# Revisit of Minimum-area Enclosing Rectangle of a Convex 

## Polygon

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#### Abstract

The problem of minimum-area enclosing rectangle of a convex polygon was first studied in [1] in 1975. We revisit this problem by providing a new complete proof via the elementary calculus and the method of rotating calipers [4], [5], [7] with transparent existence condition not revealed explicitly in [1] mainly based on geometric reasoning. The existence of minimum-area enclosing rectangle is mathematically due to monotonicy of area of enclosing rectangle with respect to the rotation angle defining its configuration re lative to an initial enclosing rectangle.


## I. Introduction

Using simpler shapes such as rectangle or circle in 2D, box or sphere in 3D to encapsulate an object of complex geometry is a widely- used idea in many applications. In [3], with additional cost (of computation and memory) associated with determining the bounding volumes, collision detection using bounding volumes for fast overlap of
two geometric objects, rejection tests is much computational cheaper than that for complex geometry. Axis-aligned Bounding Box (AABB) with the axis parallel to the axis of the coordinate frame is an example of bounding volumes, which can be optimized to obtain a minimum AABB. This optimization requires an immediate step of obtaining a minimum axis-aligned rectangle. In [2], the algorithm on scan trajectory based on square lattice for laser rapid prototyping needs to implement a minimum-area bounding rectangle (circumscribed rectangle). In 2D, a classical reference for determining such a tight fitting rectangle is [1]. This paper explains a proper re-statement of the theorem of the minimum bounding rectangle in [1] via a new proof grounded on the elementary calculus and the method of rotating calipers [4], [5], [7] with implementation detailed in [6].

## II. A New Complete Proof of Theorem of the Minimum Bounding Rectangle II. 1 Motivation [1]

In [1], the following was proved based mainly on geometric reasoning. Let $P \vee Q$ be two statements.
$P=$ There is no edge of the bounding rectangle contains more than one of the specified points on the convex polygon.
$Q=$ The bounding rectangle is not a minimum bounding rectangle.
Reference [1] proved the following two theorems regarding the minimum- area rectangle enclosing a convex polygon. The proof was refined more analytically in [7].

Theorem 1: $P \quad \Rightarrow Q$.
Given a rectangle with four points arbitrarily chosen such that no edge contains more than one point, there exists another rectangle such that each edge of the rectangle is less than that of the given rectangle.
Theorem 2: $\sim Q \quad \Rightarrow \sim P$
The rectangle of minimum- area enclosing a convex polygon has a side collinear with one of the edges of the polygon.
Theorem 1 means that $P \Rightarrow Q$; Theorem 2 is a corollary of Theorem 1 according to the logical relation " $P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$ ". However the existence of the minimum bounding rectangle is omitted, and hence Theorem 2 can't conclude that there must exist one such that it has an edge collinear with the edges of enclosed convex polygon and its area is the minimum.

