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A Family of Trivalent 1-Hamiltonian Graphs With Diameter $O(\log n)$

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In this paper, we construct a family of graphs denoted by *Eye*(*s*) that are 3-regular, 3-connected, planar, hamiltonian, edge hamiltonian, and also minimal 1-hamiltonian. Furthermore, the diameter of *Eye*(*s*) is $O(\log n)$, where n is the number of vertices in the graph and to be precise, $n = 6(2^s - 1)$ vertices.

Keywords: hamiltonian, edge hamiltonian, 1-vertex hamiltonian, 1-edge hamiltonian, 1-hamiltonian, diameter, Moore bound

1. INTRODUCTION

Given a graph $G = (V, E)$, $V(G) = V$ and $E(G) = E$ denote the vertex set and the edge set of G, respectively. All graphs considered in this paper are undirected graphs. A *simple path* (or *path* for short) is a sequence of adjacent edges (v_1, v_2), (v_2, v_3), …, (v_{m-2}, v_{m-1}), (v_{m-1}, v_m), written as $\langle v_1, v_2, v_3, \ldots, v_m\rangle$, in which all of the vertices $v_1, v_2, v_3, \ldots, v_m$ are distinct except possibly $v_1 = v_m$. The path $\langle v_1, v_2, v_3, \ldots, v_m\rangle$ is also called a *cycle* if $v_1 = v_m$ and $m \geq 3$. A cycle that traverses every vertex in the graph exactly once is called a *hamiltonian cycle*. A graph that contains a hamiltonian cycle is called a *hamiltonian graph* or said to be *hamiltonian*. A graph is *edge hamiltonian* if each edge in the graph is incident with some hamiltonian cycle in the graph. The *diameter* of graph G is the maximum *distance* among all pairs of vertices in G, where *distance* means the length of a shortest path joining two distinct vertices in G.

For $V' \subset V$ and $E' \subset E$, $G - V' - E'$ denotes the graph obtained by removing all of the vertices in V' from V and removing the edges incident with at least one vertex in V' and also all of the edges in E' from E. Let k be a positive integer. A graph G is *k-hamiltonian* if $G - V' - E'$ is hamiltonian for any set $V' \subset V$ and $E' \subset E$ with $|V'| + |E'| \leq$

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k. It is clear that every k-hamiltonian graph has at least $k + 3$ vertices. Moreover, the degree of every vertex in a k-hamiltonian graph is at least $k + 2$. A k-hamiltonian graph having n vertices is said to be *minimal* if it contains the least number of edges among all k-hamiltonian graphs having n vertices. A graph is *k-connected* if the subgraph induced by $G - V'$ is connected for any subset V' of V with $|V'| \le k - 1$. A graph is *r-regular* if the number of neighbors of each vertex is r.

Mukhopadhyaya and Sinha [6] proposed a family of 1-*hamiltonian* graphs which are also planar. These graphs are hamiltonian and have diameter of $\left\lfloor \frac{n}{6} \right\rfloor + 2$ if n is even, and $\left\lfloor \frac{n}{8} \right\rfloor + 3$ if n is odd. Harary and Hayes also studied similar problems on graphs, called *k-edge hamiltonian* and *k-vertex hamiltonian* graphs in [3] and [4], respectively. A graph G is *k-edge hamiltonian* if $G - E'$ is hamiltonian for any $E' \subset E$ with $|E'| = k$; G^* is said to be *minimal k-edge hamiltonian* if G^* contains the least number of edges among all k-edge hamiltonian graphs having the same number of vertices as G^*. A graph G is *k-vertex hamiltonian* if $G - V'$ is hamiltonian for any $V' \subset V$ with $|V'| = k$; G^* is said to be *minimal k-vertex hamiltonian* if G^* contains the least number of edges among all k-vertex hamiltonian graphs having the same number of vertices as G^*. For any positive integer k, Harary and Hayes presented families of minimal k-edge hamiltonian graphs and minimal k-vertex hamiltonian graphs in [3] and [4], respectively. In particular, the family of minimal 1-edge hamiltonian graphs proposed in [3] are identical to the family of minimal 1-vertex hamiltonian graphs proposed in [4]. Hence, this family of graphs are 1-hamiltonian. Furthermore, each graph is planar, hamiltonian, and of diameter $\left\lfloor \frac{n+2}{4} \right\rfloor + ((\frac{n}{2} + 1) \textbf{ mod } 2)$ if n is even, or $\left\lfloor \frac{n+1}{4} \right\rfloor + ((\left\lfloor \frac{n}{2} \right\rfloor + 1) \textbf{ mod } 2)$ if n is odd, where n is the number of vertices in the graph.

Harary and Hayes questioned in [4] whether their proposed minimal 1-vertex hamiltonian graphs are the only such graphs, and asked for characterization of all minimal 1-vertex hamiltonian graphs. The family of graphs proposed by Mukhopadhyaya and Sinha mentioned above are also minimal 1-vertex hamiltonian and provide a counterexample to their question. Recently, Wang, Hung, and Hsu [9] presented another family of 1-hamiltonian graphs, each of which is planar, hamiltonian, 3-regular, and of diameter $O(\sqrt{n})$ with n vertices in the graph. On the other hand, we can find minimal 1-vertex hamiltonian graphs which are non-hamiltonian. A *hypohamiltonian* graph is a non-hamiltonian 1-vertex hamiltonian graph [8]. Hypohamiltonian graphs have been extensively discussed in the literature [5], [7] and [8]. The Petersen graph and the coxeter graph are famous examples of hypohamiltonian graphs [7]. Characterization of all hypohamiltonian graphs is a difficult problem in graph theory. We are interested in finding more families of minimal 1-hamiltonian graphs that are also hamiltonian.

The three families of minimal 1-hamiltonian graphs presented in [4], [6] and [9] are planar, 3-regular and hamiltonian. It is natural to ask whether we can find such graphs with smaller diameter. This problem relates to the famous (n, d, D) problem in which we want to construct a graph of n vertices with maximum degree d such that the diameter D is minimized. When d and n are given, the lower bound on diameter D, called the *Moore bound* (on diameter), is given by $D \ge \log_{d-1} n - \frac{2}{d}$ [2]. In this paper, we propose a fam-

ily of minimal 1-hamiltonian graphs that are hamiltonian, edge hamiltonian, planar, 3-regular and 3-connected. Furthermore, the diameter of our graphs is no more than $4\log_2 \frac{n+6}{6}$, i.e., less than 4 times of the Moore bound. Since all graphs in our family are 3-regular, they are minimal 1-hamiltonian graphs.

