Natural Deduction for Propositional Logic

Bow-Yaw Wang

Institute of Information Science
Academia Sinica, Taiwan

October 7, 2020
1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
In our examples, we (informally) infer new sentences. In natural deduction, we have a collection of proof rules.

- These proof rules allow us to infer new sentences logically followed from existing ones.

Suppose we have a set of sentences: $\phi_1, \phi_2, \ldots, \phi_n$ (called premises), and another sentence $\psi$ (called a conclusion).

The notation

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

is called a sequent.

A sequent is valid if a proof (built by the proof rules) can be found.

We will try to build a proof for our examples. Namely,

$$p \land \neg q \implies r, \neg r, p \vdash q.$$
Suppose we want to prove a conclusion \( \phi \land \psi \). What do we do?

- Of course, we need to prove both \( \phi \) and \( \psi \) so that we can conclude \( \phi \land \psi \).

Hence the proof rule for conjunction is

\[
\frac{\phi \quad \psi}{\phi \land \psi} \land i
\]

- Note that premises are shown above the line and the conclusion is below. Also, \( \land i \) is the name of the proof rule.
- This proof rule is called “conjunction-introduction” since we introduce a conjunction \( (\land) \) in the conclusion.
For each connective, we have introduction proof rule(s) and also elimination proof rule(s).

Suppose we want to prove a conclusion $\phi$ from the premise $\phi \land \psi$. What do we do?

- We don’t do any thing since we know $\phi$ already!

Here are the elimination proof rules:

\[
\begin{align*}
\frac{\phi \land \psi}{\phi} \quad &\text{^e}_1 \\
\frac{\phi \land \psi}{\psi} \quad &\text{^e}_2
\end{align*}
\]

The rule $\text{^e}_1$ says: if you have a proof for $\phi \land \psi$, then you have a proof for $\phi$ by applying this proof rule.

Why do we need two rules?

- Because we want to manipulate syntax only.
Example

Prove $p \land q, r \vdash q \land r$.

Proof.

We are looking for a proof of the form:

$$
\begin{array}{c}
p \land q \\
\vdash \\
q \land r
\end{array}
$$
Example

Prove \( p \land q, r \vdash q \land r \).

Proof.

We are looking for a proof of the form:

\[
\frac{p \land q}{q} \quad \frac{r}{q \land r} \quad \frac{q \land r}{q \land r} \quad \text{\&i}
\]

We will write proofs in lines:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( p \land q )</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>( r )</td>
<td>premise</td>
</tr>
<tr>
<td>3</td>
<td>( q )</td>
<td>( &amp;e_2 ) 1</td>
</tr>
<tr>
<td>4</td>
<td>( q \land r )</td>
<td>( &amp;i ) 3, 2</td>
</tr>
</tbody>
</table>

Bow-Yaw Wang (Academia Sinica)  
Natural Deduction for Propositional Logic  
October 7, 2020 6 / 67
Suppose we want to prove $\phi$ from a proof for $\neg\neg\phi$. What do we do?

- There is no difference between $\phi$ and $\neg\neg\phi$. The same proof suffices!

Hence we have the following proof rules:

- $\phi \quad \neg\neg i$
- $\neg\neg\phi \quad \phi \quad \neg\neg e$
Example

Prove $p, \neg \neg (q \land r) \vdash \neg \neg p \land r$.

Proof.

We are looking for a proof like:

\[
p \quad \neg \neg (q \land r) \\
\vdash \neg \neg p \land r
\]
Example

Prove $p, \neg \neg (q \land r) \vdash \neg \neg p \land r$.

Proof.

We are looking for a proof like:

$$
\frac{\frac{p}{\neg\neg p \quad \neg\neg i}}{
eg\neg p} \quad \frac{\frac{\frac{q \land r}{q \land r \quad \land e_2}}{r}}{
eg\neg p \land r \quad \land i} \quad \neg\neg e
$$
Example

Prove $p, \neg\neg(q \land r) \vdash \neg\neg p \land r$.

Proof.

