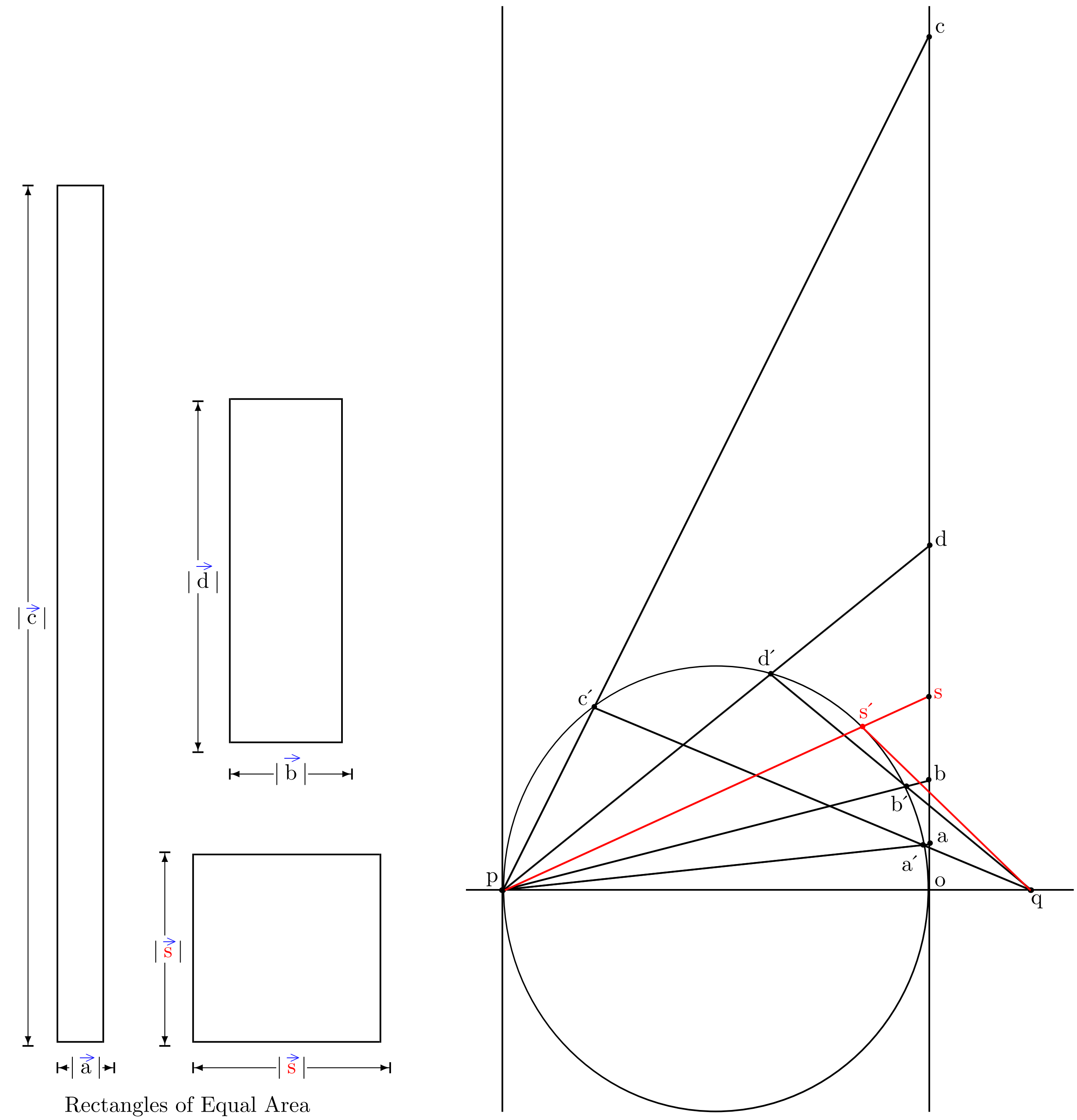
Without numbers however, the "product" of two line segments makes no sense as a line segment. It does make sense as an rectangle or, more generally, a region. One builds a rectangle with one line segment as base and the other as height. The rectangle is the **product** of the two line segments. We shall write the product of two line segments α and β as $\alpha \times \beta$. Just as a line segment has a **length**, a region has an **area**. We shall denote the area of a region A as $|A|$. The importance of H derives from the following fact: for four points a , b , c and d

Fact 3: *the area of* $\vec{a} \times \vec{c}$ *equals the area of* $\vec{b} \times \vec{d}$ *if (and only if) the pairs* {a′,c′} *and* {b′,d′} *are balanced relative to H .*

This fact is illustrated in `Figure 7` on the following page.

Fact 3 is an analogue of **Fact 2** . **Fact 1** had to do with midpoints of line segments. What a midpoint is for a line segment, is what a square root is for a rectangle. For any given region, a **square root** is a line segment of which the corresponding square has the same area as the region. As we showed in `Figure 4` how to find midpoints, we shall illustrate in `Figure 7` how to find square roots.