2. DEFINITIONS

Let $N_0 = 3$ and $N_k = 9 \cdot 2^{k-1}$ for any positive integer k. Let $[i]_m$ denote i **mod** m. An *eye* graph *Eye*(s), $s \geq 1$, is a graph with $s + 1$ layers of concentric cycles; see Fig. 1 for an illustration. These $s + 1$ cycles are denoted by $I_0, I_1, I_2, \ldots, I_{s-1}$, and O_s. In particular, O_s is the outermost cycle. The vertex set $V(Eye(s))$ is given by $\bigcup_{k=0}^{s-1} V(I_k) \cup V(O_s)$ where

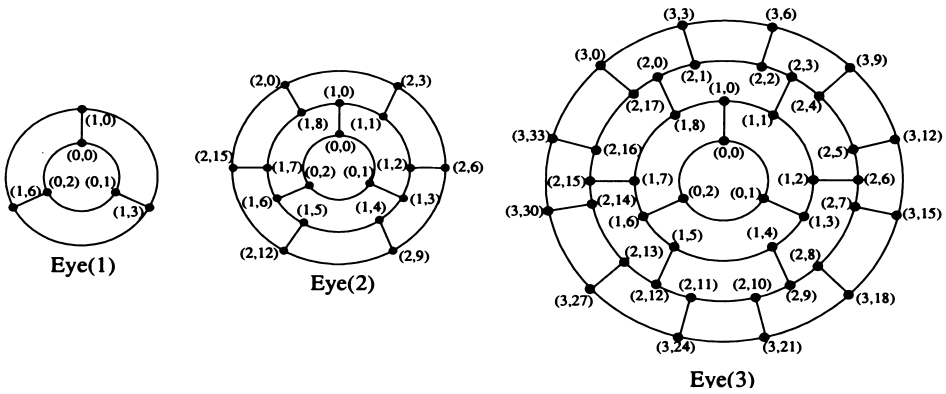


Fig. 1. Examples of *eye* graphs.

$V(I_k) = \{(k, j) \mid 0 \leq j \leq N_k - 1\}$ for $0 \leq k \leq s - 1$ and
$V(O_s) = \{(s, j) \mid 0 \leq j \leq N_s - 1$ and $[j]_3 = 0\}$.

For vertex (k, j), k and j are referred to as the first and the second coordinate, respectively. Throughout this paper all computations on the second coordinate of a vertex at the kth concentric cycle are carried out with modulo N_k. The graph *Eyes*(s) contains two types of edges, i.e., *cycle edges*, denoted by $e_{k,j}$, and *intercycle edges*, denoted by $e_{k,j}^{k+1}$, which are given as follows:

$$e_{k,j} = \begin{cases} ((k, j), (k, j+1)) & \text{for } 0 \leq k \leq s-1 \text{ and } 0 \leq j \leq N_k - 1, \\ ((k, j), (k, j+3)) & \text{for } k = s, \ 0 \leq j \leq N_s - 1 \text{ and } [j]_3 \neq 0. \end{cases}$$

$$e_{k,j}^{k+1} = \begin{cases} ((0, j), (1, 3j)) & \text{for } k = 0 \text{ and } j = 0, 1, 2 \\ ((k, j), (k+1, 2j + [j]_3)) & \text{for } 1 \leq k \leq s-1, \ 0 \leq j \leq N_k - 1 \text{ and } [j]_3 \neq 0. \end{cases}$$

The set $\{e_{k,j} \mid 0 \leq j \leq N_k - 1\}$ is denoted by $E(I_k)$ if $0 \leq k \leq s - 1$, and denoted by $E(O_s)$ if $k = s$. We use E_k^{k+1} to denote the set $\{e_{k,j}^{k+1} \mid 0 \leq j \leq N_k - 1\}$. The edge set of

Eye(*s*) is defined as $E(Eye(s)) = \bigcup_{k=0}^{s-1} E(I_k) \cup E(O_s) \cup (\bigcup_{k=0}^{s-1} E_k^{k+1})$. The graph *Eye*(*s*) is 3-regular and contains $3 + 9\sum_{k=1}^{s-1} 2^{k-1} + 3 \cdot 2^{s-1} = 6(2^s - 1)$ vertices and $9(2^s - 1)$ edges.

On the other hand, graphs *Eye*(*s* + 1), $s \geq 1$, can be recursively drawn (or constructed) from *Eye*(*s*) by performing the following two steps:

SUBDIVISION. Subdivide each edge $e_{s,j}$ of O_s, $0 \leq j \leq N_s - 1$ and $[j]_3 = 0$, into a path of length 3; i.e., replace $e_{s,j}$ with a path $\langle (s, j), (s, j + 1), (s, j + 2), (s, j + 3)\rangle$ to connect its two ends (s, j) and $(s, j + 3)$. Rename O_s as I_s.

EXTENSION. Construct graph O_{s+1} as a concentric cycle outside I_s and join every vertex (s, j) in I_s with vertex $(s + 1, 2j + [j]_3)$ in O_{s+1} for $1 \leq j \leq N_s - 1$ and $[j]_3 \neq 0$.

The above recursive construction also shows that *Eye*(*s*) is a planar graph. All vertices in each cycle can be drawn at equal distance, and all intercycle edges drawn in the normal directions of the corresponding cycles as shown in Fig. 1. Therefore, *Eye*(*s*) is invariant under the rotation of 120°. Furthermore, we can use a specified subgraph to obtain other isomorphic subgraphs by proper rotation, by which we mean rotations of 120° or 240°. To be specific, each vertex (k, i) is relabeled $(k, i + \delta N_k/3)$, where $\delta = 1$ for 120° rotation and $\delta = 2$ for 240° rotation. For example, consider the cycle H_2 of *Eye*(2) given by $H_2 = \langle (0,0), (1,0), (1,1), (2,3), (2,0), (1,8), (1,7), (2,15), (2,12), (2,9), (2,6), (1,2), (1,3), (1,4), (1,5), (1,6), (0,2), (0,1), (0,0)\rangle$. Then we can obtain another cycle C_2 by 120° rotation of H_2, which is given by $C_2 = \langle (0,1), (1,3), (1,4), (2,9), (2,6), (1,2), (1,1), (2,3), (2,0), (2,15), (2,12), (1,5), (1,6), (1,7), (1,8), (1,0), (0,0), (0,2), (0,1)\rangle$.