We are looking for a proof like:

1. $p$ premise
2. $\neg\neg(q \land r)$ premise
3. $\neg\neg p$ $\neg\neg i$ 1
4. $q \land r$ $\neg\neg e$ 2
5. $r$ $\land e_2$ 4
6. $\neg\neg p \land r$ $\land i$ 3, 5
Suppose we want to prove $\psi$ from proofs for $\phi$ and $\phi \implies \psi$. What do we do?

- We just put the two proofs for $\phi$ and $\phi \implies \psi$ together.

Here is the proof rule:

$$
\phi \quad \phi \implies \psi \quad \implies e
$$

This proof rule is also called *modus ponens*.

Here is another proof rule related to implication:

$$
\phi \implies \psi \quad \neg \psi \quad \neg \phi \quad MT
$$

This proof rule is called *modus tollens*. 
Example

Prove $p \implies (q \implies r), p, \neg r \vdash \neg q$.

Proof.

1. $p \implies (q \implies r)$ premise
2. $p$ premise
3. $\neg r$ premise
4. $q \implies r$ $\implies e \ 2, 1$
5. $\neg q$ $MT\ 4, 3$
Proof Rules for Natural Deduction – Implication

- Suppose we want to prove $\phi \implies \psi$. What do we do?
  - We assume $\phi$ to prove $\psi$. If succeed, we conclude $\phi \implies \psi$ without any assumption.
  - Note that $\phi$ is added as an assumption and then removed so that $\phi \implies \psi$ does not depend on $\phi$.

- We use “box” to simulate this strategy.

- Here is the proof rule:

\[
\begin{array}{c}
\phi \\
\vdots \\
\psi \\
\hline
\phi \implies \psi \implies \text{i}
\end{array}
\]

- At any point in a box, you can only use a sentence $\phi$ before that point. Moreover, no box enclosing the occurrence of $\phi$ has been closed.
Example

Prove \(-q \implies \neg p \vdash p \implies \neg \neg q\).

Proof.

\[
\begin{array}{c}
\neg q \implies \neg p \\
\hline
\frac{p}{\neg \neg p} \quad \quad \neg \neg i \\
\hline
\neg \neg q \quad MT \quad 1, \ 3 \\
\hline
\frac{p \implies \neg \neg q}{\implies i} \quad 2-4
\end{array}
\]

1. \(-q \implies \neg p\) \quad premise
2. \(p\) \quad assumption
3. \(\neg \neg p\) \quad \neg \neg i \quad 2
4. \(\neg \neg q\) \quad MT \quad 1, \ 3
5. \(p \implies \neg \neg q\) \quad \implies i \quad 2-4
Theorems

Example
Prove \( \vdash p \rightarrow p \).

Proof.

1. \( p \) assumption
2. \( p \rightarrow p \rightarrow i \ 1 \ - \ 1 \)

In the box, we have \( \phi \equiv \psi \equiv p \).

Definition
A sentence \( \phi \) such that \( \vdash \phi \) is called a theorem.
Example

Prove $p \land q \implies r \vdash p \implies (q \implies r)$.

Proof.

1. $p \land q \implies r$ premise
2. $p$ assumption
3. $q$ assumption
4. $p \land q$ $\land i$ 2, 3
5. $r$ $\implies e$ 4, 1
6. $q \implies r$ $\implies i$ 3-5
7. $p \implies (q \implies r)$ $\implies i$ 2-6
Proof Rules for Natural Deduction – Disjunction

- Suppose we want to prove $\phi \lor \psi$. What do we do?
  - We can either prove $\phi$ or $\psi$.
- Here are the proof rules:

  $\frac{\phi}{\phi \lor \psi} \lor i_1 \\
  \frac{\psi}{\phi \lor \psi} \lor i_2$

  - Note the symmetry with $\land e_1$ and $\land e_2$.

  $\frac{\phi \land \psi}{\phi} \land e_1 \\
  \frac{\phi \land \psi}{\psi} \land e_2$

  - Can we have a corresponding symmetric elimination rule for disjunction? Recall

  $\frac{\phi \quad \psi}{\phi \land \psi} \land i$
Suppose we want to prove $\chi$ from $\phi \lor \psi$. What do we do?