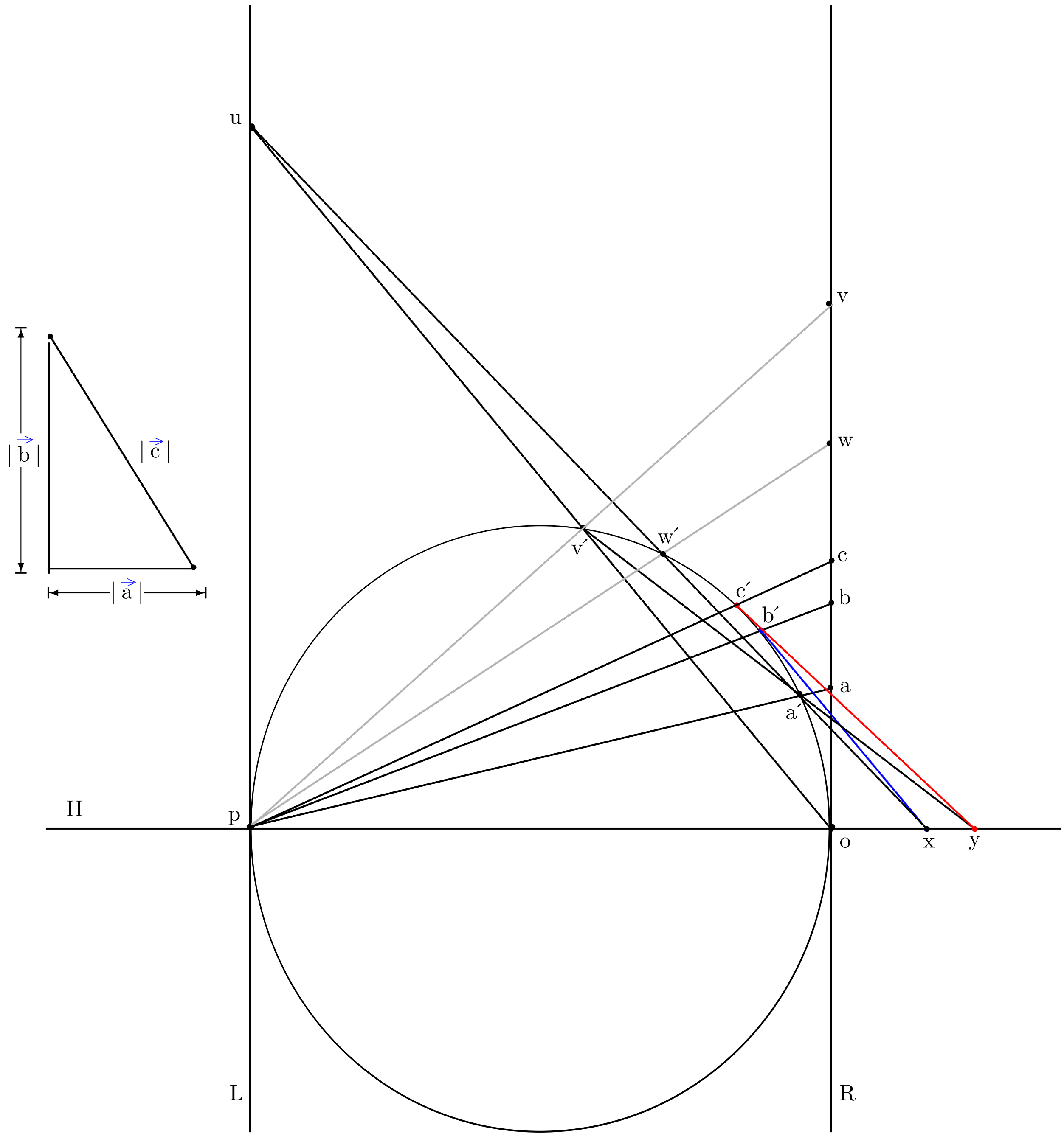
[5] There is a more direct use for the center point in the constructions of the sequel. At times one needs to draw a line from a point outside the circle which is tangent to the circle. Being tangent to the circle is the same as making a right angle with a radius drawn from the center point to the circle.



`Figure 7: Areas and Balance`

The figure above illustrates how to find square roots as follows. Suppose that one has two points a and c above o on the line R. Then there exists a rectangle with of length $|\vec{a}|$ and height $|\vec{c}|$. To find a point s such that the area of the square with side $|\vec{s}|$ is the same as the area of the rectangle, one finds the point q on H situate on the line through a´ and c´. Then he draws the line from q which is tangent to the circle.[6] The point s is the intersection with R of the line through p and that tangent point s´.

[6] Actually there are two tangent lines. Besides the one in the figure, there is one tangent on the lower part of the circle. This other one produces the negative square root.



`Figure 8:  Finding the Hypotenuse of a Right Triangle`

Here e are given a right triangle with sides $|\vec{a}|$ and $|\vec{b}|$. We draw the lines from p to a and b to obtain a´ and b´. We draw the line tangent to the circle at b´ down to the point x where it crosses the line H. From x we draw a line through a´ to the point u where it crosses L. From u we draw a line to o and call v´ the point where this line intersects the circle. We draw the line from v´ through a´ to the point y where it intersects the line H. From y we draw a line tangent to the circle, which tangent point we call c´. The line through p and c´ intersects R at a point c. The hypotenuse of the right triangle has length $|\vec{c}|$.

The construction of `Figure 8` required the drawing of 7 lines: the ones in black, red and blue.[7] We added the two gray lines only to help with the following explanation of *why* the method works.

From **Fact 3** follows that

$$|\vec{b}|\cdot|\vec{b}| = |\vec{a}|\cdot|\vec{w}|\,.$$

From **Fact 2** follows that

$$|\vec{a}|+|\vec{w}| = |\vec{v}|+|\vec{o}| = |\vec{v}|\,.$$

Solving for $|\vec{w}|$ and substituting in the first equality, we have

$$|\vec{b}|\cdot|\vec{b}| = |\vec{a}|\cdot(|\vec{v}|-|\vec{a}|)$$

which implies that

$$|\vec{a}|\cdot|\vec{a}|+|\vec{b}|\cdot|\vec{b}| = |\vec{a}|\cdot|\vec{v}|\,.$$

From **Fact 3** follows that

$$|\vec{a}|\cdot|\vec{v}| = |\vec{c}|\cdot|\vec{c}|\,.$$

These last two equalities imply that

$$|\vec{a}|\cdot|\vec{a}|+|\vec{b}|\cdot|\vec{b}| = |\vec{c}|\cdot|\vec{c}|$$

which by the Pythagorean Theorem, shows that $|\vec{c}|$ is the hypotenuse of the right triangle.

[7] We used red and blue that they might more easily be distinguished, they appearing so close together in the figure.

NUMBERS

Heretofore we have dealt with line segments, lengths of line segments, regions and their areas. All of these terms are ideas which most people understand and agree upon, but they are not numbers.

The idea of a real number is of a higher level of abstraction. A number may be defined as a ratio of the length of one line segment α to the length of another line segment β: we shall denote it by

$$|\alpha|/|\beta|\,.$$

If γ and δ are any two other line segments, then, by definition,

$$|\alpha|/|\beta| = |\gamma|/|\delta| \;\; \text{if, and only if,} \;\; |\alpha\times\delta| = |\beta\times\gamma|\,.$$

Thus a number may have many names: no matter what circle one takes, if the circle has diameter δ and circumference γ, then the number which has the name pi also has the name $|\gamma|/|\delta|$.