Note that for any vertex (k, i) in *Eye*(*s*) with $k \geq 1$ there exists a vertex $(k - 1, j)$ for some j such that the distance between (k, i) and $(k - 1, j)$ is at most 2. In other words, each vertex in I_k can reach I_{k-1} with at most two edges. It follows that the diameter of *Eye*(*s*) is at most $2(2(s - 1) + 1) + 1 = 4s - 1$. Because the number of vertices in *Eye*(*s*) is $6(2^s - 1)$, the diameter of *Eye*(*s*) is less than 4 times the Moore bound.

3. HAMILTONICITY OF *Eye*(*s*)

In this section, we prove that *Eye*(*s*) is a hamiltonian graph.

Theorem 1: *Eye*(*s*) is hamiltonian for every $s \geq 1$.

Proof: We prove this theorem by induction. For $s = 1$, $H_1 = \langle (0,0), (1,0), (1,3), (1,6), (0,2), (0,1), (0,0)\rangle$ is a hamiltonian cycle in *Eye*(1).

Assume that *Eye*(*k*) is hamiltonian for all $1 \leq k \leq s$ and $s \geq 2$. Let H_s be a hamiltonian cycle in *Eye*(*s*). Note that the degree of any vertex in *Eye*(*s*) is three, at least one edge of O_s is in H_s, and at least one edge of O_s is not in H_s. Since *Eye*(*s* + 1) can be obtained from *Eye*(*s*) by SUBDIVISION and EXTENSION, we can construct a cycle H'_s in *Eye*(*s* + 1) from H_s by subdividing the edges in $E(O_s) \cap E(H_s)$. Let U_s denote the collection of vertices in $V(I_s)$ which are not in H'_s. It follows that H'_s is a hamiltonian cycle in the graph

$Eye(s + 1) - V(O_{s+1}) - U_s$.

Observe that vertex $(s + 1, j)$ in O_{s+1} is adjacent to $(s, \left\lceil \frac{j}{2} \right\rceil - 1)$ in I_s. In order to augment O_{s+1} to include all of the vertices in U_s, we replace edge $e_{s+1, 2m+1}$ in O_{s+1} with a path $\langle (s + 1, 2m + 1), (s, m), (s, m + 1), (s + 1, 2m + 4)\rangle$ if $(s, m) \in U_s$ and $[m]_3 = 1$. In this way, we obtain a cycle O'_{s+1} that traverses each vertex of $U_s \cup V(O_{s+1})$ exactly once.

The cycles H'_s and O'_{s+1} are two disjoint cycles that traverse every vertex of $Eye(s + 1)$ exactly once. To combine these two cycles into a hamiltonian cycle, we arbitrarily choose an edge $e_{s,j}$ in H'_s with $[j]_3 = 1$, and then define a cycle H_{s+1} given by

$$(H'_s - e_{s,j}) \cup (O'_{s+1} - e_{s+1,2j+1}) \cup \left\{e^{s+1}_{s,j}, e^{s+1}_{s,j+1}\right\} \quad (1)$$

Thus, the theorem follows. ❑

In what follows, we write "the (construction) scheme (1)" to mean the construction scheme presented in the proof of Theorem 1. We use H'_s and O'_{s+1} to denote the two cycles as specified in the scheme (1). Given H'_s and an edge $e_{s,j}$ of H'_s with $[j]_3 = 1$, the hamiltonian cycle H_{s+1} is uniquely defined. Since $Eye(s + 1)$ is a 3-regular graph for $s \geq 1$, no two consecutive edges in O_{s+1} can be excluded from H_{s+1}. The following properties hold when deleting edges $e_{s,j}$ in H'_s and $e_{s+1,2j+1}$ in O'_{s+1} with $[j]_3 = 1$ to construct H_{s+1}:

(P1) The edges $e_{s+1,2j-2}$ and $e_{s+1,2j+4}$ are included in H_{s+1}.
(P2) The edge $e_{s,i} \in H_s$ with $i \neq j - 1$ if and only if $\langle (s + 1, 2i), (s + 1, 2i + 3), (s + 1, 2i + 6), (s + 1, 2i + 9)\rangle$ is included as a subpath of H_{s+1}.
(P3) Any edge $e_{s,i} \notin H_s$ implies that $e_{s,} \notin H_{s+1}$, $e_{s,i+1} \in H_{s+1}$, $e_{s,i+2} \notin H_{s+1}$ and $e_{s+1,2i+3} \notin H_{s+1}$.
(P4) Each edge in $O_{s+1} - H_{s+1}$ is given by $e_{s+1, m}$ with $[m]_6 = 3$, and on the other hand, H_{s+1} includes all of the edges $e_{s+1,l}$ with $[l]_6 = 0$.

(P1) and (P2) can be easily verified. (P3) follows from the definition of O'_{s+1}. The edge in O'_{s+1} chosen to be deleted to form H_{s+1} has the form $e_{s+1, 2j+1}$ with $[j]_3 = 1$. Therefore, (P4) follows from (P1) and (P3).

For $s \geq 2$, a hamiltonian cycle H_s constructed by scheme (1) is called the *fundamental hamiltonian cycle* of $Eye(s)$ (an example is shown in Fig. 2), if

(i) $H_1 = \langle (0,0), (1,0), (1,3), (1,6), (0,2), (0,1), (0,0)\rangle$, and
(ii) edge $e_{i,1}$ in H'_i is chosen to be deleted in the recursive construction of H_{i+1} for all $1 \leq i \leq s - 1$.