## II. 2 A new proof

The above theorem shown in [1] did not reveal the existence of minimum-area enclosing rectangle for a convex polygon. Now we present a re-statement of the
minimum-area enclosing rectangle theorem for a convex polygon with a new complete proof based on the arguments using rotating calipers proposed in [7] and calculus.
Theorem 3: Given a convex polygon, there exists a minimum-area bounding rectangle. The minimum bounding rectangle has at least one edge in coincidence with the edges of the convex polygon.
Proof. The proof applies the concept of rotating calipers method [4], [5], [7]. This method bounds the convex polygon by four lines of support through distinct extreme points (supporting vertices) of the convex polygon, as depicted in Fig. 1. Let $\left(x_{u}, y_{u}\right),\left(x_{l}, y_{l}\right),\left(x_{d}, y_{d}\right)$ and $\left(x_{r}, y_{r}\right)$ denote four supporting vertices of the convex polygon that are extreme points in the given $x$ and $y$ directions, i.e. the pairs of $\left(x_{u}, y_{u}\right),\left(x_{d}, y_{d}\right)$ vertices and $\left(x_{l}, y_{l}\right)$, $\left(x_{r}, y_{r}\right)$ vertices are farthest apart in the two orthogonal directions, respectively. These four extreme vertices form a rectangle (a minimum bounding rectangle aligned with coordinate axes). First choose an initial reference enclosing rectangle (such as a coordinate-axis aligned bounding rectangle). Then one candidate enclosing rectangle edge directions are represented, respectively, by the unit vector $\widehat{\boldsymbol{u}}=(\cos \alpha, \sin \alpha)$ and its perpendicular whose rotation direction could be the same as or the opposite to $\widehat{\boldsymbol{u}}$ [7] as $\widehat{\boldsymbol{v}}= \pm(-\sin (\alpha), \cos (\alpha))$, parametrized by $\alpha \in\left[\alpha_{l b}, \alpha_{u b}\right] \subseteq\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with respect to the $+x$ axis at the supporting vertex $\left(x_{u}, y_{u}\right)$,
where $\pm$ denotes a counterclockwise ( + ) or clockwise ( - ).of rotation. The angle limits $\alpha_{l b}, \alpha_{u b}$ are the angles of rotation that the rectangle edge coincides with one edge of the polygon. The directions $\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}$ define a collection of four edge directions of the candidate enclosing rectangle, denoted by $L_{*} \quad$ where $\quad *=u, ~ l, ~ d, r \quad$ of the candidate rectangle, where the subscript $u$, $l$, $d$, $r$ denote up-most, left-most, down-most and right-most, respectively. Then the edge vectors $\boldsymbol{u}(\alpha), \boldsymbol{v}(\alpha)$ of the candidate rectangle are obtained as the projection of the line segments connecting $\left(x_{u}, y_{u}\right),\left(x_{d}, y_{d}\right)$ and connecting $\left(x_{l}, y_{l}\right),\left(x_{r}, y_{r}\right)$ onto $\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}$, respectively, as
$\boldsymbol{u}=\left(\left(x_{r}-x_{l}\right) \cos \alpha+\left(y_{r}-y_{l}\right) \sin \alpha\right) \widehat{\boldsymbol{u}}$, $\boldsymbol{v}= \pm\left(-\left(x_{u}-x_{d}\right) \sin (\alpha)+\left(y_{u}-\right.\right.$ $\left.\left.y_{d}\right) \cos (\alpha)\right) \widehat{v}$.
We have the area of the rectangle with edges $\boldsymbol{u}, \boldsymbol{v}$ given by

$$
\begin{gathered}
\text { area }_{\text {rectangle }} \quad(\alpha)=\mid \operatorname{det}(\boldsymbol{u}(\alpha) \times \\
\boldsymbol{v}(\alpha)) \mid .
\end{gathered}
$$

This area is independent of the coordinate system chosen. Note that $\alpha=0$ corresponds to the enclosing rectangle with four lines of support as the edge axes. Without loss of generality, we assume that $\operatorname{det}(\boldsymbol{u} \times \boldsymbol{v})>0, \alpha \in\left[0, \frac{\pi}{2}\right] \geq 0$ in Fig. 1 so that
area $_{\text {rectangle }}(\alpha)= \pm$
$\left(\left(x_{r}-x_{l}\right) \cos \alpha+\left(y_{r}-\right.\right.$
$\left.\left.y_{l}\right) \sin \alpha\right)\left(-\left(x_{u}-x_{d}\right) \sin (\alpha)+\right.$
$\left(y_{u}-\right.$
$\left.\left.y_{d}\right) \cos (\alpha)\right)$
From

$$
\begin{align*}
& \partial_{\alpha} \text { rea }_{\text {rectangle }}(\alpha)=\left(-\left(x_{r}-x_{l}\right) \sin \alpha+\right. \\
& \left.\left(y_{r}-y_{l}\right) \cos \alpha\right) \cdot\left\{\mp \left(-\left(x_{u}-x_{d}\right) \cos (\alpha)-\right.\right. \\
& \left.\left.\left(y_{u}-y_{d}\right) \sin (\alpha)\right)\right\} \tag{2}
\end{align*}
$$

where $\partial_{\alpha}:=\frac{d}{d \alpha}$, we obtain

$$
\partial_{\alpha} \partial_{\alpha} \text { area }_{\text {rectangle }} \quad(\alpha)
$$