To multiply a number $\mathtt{r}$ times a length $|\chi|$ of a line segment χ, one first takes line segments such that $\mathtt{r} = \alpha/\beta$. Then one finds a line segment η such that $|\beta\times\eta| = |\alpha\times\chi|$:

$$\mathtt{r}|\chi| = |\eta|\,.$$

This concept of multiplying a number $\mathtt{r}$ times a line segment may be carried one step further. If one has a line R and a given point o on that line, then any point x on R determines a line segment $\overrightarrow{\mathrm{x}}$. We may define the product of $\mathtt{r}$ with x to be the point y on R (on the same side of o as x is) such that $|\overrightarrow{\mathrm{y}}| = \mathtt{r}|\overrightarrow{\mathrm{x}}|$. Thus, if we call this product $\mathtt{r}\mathrm{x}$, we have

$$|\overrightarrow{\mathtt{r}\mathrm{x}}| = \mathtt{r}|\overrightarrow{\mathrm{x}}|\,.$$

Recall that a square root of a region is any line segment such that the area of a square made by that line segment has the same area as the region. For any area, we define the square root of that area to be the length of any square root of a region having that area. If a is an area, we denote its square root by

$$\sqrt{\mathrm{a}}\,.$$

A **coordinate system** on a line R arises from selecting a point o on R and a point distinct from o which is usually named 1 .Thus, the point 1 is being “identified” with the number 1 .Once this is done, each other point r on R will be identified with the number which is equal to the ratio of r to the point 1.

Once a coordinate system is chosen, the computer symbol may be applied to do arithmetic on the numbers corresponding to the points on R . Furthermore, one can take square roots of numbers and so also solve any quadratic equation. Examples are give in the following three figures.