- We assume $\phi$ to prove $\chi$ and then assume $\psi$ to prove $\chi$.
- If both succeed, $\chi$ is proved from $\phi \lor \psi$ without assuming $\phi$ and $\psi$.

Here is the proof rule:

\[
\begin{array}{c}
\phi \\
\vdots \\
\chi \\
\hline
\chi \\
\hline
\phi \lor \psi \\
\hline
\psi \\
\vdots \\
\chi \\
\hline
\chi
\end{array}
\]

$\lor e$

In addition to nested boxes, we may have parallel boxes in our proofs.
Example

Recall that our syntax does not admit commutativity.

Example

Prove \( p \lor q \vdash q \lor p \).

Proof.

\[
\begin{array}{c}
p \lor q \\
\hline 
\frac{p}{q \lor p} \lor i_2 \\
\frac{q}{q \lor p} \lor i_1 \\
\hline 
q \lor p \lor e 1, 2-3, 4-5
\end{array}
\]

1. \( p \lor q \) premise
2. \( p \) assumption
3. \( q \lor p \lor i_2 2 \)
4. \( q \) assumption
5. \( q \lor p \lor i_1 4 \)
6. \( q \lor p \lor e 1, 2-3, 4-5 \)
Example

Prove $q \implies r \vdash p \lor q \implies p \lor r$.

Proof.

1. $q \implies r$  \hspace{1cm} \text{premise}
2. $p \lor q$  \hspace{1cm} \text{assumption}
3. $p$  \hspace{1cm} \text{assumption}
4. $p \lor r$  \hspace{1cm} $\lor i_1$ 3
5. $q$  \hspace{1cm} \text{assumption}
6. $r$  \hspace{1cm} $\implies e$ 5, 1
7. $p \lor r$  \hspace{1cm} $\lor i_2$ 6
8. $p \lor r$  \hspace{1cm} $\lor e$ 2, 3-4, 5-7
9. $p \lor q \implies p \lor r$  \hspace{1cm} $\lor e$ 2-8
Example

Prove $p \land (q \lor r) \vdash (p \land q) \lor (p \land r)$.

Proof.

1. $p \land (q \lor r)$ premise
2. $p$ \lande 1
3. $q \lor r$ \lande 1
4. $q$ assumption
5. $p \land q$ \landi 2, 4
6. $(p \land q) \lor (p \land r)$ \lori 5
7. $r$ assumption
8. $p \land r$ \landi 2, 7
9. $(p \land q) \lor (p \land r)$ \lori 8
10. $(p \land q) \lor (p \land r)$ \lor e 3, 4-6, 7-9
Example

Prove $(p \land q) \lor (p \land r) \vdash p \land (q \lor r)$.

Proof.

1. $(p \land q) \lor (p \land r)$ premise
2. $p \land q$ assumption
3. $p$ \&e 2
4. $q$ \&e 2
5. $q \lor r$ \lor i 4
6. $p \land (q \lor r)$ \&i 3, 5
7. $p \land r$ assumption
8. $p$ \&e 7
9. $r$ \&e 7
10. $q \lor r$ \lor i 9
11. $p \land (q \lor r)$ \&i 8, 10
12. $p \land (q \lor r)$ \lor e 1, 2-6, 7-11
Contradiction

**Definition**

Contradictions are sentences of the form $\phi \land \neg \phi$ or $\neg \phi \land \phi$.

- **Examples:**
  - $p \land \neg p$, $\neg (p \lor q \implies r) \land (p \lor q \implies r)$.

- Logically, any sentence can be proved from a contradiction.
  - If $0 = 1$, then $100 \neq 100$.

- Particularly, if $\phi$ and $\psi$ are contradictions, we have $\phi \vdash \psi$.
  - $\phi \vdash \psi$ means $\phi \vdash \psi$ and $\psi \vdash \phi$ (called provably equivalent).

- Since all contradictions are equivalent, we will use the symbol $\bot$ (called “bottom”) for them.

- We are now ready to discuss proof rules for negation.
Since any sentence can be proved from a contradiction, we have

\[ \frac{\bot}{\phi} \bot e \]

When both \( \phi \) and \( \neg \phi \) are proved, we have a contradiction.

\[ \frac{\phi \quad \neg \phi}{\bot} \neg e \]

The proof rule could be called \( \bot i \). We use \( \neg e \) because it eliminates a negation.
Example

Prove \( \neg p \lor q \vdash p \implies q \). 