$=$
$\left(-\left(x_{r}-x_{l}\right) \cos \alpha-\right.$
$\left.\left(y_{r}-y_{l}\right) \sin \alpha\right)\left\{\left(\left(x_{u}-x_{d}\right) \sin (\alpha)-\right.\right.$
$\left.\left.\left(y_{u}-y_{d}\right) \cos (\alpha)\right)\right\}$
$=$ area $_{\text {rectangle }}(\alpha)>0$,
the global minimum exists
for area ${ }_{\text {rectangle }}(\alpha)$ over $\left[0, \frac{\pi}{2}\right]$. Moreover, $\alpha=0$ corresponds to a maximum-area enclosed rectangle with fixed $\left(x_{u}, y_{u}\right),\left(x_{l}, y_{l}\right),\left(x_{d}, y_{d}\right)$ and $\left(x_{r}, y_{r}\right)$.
In (2), the term

$$
\left(x_{r}-x_{l}\right) \sin (-\alpha)+\left(y_{r}-y_{l}\right) \cos (-\alpha)
$$

is the projection of the line segment connecting vertices $\left(x_{l}, y_{l}\right),\left(x_{r}, y_{r}\right)$ onto $\widehat{\boldsymbol{v}}$, and the term
$\left(x_{u}-x_{d}\right) \cos (\alpha)+\left(y_{u}-y_{d}\right) \sin (\alpha)$
is the projection of the line segment connecting $\left(x_{u}, y_{u}\right),\left(x_{d}, y_{d}\right)$ onto $\widehat{\boldsymbol{u}}$. Both terms are either (i) of the same sign or (ii) of opposite sign for typical configuration in Fig. 1 with fixed extreme vertices. Thus, in any case it is guaranteed that

$$
\partial_{\alpha} \text { area }_{\text {rectangle }}(\alpha)<0
$$

by rotating the lines of support counterclockwise or clockwise (by choosing
appropriate sign of $\widehat{\boldsymbol{v}}$ ). Therefore, $\operatorname{area}_{\text {rectangle }}(\alpha)$ is a monotone decreasing smooth function over the interval $\left[0, \frac{\pi}{2}\right]$ (or over $\left[-\frac{\pi}{2}, 0\right]$ ). The monotonicy of the area $_{\text {rectangle }}(\alpha)$ implies that its minimum is at $\alpha^{*}=\min \left(\left|\alpha_{l b}\right|,\left|\alpha_{u b}\right|\right)$. By a rotation $\alpha^{*}$ of both of the reference enclosing rectangle edge axes, the edge $\boldsymbol{u}\left(\alpha^{*}\right)$ or its perpendicular $\boldsymbol{v}\left(\alpha^{*}\right)$ of the new enclosing rectangle is in coincidence with an edge of the convex polygon at the supporting vertex. Thus the theorem has been proved. Q.E.D.

Note that it may happen that the line direction coincides with a polygon edge so that the extreme vertex can be chosen as one of the vertex of the coinc ident polygon edge. Or it may happen that a polygon vertex is a corner of the rectangle, so that the four extreme points may degenerate to three or two distinct extreme points with one or two duplicate extreme points [6]. These special configurations, though complicated the geometry reasoning proof in [1], do not affect the rectangle area computation.