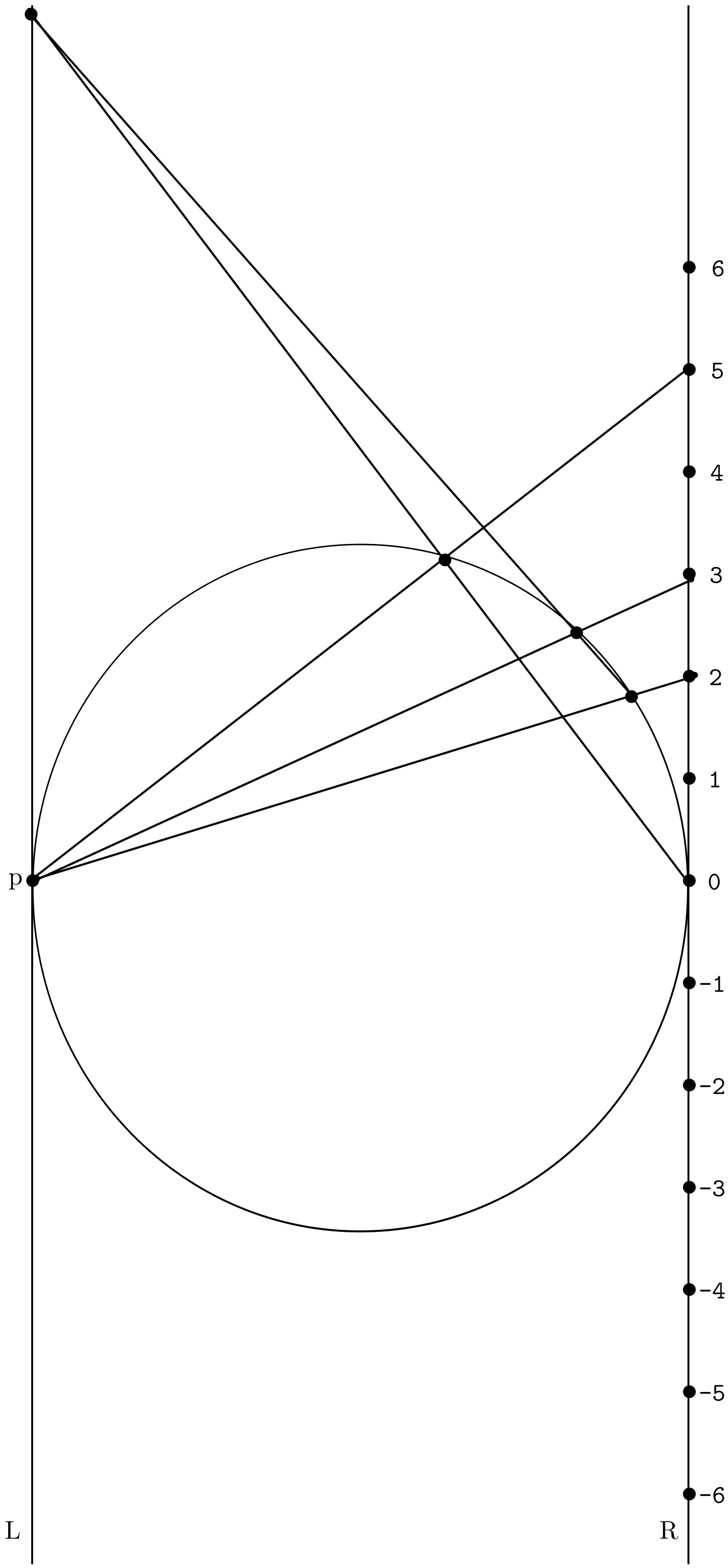


Figure 9: 2+3 = 5

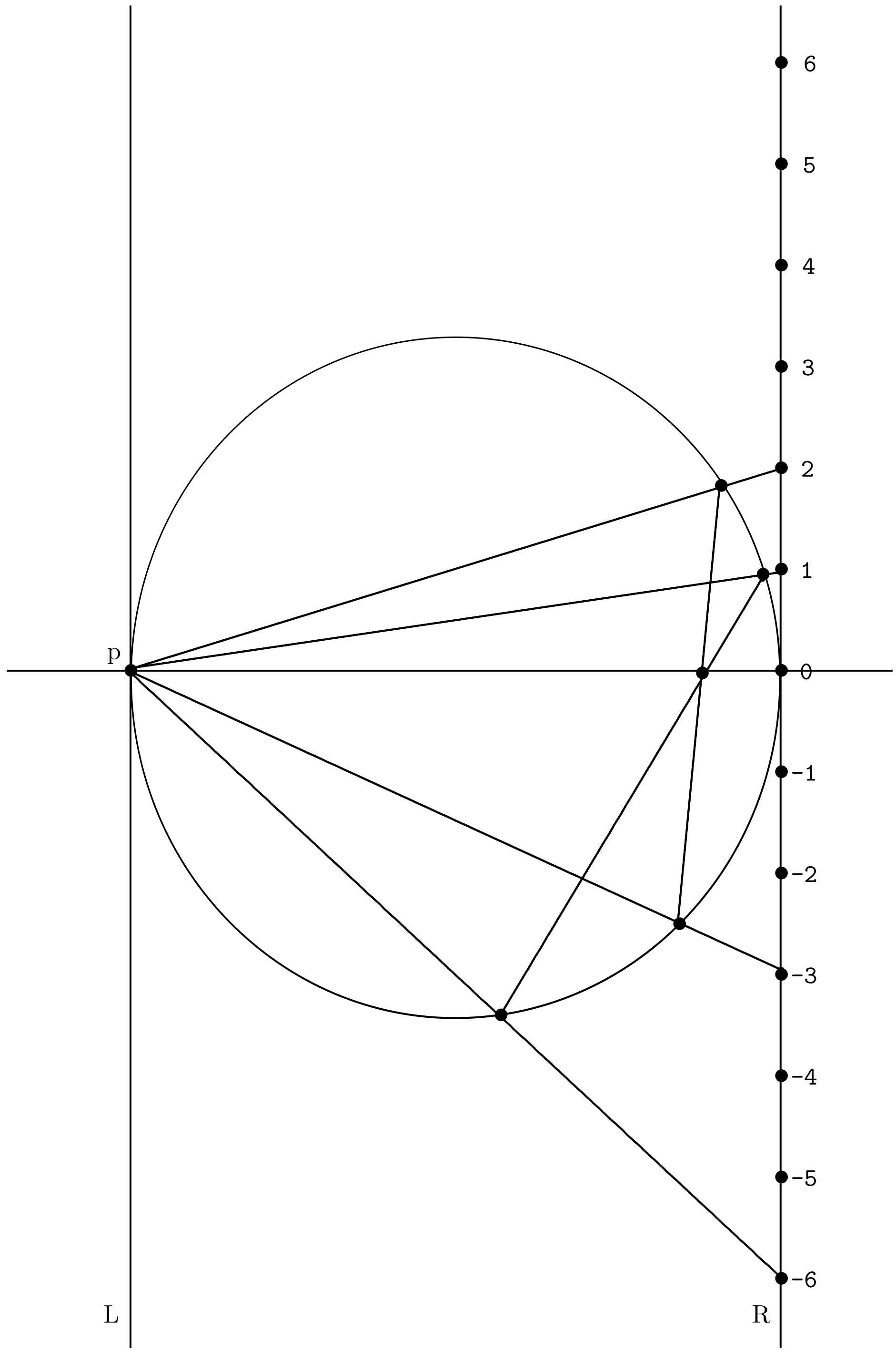


Figure 10: 2 times -3 equals -6

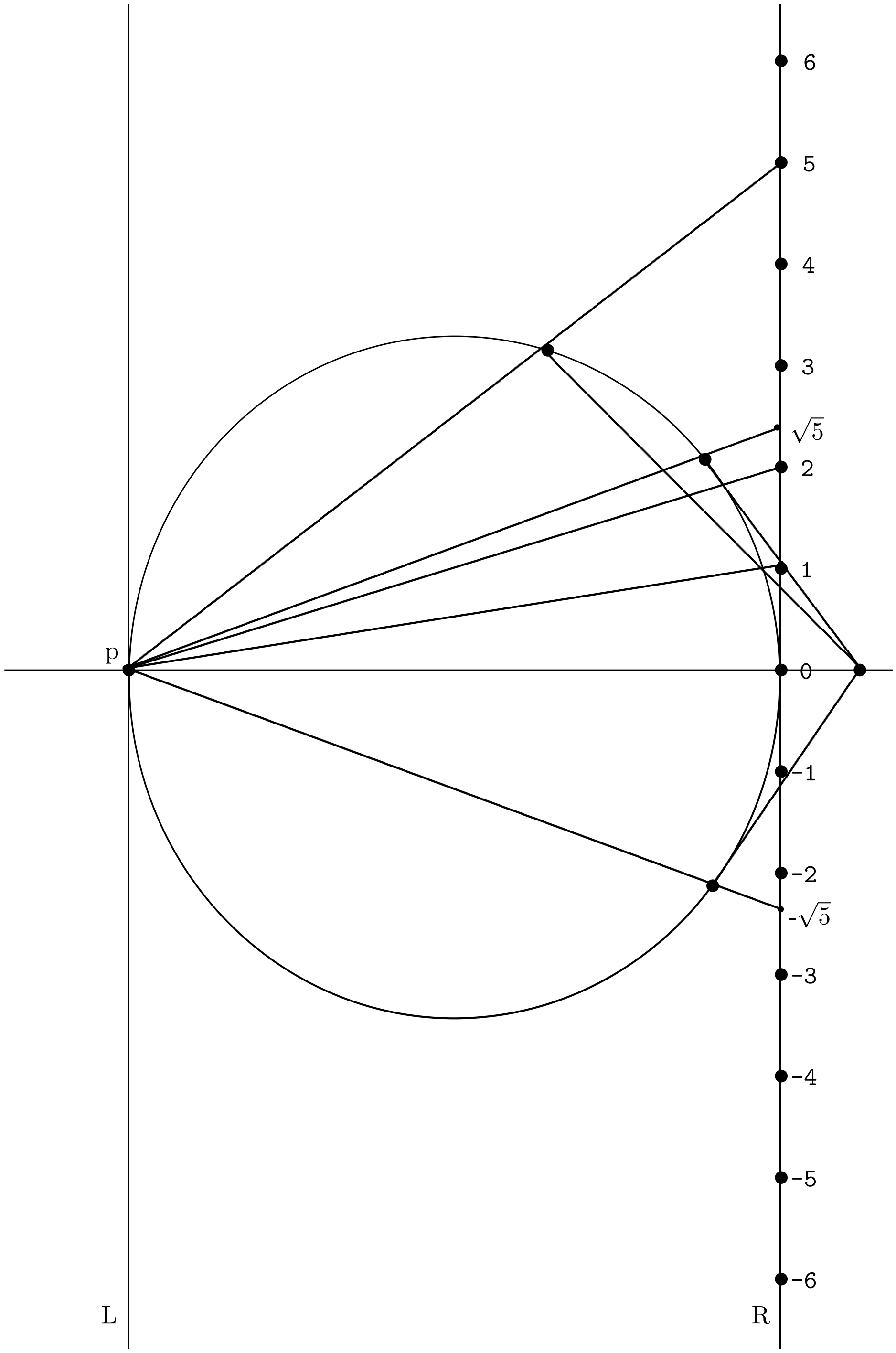


Figure 11: Square Roots of 5

HOW TO BUILD THE GREAT PYRAMID OF GIZA

Actually, we show how to build a pyramid **similar** to the great pyramid, leaving out the intricacies of the inner passages and secret spaces. It was built in the 26th century BC and was constructed as a burial place for Khufu (Cheops), the second king of the fourth Egyptian dynasty. It is the oldest and largest of the pyramids.

There are many remarkable things about this pyramid. Not the least of those is that the ratio of the area of the surface to the area of the base is very close to what is now know as the "golden ratio". The **golden ratio** is that number $|\sigma|/|\beta|$ such that a pair of line segments σ and β defining it satisfy the equation

$$|\sigma\times\sigma| = |\beta\times\sigma| + |\beta\times\beta|\,. \tag{-1-}$$

The golden ratio often is denominated by the symbol φ.

Given any line segment σ, there is exactly on other line segment β such that

$$\varphi = |\sigma|/|\beta|\,.$$

In fact, equation (-1-) may be solved (via the quadratic formula) and the solution for $|\beta|$ in terms of σ is given by

$$|\beta| = \frac{|\sigma| + \sqrt{5|\sigma\times\sigma|}}{2}\,. \tag{-2-}$$