Proof.

1. \( \neg p \lor q \) premise
2. \( \neg p \) assumption
3. \( p \) assumption
4. \( \bot \) \( \neg e \) 3, 2
5. \( q \) \( \bot e \) 4
6. \( p \implies q \) \( \implies i \) 3-5
7. \( q \) assumption
8. \( p \) assumption
9. \( q \) copy 7
10. \( p \implies q \) \( \implies i \) 8-9
11. \( p \implies q \) \( \lor e \) 1, 2-6, 7-10
Suppose we want to prove $\neg \phi$. What do we do?

- We assume $\phi$ and try to prove a contradiction. If succeed, we prove $\neg \phi$.

Here is the proof rule:

\[
\begin{array}{c}
\phi \\
\vdots \\
\bot \\
\hline
\neg \phi \\
\neg i
\end{array}
\]
Example

Prove \( p \implies q, \ p \implies \neg q \vdash \neg p \).

Proof.

1. \( p \implies q \) premise
2. \( p \implies \neg q \) premise
3. \( p \) assumption
4. \( q \implies e \ 3, 1 \)
5. \( \neg q \implies e \ 3, 2 \)
6. \( \bot \implies \neg e \ 4, 5 \)
7. \( \neg p \implies \neg i \ 3-6 \)
Example

Prove $p \land \neg q \implies r, \neg r, p \vdash q$.

Proof.

1. $p \land \neg q \implies r$  
   premise
2. $\neg r$  
   premise
3. $p$  
   premise
4. $\neg q$  
   assumption
5. $p \land \neg q$  
   $\land i$ 3, 4
6. $r$  
   $\implies e$ 5, 1
7. $\bot$  
   $\neg e$ 6, 2
8. $\neg \neg q$  
   $\neg i$ 4-7
9. $q$  
   $\neg \neg e$ 8
Some rules can actually be derived from others.

**Examples**

Prove $p \rightarrow q, \neg q \vdash \neg p$ (modus tollens).

**Proof.**

1. $p \rightarrow q$ premise
2. $\neg q$ premise
3. $p$ assumption
4. $q \rightarrow e 3, 1$
5. $\bot \neg e 4, 2$
6. $\neg p \neg i 3-5$
Examples

Prove $p \vdash \neg
\neg p$ ($\neg \neg i$)

Proof.

1. $p$ premise
2. $\neg p$ assumption
3. $\bot$ \neg e 1, 2
4. $\neg \neg p$ $\neg i$ 2-3

- These rules can be replaced by their proofs and are not necessary.
  - They are just macros to help us write shorter proofs.
Example

Prove \( \neg p \implies \bot \vdash \neg \neg p \) (RAA).

Proof.

1. \( \neg p \implies \bot \) premise
2. \( \neg p \) assumption
3. \( \bot \implies e \ 2, \ 1 \)
4. \( \neg \neg p \neg i \ 2-3 \)
5. \( p \) \( \neg \neg e \ 4 \)
Example

Prove $\vdash p \lor \neg p$.

Proof.

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\neg (p \lor \neg p)$</td>
<td>assumption</td>
</tr>
<tr>
<td>2</td>
<td>$p$</td>
<td>assumption</td>
</tr>
<tr>
<td>3</td>
<td>$p \lor \neg p$</td>
<td>$\lor i_1$ 2</td>
</tr>
<tr>
<td>4</td>
<td>$\bot$</td>
<td>$\neg e$ 3, 1</td>
</tr>
<tr>
<td>5</td>
<td>$\neg p$</td>
<td>$\neg i$ 2-4</td>
</tr>
<tr>
<td>6</td>
<td>$p \lor \neg p$</td>
<td>$\lor i_2$ 5</td>
</tr>
<tr>
<td>7</td>
<td>$\bot$</td>
<td>$\neg e$ 6, 1</td>
</tr>
<tr>
<td>8</td>
<td>$\neg \neg (p \lor \neg p)$</td>
<td>$\neg i$ 1-7</td>
</tr>
<tr>
<td>9</td>
<td>$p \lor \neg p$</td>
<td>$\neg \neg e$ 8</td>
</tr>
</tbody>
</table>
Conjunction ($\land$)