In particular, we use FH_s to denote the fundamental hamiltonian cycle in $Eye(s)$. Since $O_1 - H_1$ consists of only one edge and we delete one edge from O'_{i+1} in the recursive construction of H_{i+1}, $FH_s \cap O_s$ is a set of s disjoint paths which are denoted by P_0, P_1, …, P_{s-1}. To be precise, for $s \geq 2$,

$P_0 = \langle (s, 0), (s, 3)\rangle$,
$P_i = \langle (s, 3\cdot 2^i), (s, 3\cdot 2^i + 3), (s, 3\cdot 2^i + 6), \ldots, (s, 32^{i+1} - 3)\rangle$ for $1 \leq i \leq s - 2$,
$P_{s-1} = \langle (s, 3\cdot 2^{s-1}), (s, 3\cdot 2^{s-1} + 3), (s, 3\cdot 2^{s-1} + 6), \ldots, (s, N_s - 3)\rangle$.

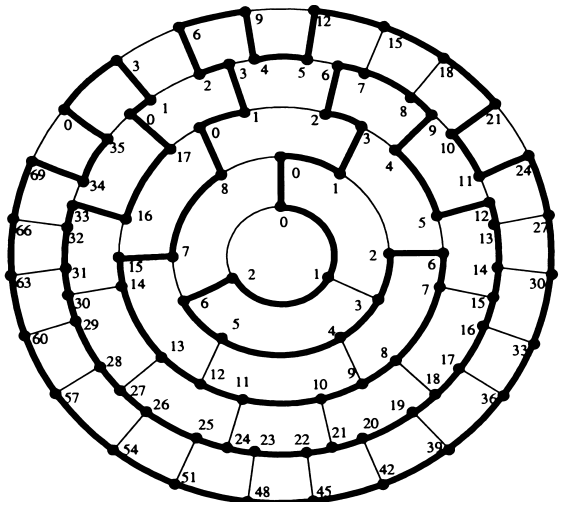


Fig. 2. The fundamental hamiltonian cycle FH_4 of $Eye(4)$. (The first coordinate of each vertex is omitted.)

Path P_i has $2^i - 1$ edges for $1 \le i \le s-2$, and $2^s - 1$ edges for $i = s-1$. Furthermore, there is a set of $2^s - 2$ intercycle edges, given by $T_s = \{ e^s_{s-1,3\cdot 2^{s-2}+1}, e^s_{s-1,3\cdot 2^{s-2}+2}, e^s_{s-1,3\cdot 2^{s-2}+4}, \ldots, e^s_{s-1,N_{s-1}-4} \}$, which are not included in FH_s.

Remark 1: Since $\frac{1}{2} \le \frac{|P_{s-1}|}{|E(O_s)|} < \frac{2}{3}$ for $s \ge 2$, we can always obtain a hamiltonian cycle in $Eye(s)$ containing any specific edge $e \in E(O_s)$ by proper rotation of FH_s. Moreover, since $\frac{1}{3} \le \frac{|T_s|}{|E^s_{s-1}|} < \frac{2}{3}$ we can always obtain a hamiltonian cycle in $Eye(s) - e$ by rotating FH_s properly for any edge $e \in E^s_{s-1}$ and $s \ge 2$.

We use Fig. 2 to illustrate Remark 1. Given the hamiltonian cycle FH_4 which contains the edges $((4, 3i),(4, 3i + 3))$ for $8 \le i \le 22$, we can obtain a hamiltonian cycle from FH_4 by 120° rotation, which contains the edges $((4, 3i), (4, 3i + 3))$ for $16 \le i \le 23$ and $((4, 3i), (4, 3i + 3))$ for $0 \le i \le 6$. Furthermore, by rotating FH_4 240°, we can obtain a hamiltonian cycle that contains the edges $((4, 3i), (4, 3i + 3))$ for $0 \le i \le 14$.

4. 1-EDGE HAMILTONICITY OF *Eye*(*s*)

The hamiltonian cycle $H_1 = \langle (0,0), (1,0), (1,3), (1,6), (0,2), (0,1), (0,0) \rangle$ of $Eye(1)$ does not include edges $e_{1,6}$, $e_{0,2}$ and $e^1_{0,1}$. By proper rotation of H_1, we can obtain other two distinct hamiltonian cycles $H^1_1 = \langle (0,1), (1,3), (1,6), (1,0), (0,0), (0,2), (0,1) \rangle$ and $H^2_1 = \langle (0,2), (1,6), (1,0), (1,3), (0,1), (0,0), (0,2) \rangle$ such that edges $e_{1,0}$, $e_{0,0}$ and $e^1_{0,2}$ are not

in H_1^1, and edges $e_{1,3}$, $e_{0,1}$ and $e_{0,0}^1$ are not in H_1^2. Thus, $Eye(1)$ is a 1-edge hamiltonian graph. For $Eye(2)$, it can be verified by exhaustive construction that $Eye(2) - e_{2,0}$ is not hamiltonian. Hence, $Eye(2)$ is not 1-edge hamiltonian. Furthermore, $Eye(2) - e$ is not hamiltonian if $e \in \{e_{2,0}, e_{2,6}, e_{2,12}\}$, but is hamiltonian otherwise.

Let $E_s^1 = \{e_{s,m} \mid [m]_6 = 0, 0 \le m < N_s\}$ and $E_s^2 = \{e_{k,m} \mid [m]_6 = 0 \text{ or } 2, 0 \le m < N_k, 2 \le k \le s-1\}$ be two subsets of $E(Eye(s))$. By definition, we have $E_1^2 = E_2^2 = \varnothing$.

Lemma 1: $Eye(s) - e$ is hamiltonian for all $e \in E(Eye(s)) - (E_s^1 \cup E_s^2)$.

Proof: The statement is true for $s = 1, 2$. Assume that $Eye(k) - e$ is hamiltonian for $1 \le k \le s-1$ and $s \ge 2$. It is suffices to show that $Eye(s) - e$ is hamiltonian where $e \in (E(I_{s-1}) \cup E(O_s) \cup E_{s-1}^s) - (E_s^1 \cup E_s^2)$.

First, let $e \in E(O_s) - E_s^1$, say $e = e_{s,m}$ with $[m]_6 = 3$. It follows from Remark 1 that we have a hamiltonian cycle H_{s-1} such that H_{s-1} contains $e_{s-1,\left\lceil \frac{m}{2} \right\rceil - 2}$. To construct a hamiltonian cycle H_s from H_{s-1} by using the scheme (1), we choose to delete $e_{s-1,\left\lceil \frac{m}{2} \right\rceil - 1}$ in H'_{s-1}. Therefore, H_s does not contain the edge $e_{s,m}$, and $Eye(s) - e$ is hamiltonian.