Suppose now that β is the base region of the pyramid at Giza and that σ is a region with the same area as the surface of the pyramid. Let $\mathfrak{l}$ be one of the four line segments bounding the base β, let $\mathfrak{h}$ be the line segment from the center of the base up to the top of the pyramid, and let $\mathfrak{d}$ be a line segment from the center of $\mathfrak{l}$ up to the top of the pyramid: thus the area of one side of the pyramid is one half of $|\mathfrak{d}\times\mathfrak{l}|$.

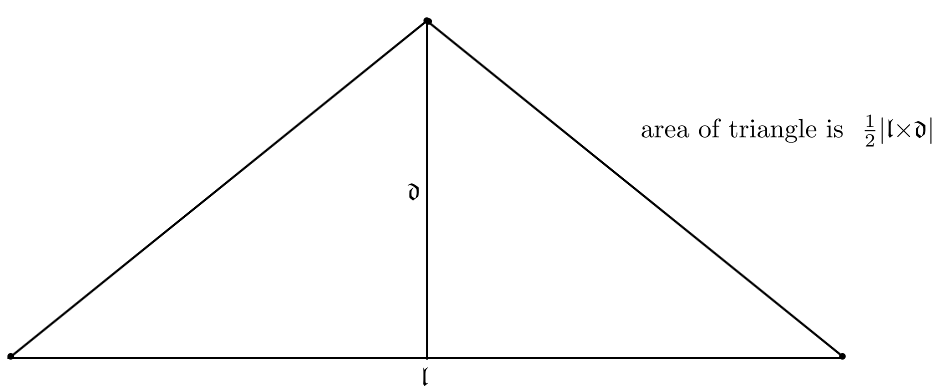


Figure 12: One Face of a Pyramid

It follows that the area of the pyramid surface is $2|\mathfrak{l}\times\mathfrak{d}|$. Since the area of the base is $|\mathfrak{l}\times\mathfrak{l}|$, we know that

$$\varphi = \frac{2|\mathfrak{l}\times\mathfrak{d}|}{|\mathfrak{l}\times\mathfrak{l}|}\,. \tag{-3-}$$

If we take a vertical slice of the pyramid through the topmost point and the midpoint of $\mathfrak{l}$,we get a triangle of height $|\mathfrak{h}|$ and with base length $|\mathfrak{l}|$.

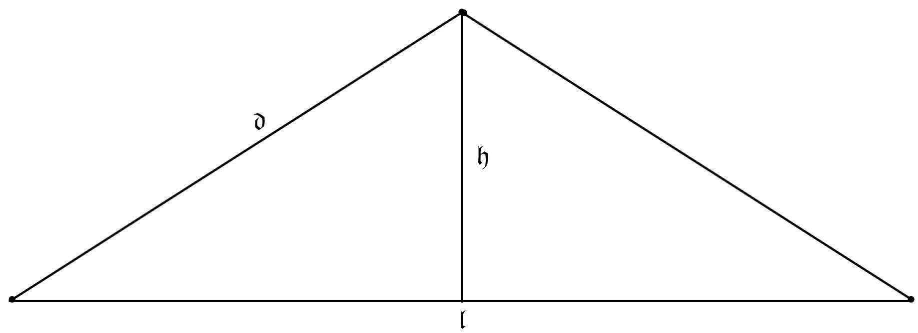


Figure 13: Cross-section of a Pyramid

Applying the pythagorean theorem to the left half of the triangle we obtain

$$|\mathfrak{d}\times\mathfrak{d}| = |\mathfrak{h}\times\mathfrak{h}| + |\tfrac{1}{2}\mathfrak{l}\times\tfrac{1}{2}\mathfrak{l}|\,. \qquad \text{(-4-)}$$

If we eliminate $\mathfrak{d}$ from the simultaneous equations (-3-) and (-4-), we obtain

$$|\mathfrak{h}| = \tfrac{1}{2}\sqrt{\varphi}\,|\mathfrak{l}|\,. \qquad \text{(-5)}$$

which describes the height of the great pyramid in terms of the base.