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad \land i
\]

Disjunction ($\lor$)

\[
\frac{\phi}{\phi \lor \psi} \quad \lor i_1 \quad \frac{\psi}{\phi \lor \psi} \quad \lor i_2
\]

Implication ($\implies$)

\[
\frac{\phi \quad \psi}{\phi \implies \psi} \quad \implies i \quad \frac{\phi}{\phi \quad \phi \implies \psi} \quad \implies e
\]

\[
\frac{\phi \lor \psi}{\chi} \quad \lor e
\]
### Negation ($\neg$)

$$
\begin{array}{c}
\phi \\
\vdots \\
\bot
\end{array}
\quad \frac{\bot}{\neg \phi} \ \neg i
$$

### Contradiction ($\bot$)

(no introduction rule)

$$
\frac{\bot}{\phi} \quad \bot e
$$

### Double negation ($\neg\neg$)

(no introduction rule)

$$
\frac{\neg\neg \phi}{\phi} \quad \neg\neg e
$$
Useful Derived Proof Rules

\[
\begin{align*}
\frac{\phi \iff \psi}{\neg \psi} \quad & \quad MT \\
\frac{\neg \phi}{\neg \phi} \quad & \quad \neg \neg \neg \neg \phi \quad \neg \neg i \\
\frac{\neg \phi}{\phi} \quad & \quad RAA \\
\frac{\phi \lor \neg \phi}{LEM}
\end{align*}
\]
Recall $p \dashv \vdash q$ means $p \vdash q$ and $q \vdash p$.

Here are some provably equivalent sentences:

- $(\neg (p \land q)) \vdash \neg q \lor \neg p$
- $(\neg (p \lor q)) \vdash \neg q \land \neg p$
- $p \rightarrow \rightarrow q \vdash \neg q \rightarrow \rightarrow \neg p$
- $p \rightarrow \rightarrow q \vdash \neg p \lor q$
- $p \land q \rightarrow \rightarrow p \vdash r \lor \neg r$
- $p \land q \rightarrow \rightarrow r \vdash p \rightarrow \rightarrow (q \rightarrow \rightarrow r)$

Try to prove them.
Proof by Contradiction

- Although it is very useful, the proof rule RAA is a bit puzzling.

\[ \neg \phi \]
\[ \vdots \]
\[ \bot \]
\[ \phi \]

RAA

- Instead of proving \( \phi \) directly, the proof rule allows indirect proofs.
  - If \( \neg \phi \) leads to a contradiction, then \( \phi \) must hold.

- Note that indirect proofs are not “constructive.”
  - We do not show why \( \phi \) holds; we only know \( \neg \phi \) is impossible.

- In early 20th century, some logicians and mathematicians chose not to prove indirectly. They are intuitionistic logicians or mathematicians.

- For the same reason, intuitionists also reject

\[ \phi \lor \neg \phi \] \quad LEM

\[ \frac{\neg \neg \phi}{\phi} \quad \neg \neg e \]
## Proof by Contradiction

### Theorem

*There are $a, b \in \mathbb{R} \setminus \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.*

### Proof.

Let $b = \sqrt{2}$. There are two cases:

- If $b^b \in \mathbb{Q}$, we are done since $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

- If $b^b \notin \mathbb{Q}$, choose $a = b^b = \sqrt{2}^{\sqrt{2}}$. Then $a^b = (b^b)^b = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$. Since $\sqrt{2}^{\sqrt{2}}, \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, we are done. $\square$

- An intuitionist would criticize the proof since it does not tell us what $a, b$ give $a^b \in \mathbb{Q}$.
  - We know $(a, b)$ is either $(\sqrt{2}, \sqrt{2})$ or $(\sqrt{2}^{\sqrt{2}}, \sqrt{2})$. 