Next, let $e \in E(I_{s-1}) - E_s^2$, say $e = e_{s-1,m}$. For $[m]_6 = 1$ or 4, by Remark 1 we can always find a Hamiltonian cycle H_{s-1} containing $e_{s-1,\, m-[m]_3}$ in $Eye(s-1)$. Using scheme (1) and choosing to delete $e_{s-1,\, m-[m]_3+1}$, we obtain a hamiltonian cycle H_s in $Eye(s)$ that does not include e. For $[m]_6 = 3$ or 5, by the induction assumption we have a hamiltonian cycle H_{s-1} in $Eye(s-1) - e_{s-1,\, m-[m]_3}$. It follows from (P3) that hamiltonian cycles in $Eye(s)$ constructed from H_{s-1} by scheme (1) do not contain edge $e_{s-1,m}$.

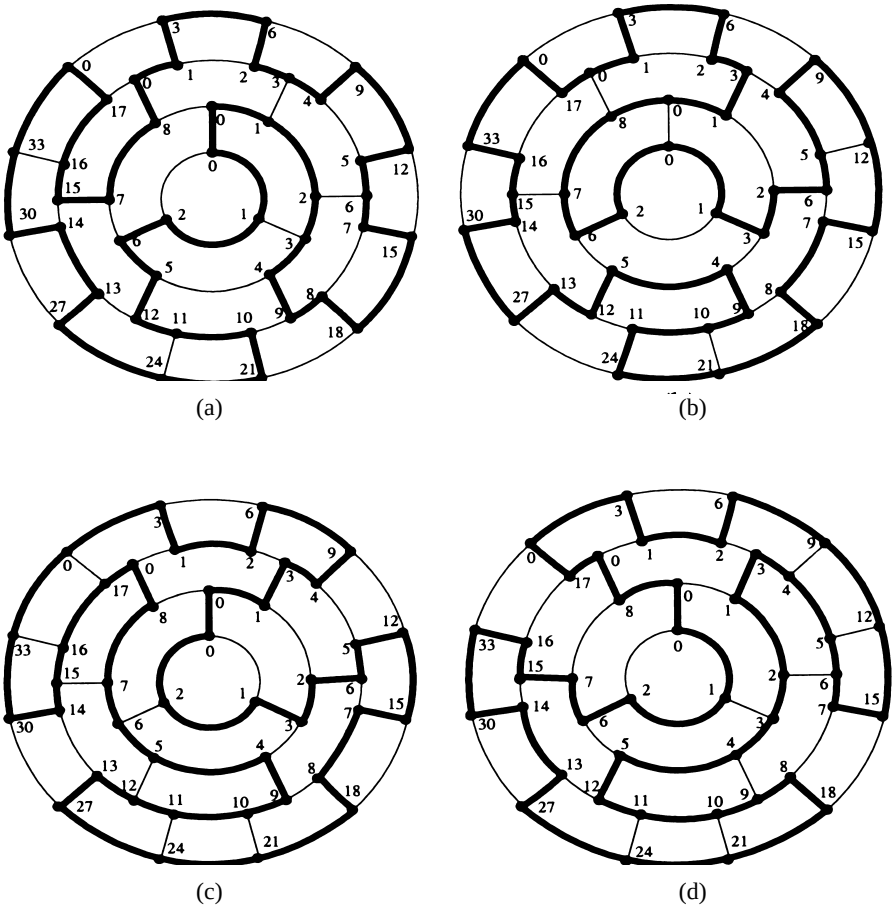
Following from Remark 1 when e is an intercycle edge, $Eye(s) - e$ is hamiltonian. Hence, the lemma follows: ❑

Lemma 2: $Eye(s) - e$ is hamiltonian for $e \in E_s^1$ and $s \ge 3$.

Proof: For $s = 3$, we illustrate the existence of hamiltonian cycles in $Eye(3) - e$ in Fig. 3(a) and (b). Consider $s \ge 4$. Let $e = e_{s,12m}$ or $e_{s,12m-6}$, where $m \ge 1$. By proper rotation of the fundamental hamiltonian cycle, we can obtain a hamiltonian cycle H_{s-1} in $Eye(s-1)$ such that $\langle (s-1, 6m-6), (s-1, 6m-3), (s-1, 6m), (s-1, 6m+3) \rangle$ is included as a subpath of H_{s-1}. Again, we perform SUBDIVISION and EXTENSION and attain two disjoint cycles including all vertices in $Eye(s)$, i.e., H'_{s-1} and O'_s. Since $\langle (s-1, 6m-6), (s-1, 6m-3), (s-1, 6m), (s-1, 6m+3) \rangle$ included in H_{s-1} implies $e_{s-2,3m-3} \in H_{s-2}$ by (P2), where H_{s-2} is a hamiltonian cycle in $Eye(s-2)$, it follows that $e_{s-2,\,3m-2}$ is included in H'_{s-1}. To combine the two disjoint cycles H'_{s-1} and O'_s into a hamiltonian cycle, we replace $e_{s,12m}$, $e_{s,12m-6}$ in O'_s and $e_{s-1,6m}$, $e_{s-1,6m-2}$, $e_{s-1,6m-4}$, $e_{s-2,3m-2}$ in H'_{s-1} with $e_{s-2,3m-2}^{s-1}, e_{s-2,3m-1}^{s-1}$, $e_{s-1,6m-4}^{s}, e_{s-1,6m-2}^{s}, e_{s-1,6m-1}^{s}, e_{s-1,6m+1}^{s}$. Thus, we obtain a hamiltonian cycle that excludes $e_{s,12m}$ and $e_{s,12m-6}$. The lemma follows. ❑

Lemma 3: $Eye(s) - e$ is hamiltonian for $e \in E_s^2$ and $s \ge 3$.

Proof: For $s = 3$, we illustrate the existence of a hamiltonian cycle in $Eye(3) - e$ in Figs. 3(c) and (d).



(a) (b) (c) (d)

Fig. 3. A hamiltonian cycle of $Eye(3) - e_{3,m}$ for m = 0,6,12,18 in (a) and m = 24,30 in (b). And a hamiltonian cycle of $Eye(3) - e_{2,m}$ for m = 0,2,6,8 in (c) and m = 12,14 in (d). (The first coordinate of each vertex is omitted.)