We purpose to build the pyramid by using blocks with a square base and a height which divides $|\mathfrak{h}|$ evenly. For the purpose of visualization, we shall assume the pyramid is three blocks high. Then the drawing in `Figure 13` becomes

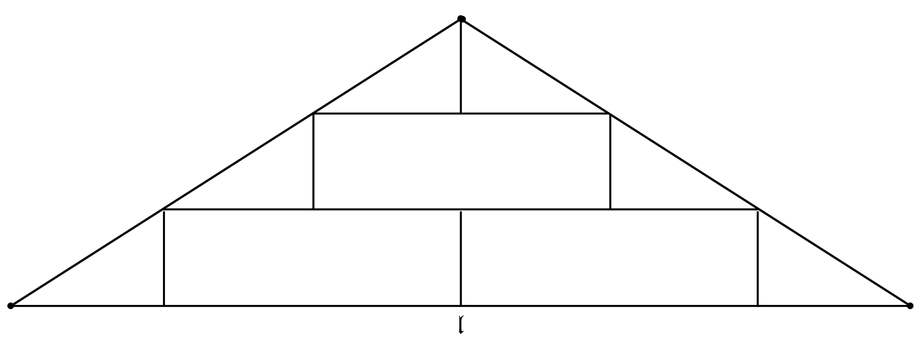


Figure 14: Cross-section of a Pyramid Three Blocks High

We note from `Figure 14` that the triangular surface blocks have base length just one half of the base length of the inner blocks. This would be the case, no matter how many blocks high we decided to make the pyramid.

If l denotes the length of the base of a block and h the height of a block, it is evident from the figure that

$$|l|/|h| = |\mathfrak{l}|/|\mathfrak{h}|\,.$$

From this and equation (-5-) follows that

$$|h| = \tfrac{1}{2}\sqrt{\varphi}\,|l|\,. \qquad \text{(-6-)}$$

It follows that, to know what size to cut the building blocks of the pyramid, we need to know, given a block base length l, how to calculate its height length $\frac{1}{2}\sqrt{\varphi}\,|l|$. We demonstrate how to do this in the following final four figures.

In these figures we shall use two different circles. The reason for this is in some of the figures, using one or the other would place part of a line off the page, or would put several of the lines so close together as to be indistinct.

In `Figure 15` we compute $5l$ using the smaller circle by first adding l to l, to obtain $2l$; then adding this $2l$ to l, to obtain $3l$; and finally adding $3l$ to $2l$.

In `Figure 16` we use the larger circle first to compute the square root of $5l{\times}l$, then to add l to it, and finally to divide by 2. We perform this last task by finding the midpoint of the line segment originating at o and terminating at $l{+}\sqrt{5l{\times}l}$.

In `Figure 17` we use the smaller circle again to find $\sqrt{\varphi l}$.

In `Figure 18` we use the larger circle finally to obtain $h = \frac{1}{2}\sqrt{\varphi l}$.If we had attempted to use the midpoint method here, with either circle, the lines would have been off the paper. Hence we use the line H actually to divide by 2.