Outline

1 Natural Deduction

2 Propositional logic as a formal language

3 Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
A well-formed formula is constructed by applying the following rules finitely many times:

- **atom**: Every propositional atom $p, q, r, \ldots$ is a well-formed formula;
- $\neg$: If $\phi$ is a well-formed formula, so is $(\neg \phi)$;
- $\land$: If $\phi$ and $\psi$ are well-formed formulae, so is $(\phi \land \psi)$;
- $\lor$: If $\phi$ and $\psi$ are well-formed formulae, so is $(\phi \lor \psi)$;
- $\rightarrow$: If $\phi$ and $\psi$ are well-formed formulae, so is $(\phi \rightarrow \psi)$.

More compactly, well-formed formulae are defined by the following grammar in Backus Naur form (BNF):

$$\phi ::= p \mid (\neg \phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow \phi)$$
Inversion Principle

How do we check if \(((\neg p) \land q) \Rightarrow (p \land (q \lor (\neg r))))\) is well-formed?

Although a well-formed formula needs five grammar rules to construct, the construction process can always be inverted.

- This is called inversion principle.

To show \(((\neg p) \land q) \Rightarrow (p \land (q \lor (\neg r))))\) is well-formed, we need to show both \(((\neg p) \land q)\) and \((p \land (q \lor (\neg r))))\) are well-formed.

To show \(((\neg p) \land q)\) is well-formed, we need to show both \((\neg p)\) and \(q\) are well-formed.

- \(q\) is well-formed since it is an atom.

To show \((\neg p)\) is well-formed, we need to show \(p\) is well-formed.

- \(p\) is well-formed since it is an atom.

Similarly, we can show \((p \land (q \lor (\neg r))))\) is well-formed.
The easiest way to decide whether a formula is well-formed is perhaps by drawing its parse tree.
Subformulae

- Given a well-formed formula, its subformulae are the well-formed formulae corresponding to its parse tree.
- For instance, the subformulae of the well-formed formulae 

\[ (((\neg p) \land q) \rightarrow (p \land (q \lor (\neg r)))) \] 

are:

- \( p \)
- \( q \)
- \( r \)
- \( (\neg p) \)
- \( (\neg r) \)
- \( (((\neg p) \land q) \) 
- \( (q \lor (\neg r)) \)
- \( (p \land (q \lor (\neg r))) \)
- \( (((\neg p) \land q) \rightarrow (p \land (q \lor (\neg r)))) \)
Outline

1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
We have developed a calculus to determine whether \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) is valid.
- That is, from the premises \( \phi_1, \phi_2, \ldots, \phi_n \), we can conclude \( \psi \).
- Our calculus is syntactic. It depends on the syntactic structures of \( \phi_1, \phi_2, \ldots, \phi_n \), and \( \psi \).

We will introduce another relation between premises \( \phi_1, \phi_2, \ldots, \phi_n \) and a conclusion \( \psi \).

\[
\phi_1, \phi_2, \ldots, \phi_n \models \psi.
\]

- The new relation is defined by ‘truth values’ of atomic formulae and the semantics of logical connectives.
Definition

The set of truth values is \{F, T\} where F represents ‘false’ and T represents ‘true.’

Definition

A valuation or model of a formula \(\phi\) is an assignment from each proposition atom in \(\phi\) to a truth value.
## Truth Values of Formulae

**Definition**

Given a valuation of a formula $\phi$, the truth value of $\phi$ is defined inductively by the following truth tables:

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$\phi \land \psi$</th>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$\phi \lor \psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$\phi \Rightarrow \psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\neg \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\top$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\bot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
</tr>
</tbody>
</table>
Example

- $\phi \land \psi$ is T when $\phi$ and $\psi$ are T.
- $\phi \lor \psi$ is T when $\phi$ or $\psi$ is T.
- $\bot$ is always F; $\top$ is always T.
- $\phi \implies \psi$ is T when $\phi$ “implies” $\psi$.

Example

Consider the valuation $\{q \leftrightarrow T, p \leftrightarrow F, r \leftrightarrow F\}$ of $(q \land p) \implies r$. What is the truth value of $(q \land p) \implies r$?

Proof.