Consider $s \geq 4$. First, let $e \in \{e_{2,0}, e_{2,2}, e_{2,6}, e_{2,8}, e_{2,12}, e_{2,14}\}$. Hamiltonian cycles in $Eye(s) - e$ can be obtained by first applying the construction of Fig. 3(c) or Fig. 3(d) and followed by iteratively performing construction scheme (1).

Next, let $e = e_{k,m}$ with $3 \leq k \leq s - 1$ and $[m]_6 = 0$ or 2. Let H_k be a hamiltonian cycle of $Eye(k) - e_{k,m-[m]_3}$ obtained by the construction scheme presented in the proof of Lemma 2. Using construction scheme (1) on H_k iteratively, we can attain a hamiltonian cycle H_s in $Eye(s)$. Following from (P3), edges $e_{k,m-[m]_3}$ and $e_{k,m-[m]_3+2}$ are not included in the hamiltonian cycle H_s. ❑

Lemmas 1, 2, 3 can immediately lead to the following theorem.

Theorem 2: *Eye*(*s*) is 1-edge hamiltonian for every $s \geq 1$ and $s \neq 2$.

Because the degree of every vertex in *Eye*(*s*) is three, we have the following corollary:

Corollary 1: *Eye*(*s*) is minimal 1-edge hamiltonian for $s \neq 2$.

To verify the relationship between 1-edge hamiltonicity and edge hamiltonicity, we have the following theorem.

Theorem 3: Any 3-regular 1-edge hamiltonian graph is edge hamiltonian.

Proof: Let G be a 3-regular 1-edge hamiltonian graph. Let e be an edge in G and e' be one of the four edges that are incident with e. Since G is 1-edge hamiltonian and 3-regular, there exists a hamiltonian cycle C in $G - e'$ such that e is in C. Thus, G is edge hamiltonian. ❑

Theorem 4: *Eye*(*s*) is edge hamiltonian for every *s*.

Proof: It can be verified that *Eye*(2) is edge hamiltonian. Furthermore, it follows from Theorem 3 that *Eye*(*s*) is edge hamiltonian for every *s*. ❑

5. 1-VERTEX HAMILTONICITY OF *Eye*(*s*)

In this section, we first show that *Eye*(*s*) $- x$, $s \geq 2$, is hamiltonian, where $x \in V(Eye(s)) - V(Eye(s-1))$. The set $V(Eye(s)) - V(Eye(s-1))$ for $s \geq 2$ can be partitioned into the following four subsets:

$$V_s^1 = \{(s,m) \mid [m]_{12} = 3 \text{ or } 9,\ 0 \leq m \leq N_s - 1\},$$
$$V_s^2 = \{(s,m) \mid [m]_{12} = 0 \text{ or } 6,\ 0 \leq m \leq N_s - 1\},$$
$$V_s^3 = \{(s-1,m) \mid [m]_6 = 1 \text{ or } 4,\ 0 \leq m \leq N_{s-1} - 1\}, \text{ and}$$
$$V_s^4 = \{(s-1,m) \mid [m]_6 = 2 \text{ or } 5,\ 0 \leq m \leq N_{s-1} - 1\}.$$

To prove that *Eye*(*s*) $- x$ is hamiltonian for $s \geq 2$ and $x \in V(Eye(s)) - V(Eye(s-1))$, we first construct a hamiltonian cycle H_{s-1} in *Eye*($s - 1$) and then augment it with O_s without traversing vertex x. Since *Eye*(2) $- e$ is not hamiltonian if $e \in \{e_{2,0}, e_{2,6}, e_{2,12}\}$, it will yield a special case of x in each of V_s^1, V_s^2, V_s^3, and V_s^4 for the construction of H_{s-1} as stated in the proofs of the following lemmas.

Lemma 4: *Eye*(*s*) $- x$ is hamiltonian for $x \in V_s^1$ and $s \geq 2$.

Proof: The special case is given by $s = 3$ and $x \in \{(3,3), (3,15), (3,27)\}$. *Eye*(3) $- x$ is hamiltonian as illustrated in Fig. 4(a) with proper rotation.

For the cases of $s = 2$ and $x = (s, m)$, we define H_{s-1} as H_1^1 for $m = 3$, as H_1^2 for $m = 9$, and as H_1 for $m = 15$, where H_1, H_1^1, and H_1^2 are defined in the first paragraph of Section 4. For the remaining cases of s and $x = (s, m)$, let H_{s-1} denote a hamiltonian cycle of $Eye(s-1) - e_{s-1,\lfloor \frac{m}{2} \rfloor - 2}^{s-1,\lceil \frac{m}{2} \rceil - 2}$ obtained by applying the construction scheme (1) if $e_{s-1,\lfloor \frac{m}{2} \rfloor - 2}^{s-1,\lceil \frac{m}{2} \rceil - 2} \notin E_{s-1}^1$, and by applying the construction presented in the proof of Lemma 2 if $e_{s-1,\lfloor \frac{m}{2} \rfloor - 2}^{s-1,\lceil \frac{m}{2} \rceil - 2} \in E_{s-1}^1$. Since deleting vertex x from $Eye(s)$ reduces the degree of each vertex of $(s, m-3)$, $(s, m+3)$ and $(s-1, \lceil \frac{m}{2} \rceil - 1)$ by one, it follows that any hamiltonian cycle in $Eye(s) - x$ must traverse edges $e_{s,m-6}, e^s_{s-1,\lceil \frac{m}{2} \rceil - 3}, e^s_{s-1,\lceil \frac{m}{2} \rceil}, e_{s,m+3}, e_{s-1,\lceil \frac{m}{2} \rceil - 2}$, and $e_{s-1,\lceil \frac{m}{2} \rceil - 1}$. Let P denote the path $(O'_s - e_{s,m-3} - e^s_{s-1,\lceil \frac{m}{2} \rceil - 1}) \cup \{e_{s-1,\lceil \frac{m}{2} \rceil - 2}, e^s_{s-1,\lceil \frac{m}{2} \rceil - 3}\}$. Then P contains all vertices in $(V(Eye(s)) - V(H'_{s-1}) - x) \cup \{(s-1, \lceil \frac{m}{2} \rceil - 2), (s-1, \lceil \frac{m}{2} \rceil - 3)\}$. Furthermore, P and $H'_{s-1} - e_{s-1,\lceil \frac{m}{2} \rceil - 3}$ are two disjoint paths with common end vertices $(s-1, \lceil \frac{m}{2} \rceil - 2)$, $(s-1, \lceil \frac{m}{2} \rceil - 3)$ and contain all vertices in $Eye(s) - x$. Thus, $P \cup (H'_{s-1} - e_{s-1,\lceil \frac{m}{2} \rceil - 3})$, defines a hamiltonian cycle of $Eye(s) - x$. ❑

Lemma 5: $Eye(s) - x$ is hamiltonian for $x \in V_s^2$ and $s \geq 2$.