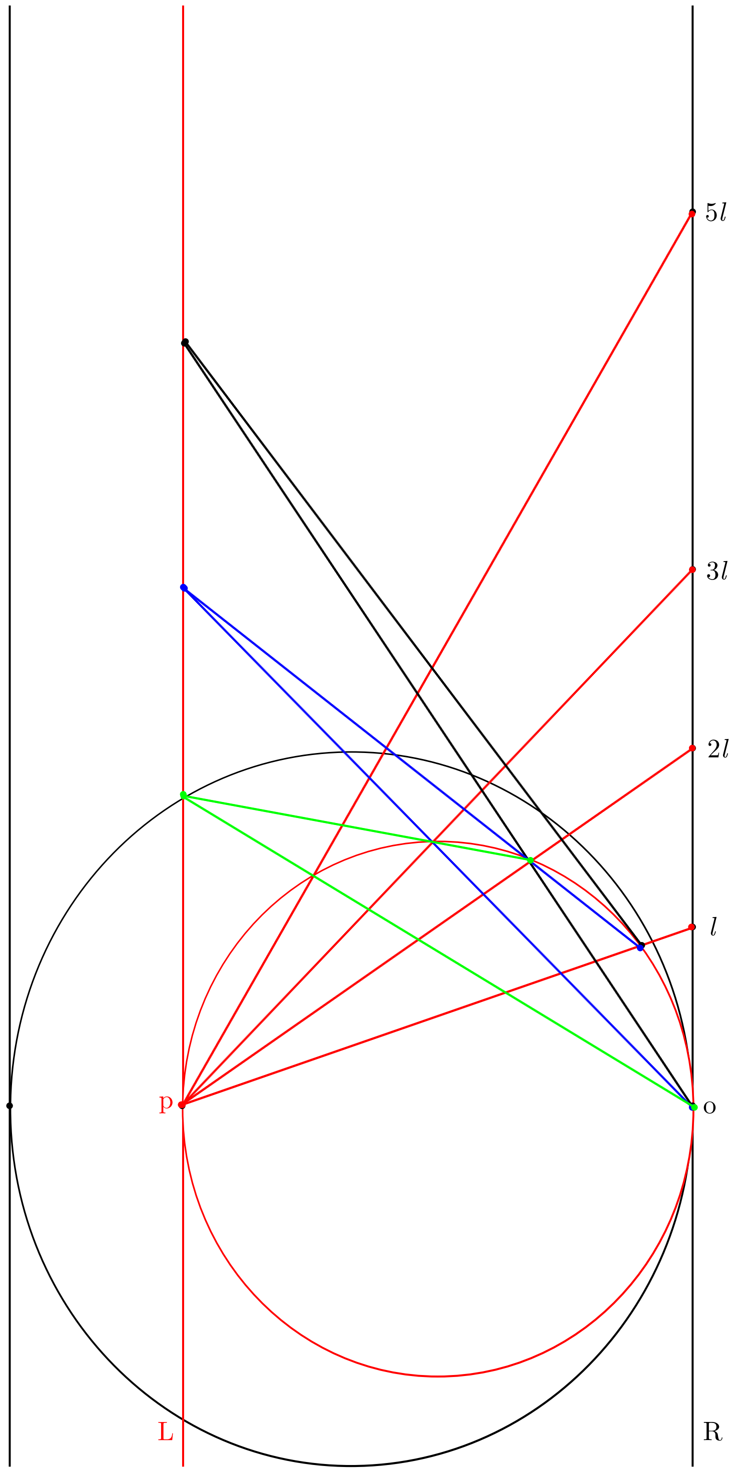


Figure 15: $2l = l + l$, $3l = l + 2l$ and $5l = 2l + 3l$

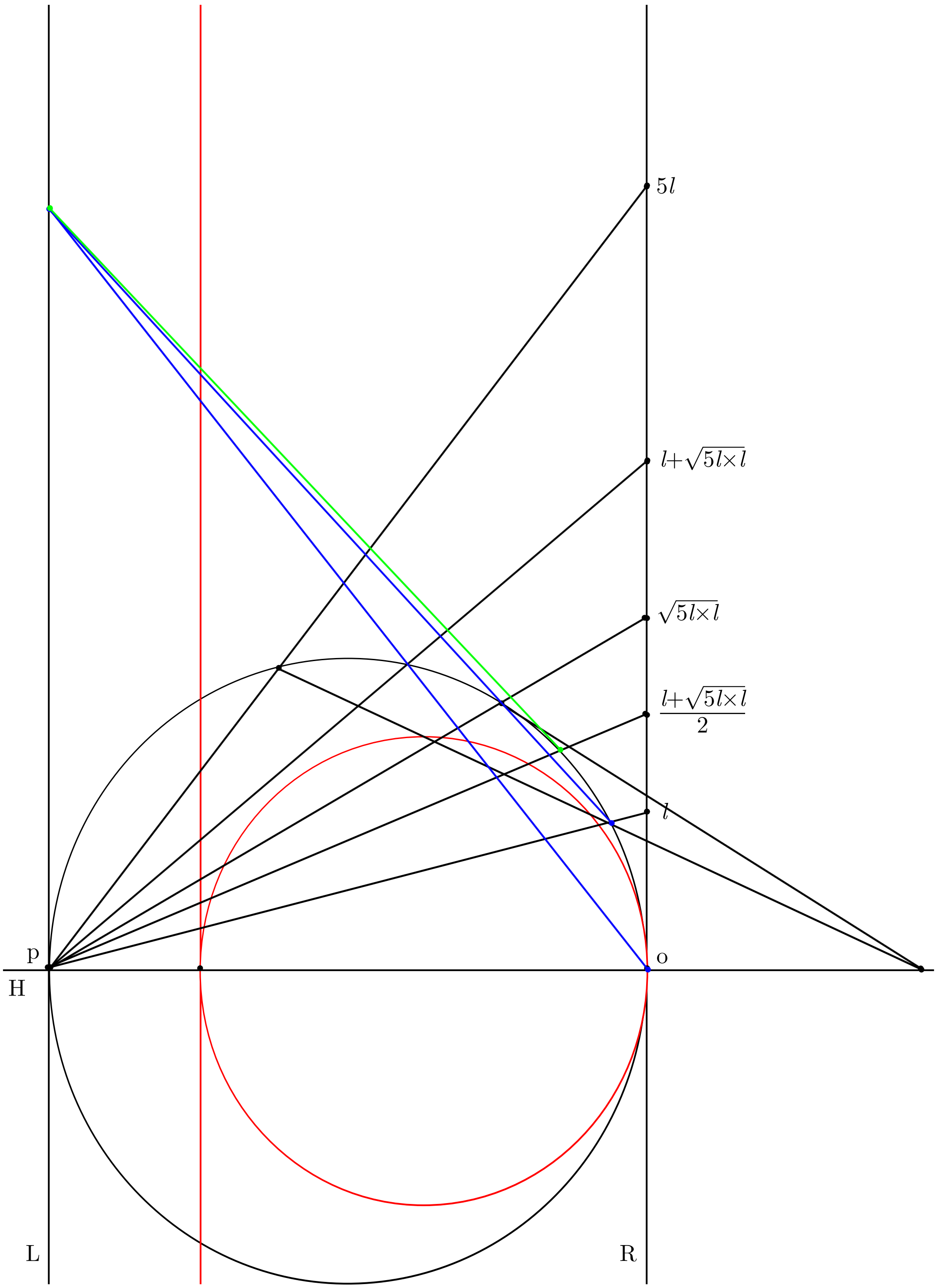


Figure 16: $\sqrt{5l\times l}$, $l+\sqrt{5l\times l}$ and $\phi\, l=\frac{l+\sqrt{5l\times l}}{2}$