Since the truth values of $q$ and $p$ are T and F respectively, the truth value of $q \land p$ is F. Moreover, the truth value of $r$ is F. The truth value of $(q \land p) \implies r$ is T.
Given a formula \( \phi \) with propositional atoms \( p_1, p_2, \ldots, p_n \), we can construct a truth table for \( \phi \) by listing \( 2^n \) valuations of \( \phi \).

**Example**

Find the truth table for \((p \implies \neg q) \implies (q \lor \neg p)\).

**Proof.**

\[
\begin{array}{cccccccc}
  p & q & \neg p & \neg q & p \implies \neg q & q \lor \neg p & (p \implies \neg q) \implies (q \lor \neg p) \\
  \hline
  F & F & T & T & T & T & T \\
  F & T & T & F & T & T & T \\
  T & F & F & T & T & F & F \\
  T & T & F & F & F & T & T \\
\end{array}
\]
Outline

1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic
Validity of Sequent Revisited

- Informally $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid if we can derive $\psi$ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$.
  - We have formalized “deriving $\psi$ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$” by “constructing a proof in a formal calculus.”
- We can give another interpretation by valuations and truth values.
- Consider a valuation $\nu$ over all propositional atoms in $\phi_1, \phi_2, \ldots, \phi_n, \psi$.
  - By “assumptions $\phi_1, \phi_2, \ldots, \phi_n$,” we mean “$\phi_1, \phi_2, \ldots, \phi_n$ are T under the valuation $\nu$.
  - By “deriving $\psi$,” we mean $\psi$ is also T under the valuation $\nu$.
- Hence, “we can derive $\psi$ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$” actually means “if $\phi_1, \phi_2, \ldots, \phi_n$ are T under a valuation, then $\psi$ must be T under the same valuation.”
Semantic Entailment

Definition

We say \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \) holds if for every valuations where \( \phi_1, \phi_2, \ldots, \phi_n \) are T, \( \psi \) is also T. In this case, we also say \( \phi_1, \phi_2, \ldots, \phi_n \) semantically entail \( \psi \).

Examples

- \( p \land q \models p \). For every valuation where \( p \land q \) is T, \( p \) must be T. Hence \( p \land q \models p \).
- \( p \lor q \not\models q \). Consider the valuation \( \{ p \mapsto T, q \mapsto F \} \). We have \( p \lor q \) is T but \( q \) is F. Hence \( p \lor q \not\models q \).
- \( \neg p, p \lor q \not\models q \). Consider any valuation where \( \neg p \) and \( p \lor q \) are T. Since \( \neg p \) is T, \( p \) must be F under the valuation. Since \( p \) is F and \( p \lor q \) is T, \( q \) must be T under the valuation. Hence \( \neg p, p \lor q \not\models q \).

The validity of \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \) is defined by syntactic calculus. \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \) is defined by truth tables. Do these two relations coincide?
Soundness Theorem for Propositional Logic

**Theorem (Soundness)**

Let $\phi_1, \phi_2, \ldots, \phi_n$ and $\psi$ be propositional logic formulae. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ holds.

**Proof.**

Consider the assertion $M(k)$:

“For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi (n \geq 0)$ that have a proof of length $k$, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ holds.”

$k = 1$. The only possible proof is of the form

```
1  \phi  \text{ premise}
```

This is the proof of $\phi \vdash \phi$. For every valuation such that $\phi$ is T, $\phi$ must be T. That is, $\phi \models \phi$. 

Proof (cont’d).

Assume $M(i)$ for $i < k$. Consider a proof of the form

1. $\phi_1$ premise
2. $\phi_2$ premise
   ...
$n$. $\phi_n$ premise
   ...
k. $\psi$ justification

We have the following possible cases for justification:

$i$ $\land$ $i$. Then $\psi$ is $\psi_1 \land \psi_2$. In order to apply $\land$ $i$, $\psi_1$ and $\psi_2$ must appear in the proof. That is, we have $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_2$. By inductive hypothesis, $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_2$. Hence $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1 \land \psi_2$ (Why?).
Proof (cont’d).

ii  \( \lor e \). Recall the proof rule for \( \lor e \):

\[
\begin{array}{c}
\eta_1 \\
\vdots \\
\psi \\
\eta_2 \\
\vdots \\
\psi \\
\hline
\psi \\
\hline
\eta_1 \lor \eta_2 \\
\hline
\lor e
\end{array}
\]