Proof: Consider the special case of $s = 3$ and $x \in \{(3,6), (3,18), (3,30)\}$. $Eye(3) - x$ is hamiltonian as illustrated in Fig. 4(b) with proper rotation.

For the cases of $s = 2$ and $x = (s, m)$, we define H_{s-1} as H_1^1 for $m = 6$, as H_1^2 for $m = 12$, and as H_1 for $m = 0$. For the remaining cases of s and $x = (s, m)$, let H_{s-1} denote a hamiltonian cycle of $Eye(s-1) - e_{s-1,\frac{m}{2}-3}^{s-1,\frac{m}{2}-3}$ obtained by applying construction scheme (1) if $e_{s-1,\frac{m}{2}-3}^{s-1,\frac{m}{2}-3} \notin E_{s-1}^1$, and by applying the construction presented in the proof of Lemma 2 if $e_{s-1,\frac{m}{2}-3}^{s-1,\frac{m}{2}-3} \in E_{s-1}^1$. Deleting x from O'_s causes each hamiltonian cycle in $Eye(s) - x$ to traverse edges $e_{s,m-6}, e^s_{s-1,\frac{m}{2}-2}, e_{s-1,\frac{m}{2}-2}, e_{s-1,\frac{m}{2}-1}, e^s_{s-1,\frac{m}{2}+1}$, and $e_{s,m+3}$. Let P denote the path $(O'_s - e_{s.m} - e^s_{s-1,\frac{m}{2}-1}) \cup \{e_{s-1,\frac{m}{2}-1}, e^s_{s-1,\frac{m}{2}+1}\}$. Then $P \cup (H'_{s-1} - e_{s-1,\frac{m}{2}})$ defines a hamiltonian cycle in $Eye(s) - x$. ❑

Lemma 6: $Eye(s) - x$ is hamiltonian for $x \in V_s^3$ and $s \geq 2$.

***Proof*:** Let $x = (s-1, m)$. The special case is given by $s = 3$ and $x \in \{(2,1), (2,7), (2,13)\}$. $Eye(3) - x$ is hamiltonian as illustrated in Fig. 4(c) with proper rotation.

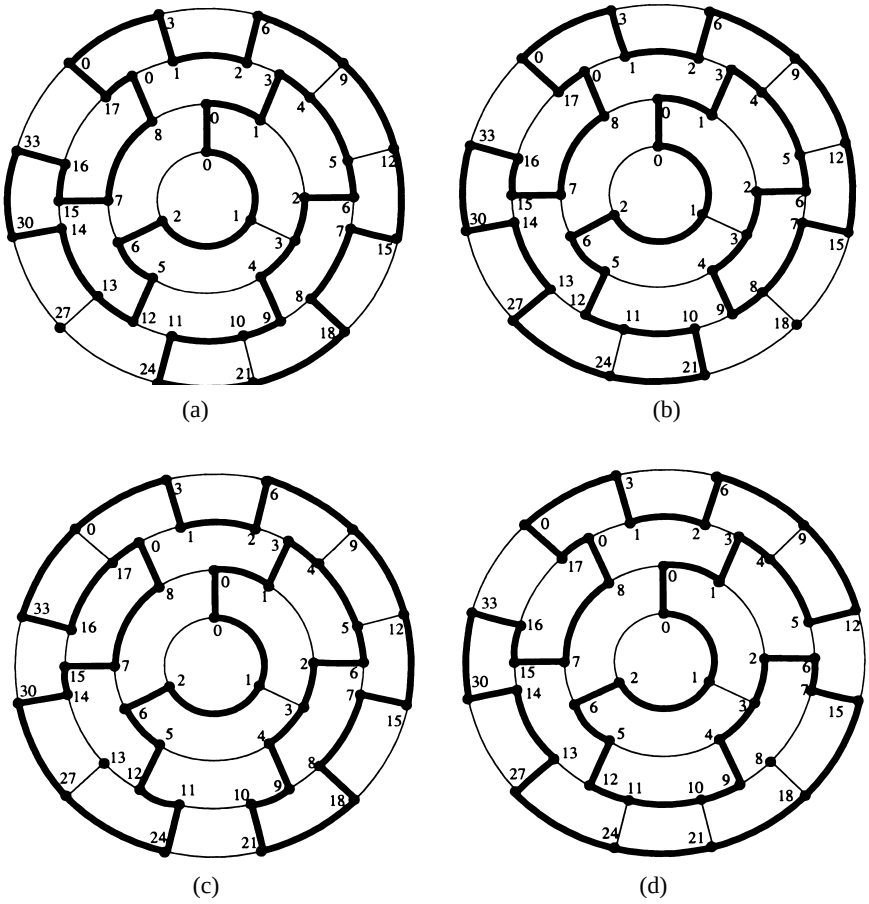


Fig. 4. (a), (b), (c), and (d) are the special cases of Lemma 4, 5, 6, and 7, respectively. (The first coordinate of each vertex is omitted.)