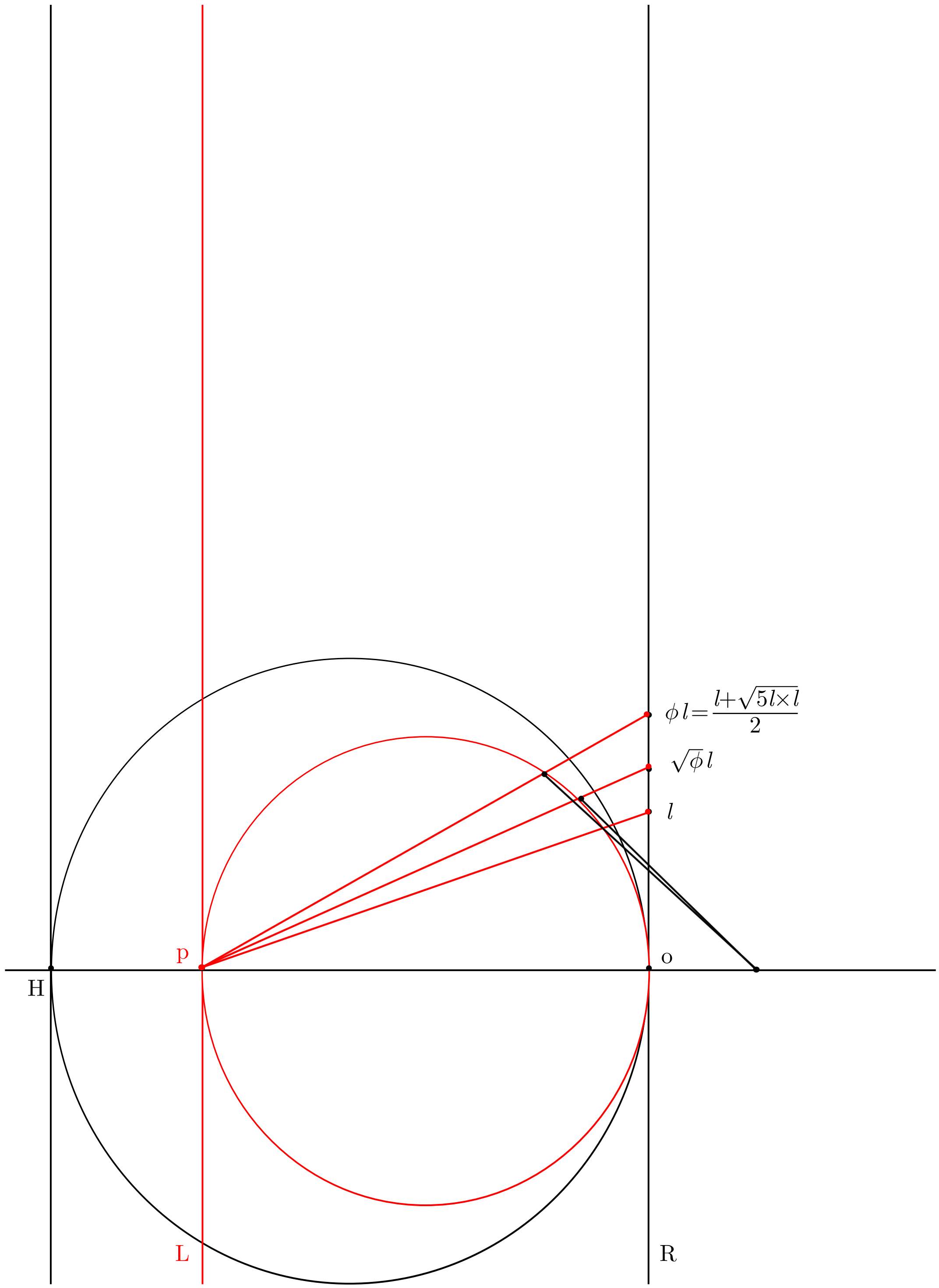


Figure 17: $\phi\, l$ and $\sqrt{\phi}\, l = \sqrt{\phi\, l \times l}$

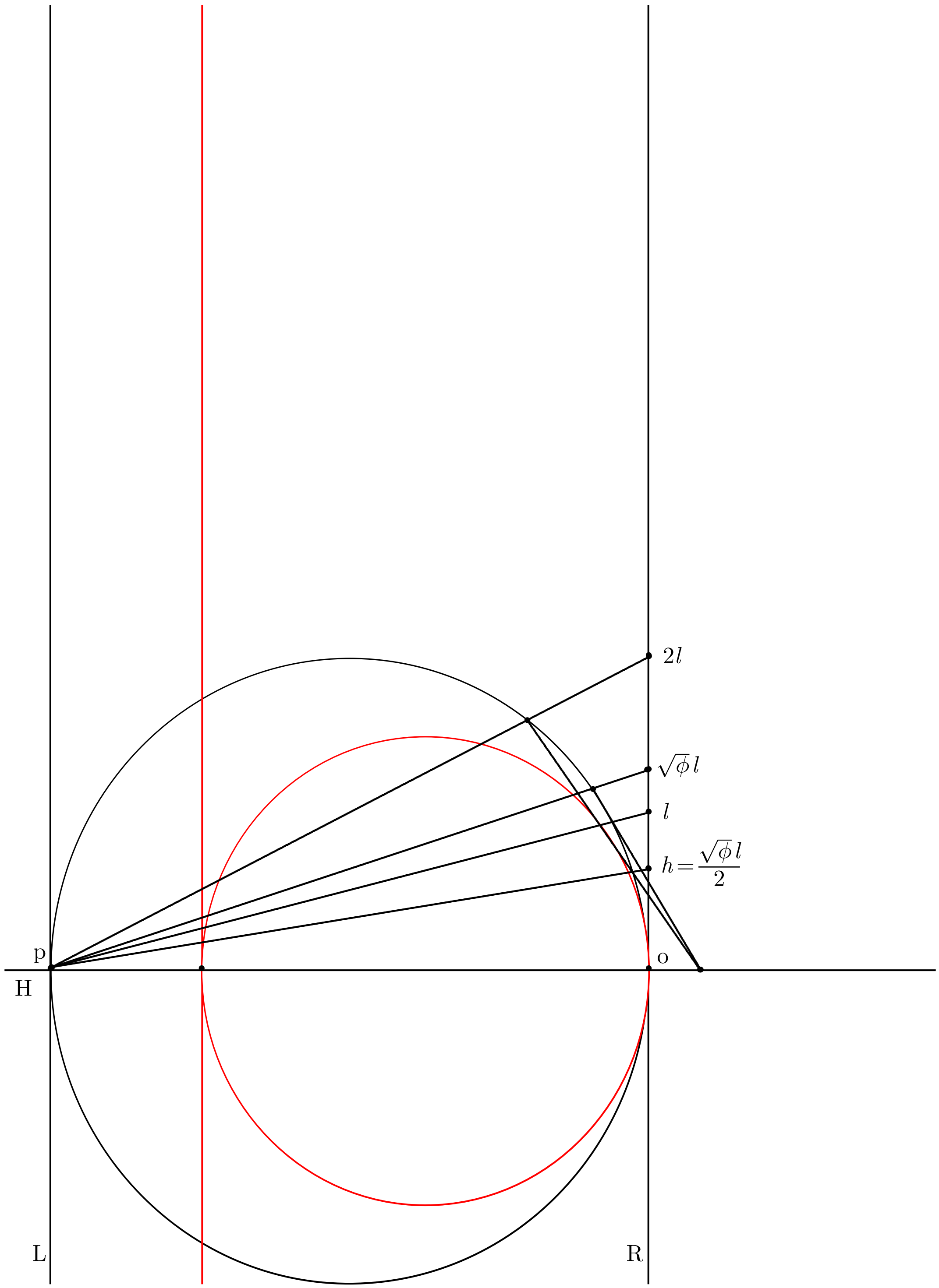


Figure 18: $h = \dfrac{\sqrt{\phi}\,l}{2}$