The area computation for each candidate enclosing rectangle could be done sequentially in counterclockwise order of vertices, starting from selecting an edge of the convex polygon as one edge direction $\widehat{\boldsymbol{u}}$ of the rectangle. For the selected rectangle edge axis direction $\widehat{\boldsymbol{u}}$ and its orthogonal direction $\widehat{\boldsymbol{v}}$, a bounding rectangle with one edge and three supporting vertices from the
convex polygon is formed as depicted in typical configuration of Fig. 1. This $x$-direction can eb set as a bottom edge of the rectangle in coincidence with the polygon edge. Then project all the polygon vertices onto the two directions and compute the rectangle area. This process is iterated for each edge of the convex polygon during the rotation of the rectangle edge lines $\widehat{\boldsymbol{u}}$, (or equivalently $\widehat{\boldsymbol{v}}$ ), the succeeding polygon edge selected is the one that makes the smallest angle with rectangle edges. The edge (and its orthogonal direction) of the rectangle that achieves the smallest area among the candidate rectangles is the edge direction of the minimum-area rectangle. This $\mathrm{O}\left(n^{2}\right)$ algorithm is implemented in [3] p111, where $n$ is the number of vertices of the convex polygon. On the other hand, starting from a candidate rectangle of the smallest enclosing rectangle with one edge (e.g. the bottom edge of the rectangle [6]) coincident with a polygon edge (e.g. the longest edge of polygon [8] from an easy implementation point of view), an $\mathrm{O}\left(n^{2}\right)$ algorithm for computing the minimum-area enclosing rectangle was developed in [9]. Furthermore, applying the method of rotating calipers, the determination of minimum-area enclosing rectangle requires only $\mathrm{O}(n)$ time complexity [5], [7], since the candidate rectangle with one edge direction coincident with one edge of polygon can be generated at constant-time complexity and the (at most) three extreme vertices are found by a linear scan of all the
vertices. Note that if the convex polygon has pairs of parallel edge, then the number of candidate rectangles is reduced.

There are other special configurations noted in [6] that require special handling in algorithm implementation. However, the area computation (1) is the same for any configuration of rectangle and convex polygon. In addition, the angles $\alpha^{*}$ that make (2) zero also show that the calipers (supporting lines) should rotate simultaneously by an angle $\alpha^{*}$ so that a rectangle edge $\boldsymbol{u}\left(\alpha^{*}\right)$ or its perpendicular $\boldsymbol{v}\left(\alpha^{*}\right)$ coincides with at least one polygon edge. The minimum angle can be computed by Newton's method for zeros of nonlinear function (2). The minimum of the (at most four) angles of the polygon edge with respect to the rectangle edge at the support vertex is the required rotated angle $\alpha^{*}$ for the current configuration defined by the four extreme vertices or edge directions of rectangle.

## III. Conclusion

This paper gives a new informative and complete proof of the minimum-area rectangle enclosing a convex polygon initially shown in [1]. The comparison of proof method is summarized in Table 1.

## REFERENCES

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Fig. 1 A typical configuration for enclosing rectangle area computation of a convex polygon parametrized by $\alpha$, the rotation angle with respect to an initial re ference enclosing rectangle with supporting vertices $\left(x_{u}, y_{u}\right),\left(x_{d}, y_{d}\right),\left(x_{l}, y_{l}\right),\left(x_{r}, y_{r}\right)$. The four supporting lines $L_{u}, L_{l}, L_{d}, L_{r}$ bounding the convex polygon form a new enclosing rectangle. The red lines with double arrows are rectangle edge directions, the black lines with corner vertices are polygon edges.

Table 1 Comparison of our rev isit with [1]. $n$ is the number of vertices/edges of a given convex polygon

|  | H. Freeman and R. Shapira [1] | This paper |
| :---: | :---: | :---: |
| Proof method | Geometric reasoning | Calculus |
| Existence of minimum | Not included explicitly | Included |
| Simplicity | Separate into two (but not complete) cases with fixed supporting vertices $(\alpha>0, \alpha<0)[7]$ | One complete case via method of rotating calipers with two orthogonal supporting lines |
| Intuition | Calculate the area difference between the initial enclosing rectangle and its rotated rectangle that passing through four points with one point on each edge. | Calculate the area area $_{\text {rectangle }}(\alpha)$ of enclosing rectangle parametrized by the rotation angle $\alpha$ with respect to the initial enclosing rectangle directly. |
| Informative | There is (at least) one edge of the convex polygon in common with the minimum-area rectangle. | $\operatorname{area}_{\text {rectangle }}(\alpha)$ is monotonic. <br> At least one edge of minimum-area rectangle is in coincidence with a poly gon edge. |
| Implementation | $\mathrm{O}\left(n^{2}\right)$ | $\mathrm{O}(n)$ |