For the cases of $s = 2$ and $x = (s - 1, m)$, we define H_{s-1} as H_1^1 for $m = 1$, as H_1^2 for $m = 4$, and as H_1 for $m = 7$. For the remaining cases of s and x, let H_{s-1} denote a hamiltonian cycle of $Eye(s-1) - e_{s-1,m-1}$ obtained from applying construction scheme (1) if $e_{s-1,m-1} \notin E_{s-1}^1$, and applying the construction presented in the proof of Lemma 2 if $e_{s-1,m-1} \in E_{s-1}^1$. Deleting x from $Eye(s)$ causes edges $e_{s-1,m+1}, e_{s-1,m+1}^s, e_{s,2m+[m]_3}$, and $e_{s,2m+[m]_3-3}$ to be included in each hamiltonian cycle of $Eye(s) - x$. Let P denote the path $(\mathcal{O}_s - e_{s-1,m}^s - e_{s-1,m} - e_{s,2m+[m]_3+3}) \cup \{e_{s,2m+[m]_3}, e_{s-1,m+1}, e_{s-1,m+3}^s\}$. Then, $P \cup (H'_{s-1} - e_{s-1,m+2})$ defines a hamiltonian cycle of $Eye(s) - x$. ❑

Lemma 7: $Eye(s) - x$ is hamiltonian for $x \in V_s^4$ and $s \geq 2$.

Proof: Let $x = (s-1, m)$. The special case is given by $s = 3$ and $x \in \{(2,2), (2,8), (2,14)\}$. $Eye(3) - x$ is hamiltonian as illustrated in Fig. 4(d) with proper rotation.

For the cases of $s = 2$ and $x = (s-1, m)$, we define H_{s-1} as H_1^1 for $m = 2$, as H_1^2 for $m = 5$, and as H_1 for $m = 8$. For the remaining cases of s and x, let H_{s-1} denote a Hamiltonian cycle of $Eye(s-1) - e_{s-1,m-2}$ obtained by applying the construction scheme (1) if $e_{s-1,m-2} \notin E_{s-1}^1$, and by applying the construction presented in the proof of Lemma 2 if $e_{s-1,m-2} \in E_{s-1}^1$. Let P denote the path $(O'_s - e_{s,2m+[m]_{3-6}} - e_{s-1,m-1} - e_{s-1,m-1} - e_{s-1,m}^s) \cup \{e_{s-1,m-3}^s, e_{s-1,m-2}, e_{s,2m+[m]_{3-3}}\}$. Then, $P \cup (H'_{s-1} - e_{s-1,m-3})$ defines a hamiltonian cycle of $Eye(s) - x$. ❑

Theorem 5: *Eye*(s) is 1-vertex hamiltonian for every $s \geq 1$.

Proof: For *Eye*(1), the cycle $\langle(0,0), (0,1), (1,3), (1,6), (0,2), (0,0)\rangle$ does not include vertex (1,0). Since this cycle is unique of length five up to isomorphism, it follows that *Eye*(1) is 1-vertex hamiltonian. Now consider $s \geq 2$. Let $x = (k, m) \in Eye(s)$, where $0 \leq k \leq s$. If $k = s$, it follows from Lemmas 4, 5, 6, and 7 that $Eye(s) - x$ is hamiltonian. If $k \leq s-1$, then we distinguish the following four cases of x.

case 1: Let $k = 0$. Since *Eye*(1) is 1-vertex hamiltonian, we can obtain a hamiltonian cycle, denoted by NH_1^1, in $Eye(1) - x$.

case 2: Let $k = 1$ and $[m]_3 = 0$. Similarly, we can obtain a hamiltonian cycle, denoted by NH_2, in $Eye(1) - x$.

case 3: Let $k \geq 2$ and $[m]_3 = 0$. Since $x \in V_k^1 \cup V_k^2$, it follows from Lemma 4 and Lemma 5 that we can obtain a hamiltonian cycle, denoted by NH_k, in $Eye(k) - x$.

case 4: Let $k \geq 1$ and $[m]_3 \neq 0$. Since $x \in V_{k+1}^3 \cup V_{k+1}^4$, , it follows from Lemma 6 and Lemma 7 that we also obtain a hamiltonian cycle, denoted by NH_{k+1}^4, in $Eye(k+1) - x$.

In case 3, NH_k is a cycle that traverses all vertices in $Eye(k)$ except x exactly once. Let NH'_k denote the cycle obtained from NH_k by subdividing all edges in $(O_k - x) \cap NH_k$. By performing SUBDIVISION and EXTENSION on $Eye(k)$, NH'_k will not include vertices $(k, m-2)$, $(k, m-1)$, (k, m), $(k, m+1)$ and $(k, m+2)$. To construct a cycle, denoted by NH_{k+1}^3, in $Eye(k+1)$ to traverse all vertices excluding x exactly once, we use the scheme (1) and replace the path $\langle(k+1, 2m-3), (k+1, 2m), (k+1, 2m+3), (k+1, 2m+6)\rangle$ in O'_{k+1} with path $\langle(k+1, 2m-3), (k, m-2), (k, m-1), (k+1, 2m), (k+1, 2m+3), (k, m+1), (k, m+2), (k+1, 2m+6)\rangle$. In case 2, we can obtain a hamiltonian cycle in $Eye(2) - x$, denoted by NH_2^2, using the same approach as in case 3.

We then apply scheme (1) on NH_1^1, NH_2^2, NH_{k+1}^3 or NH_{k+1}^4, respectively, to construct a hamiltonian cycle in $Eye(s) - x$. ❑

Since *Eye*(s) is 1-vertex hamiltonian and 3-regular, the following corollaries hold.

Corollary 2: *Eye*(s) is 3-connected for every s.

Corollary 3: *Eye*(s) is minimal 1-vertex hamiltonian for every s.

Corollary 1 and Corollary 3 immediately lead to the following theorem.

Theorem 6: *Eye*(s) is minimal 1-hamiltonian for $s \neq 2$.

6. CONCLUDING REMARKS

In summary, *Eye*(*s*) is minimal 1-edge hamiltonian for $s \neq 2$, minimal 1-vertex hamiltonian for every *s*, and minimal 1-hamiltonian for $s \neq 2$. Furthermore, *Eye*(*s*) is edge hamiltonian for every *s*. *Eye*(*s*) indeed has nice properties on hamiltonicity and is planar. This graph has diameter of $O(\log n)$ and, to be precise, bounded by $4\log\frac{n+6}{6}-1$, where $n = 6(2^s - 1)$. It would be interesting to construct a family of planar minimal 1-hamiltonian graphs which has properties as nice as *Eye*(*s*) and has an even smaller diameter than *Eye*(*s*).

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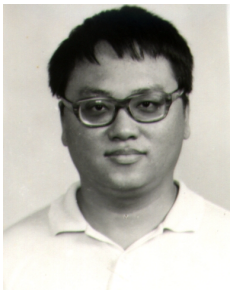

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