In order to apply \( \lor e \), \( \eta_1 \lor \eta_2 \) must appear in the proof. We have \( \phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \lor \eta_2 \). By turning “assumptions” \( \eta_1 \) and \( \eta_2 \) to “premises,” we obtain proofs for \( \phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi \) and \( \phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi \). By inductive hypothesis, \( \phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \lor \eta_2 \), \( \phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi \), and \( \phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi \). Consider any valuation such that \( \phi_1, \phi_2, \ldots, \phi_n \) evaluates to \( T \). \( \eta_1 \lor \eta_2 \) must be \( T \). If \( \eta_1 \) is \( T \) under the valuation, \( \psi \) is also \( T \) (Why?). Similarly for \( \eta_2 \) is \( T \). Thus \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \).
Soundness Theorem for Propositional Logic

Proof (cont’d).

iii Other cases are similar. Prove the case of $\rightarrow e$ to see if you understand the proof.

- The soundness theorem shows that our calculus does not go wrong.
- If there is a proof of a sequent, then the conclusion must be true for all valuations where all premises are true.
- The theorem also allows us to show the non-existence of proofs.
- Given a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, how do we prove there is no proof for the sequent?
  - Try to find a valuation where $\phi_1, \phi_2, \ldots, \phi_n$ are T but $\psi$ is F.
1 Natural Deduction

2 Propositional logic as a formal language

3 Semantics of propositional logic
   • The meaning of logical connectives
   • Soundness of Propositional Logic
   • Completeness of Propositional Logic
"\(\phi_1, \phi_2, \ldots, \phi_n \vdash \psi\) is valid" and "\(\phi_1, \phi_2, \ldots, \phi_n \vDash \psi\) holds" are very different.

- "\(\phi_1, \phi_2, \ldots, \phi_n \vdash \psi\) is valid" requires proof search (syntax);
- "\(\phi_1, \phi_2, \ldots, \phi_n \vDash \psi\) holds" requires a truth table (semantics).

If "\(\phi_1, \phi_2, \ldots, \phi_n \vDash \psi\) holds" implies "\(\phi_1, \phi_2, \ldots, \phi_n \vdash \psi\) is valid,“ then our natural deduction proof system is complete.

The natural deduction proof system is both sound and complete. That is

\[\phi_1, \phi_2, \ldots, \phi_n \vdash \psi\) is valid iff \(\phi_1, \phi_2, \ldots, \phi_n \vDash \psi\) holds.\]
We will show the natural deduction proof system is complete.

That is, if $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ holds, then there is a natural deduction proof for the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.

Assume $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$. We proceed in three steps:

1. $\vdash \phi_1 \implies (\phi_2 \implies (\ldots (\phi_n \implies \psi))))$ holds;
2. $\vdash \phi_1 \implies (\phi_2 \implies (\ldots (\phi_n \implies \psi))))$ is valid;
3. $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.
Completeness Theorem for Propositional Logic (Step 1)

**Lemma**

If \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \) holds, then \( \models \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi))) \) holds.

**Proof.**

Suppose \( \models \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi))) \) does not hold. Then there is valuation where \( \phi_1, \phi_2, \ldots, \phi_n \) is T but \( \psi \) is F. A contradiction to \( \phi_1, \phi_2, \ldots, \phi_n \models \psi \).

**Definition**

Let \( \phi \) be a propositional logic formula. We say \( \phi \) is a **tautology** if \( \models \phi \).

- A tautology is a propositional logic formula that evaluates to T for all of its valuations.
Our goal is to show the following theorem:

**Theorem**

If $\models \eta$ holds, then $\vdash \eta$ is valid.

Similar to tautologies, we introduce the following definition:

**Definition**

Let $\phi$ be a propositional logic formula. We say $\phi$ is a **theorem** if $\vdash \phi$.

Two types of theorems:

- If $\vdash \phi$, $\phi$ is a theorem proved by the natural deduction proof system.
- The soundness theorem for propositional logic is another type of theorem proved by mathematical reasoning (less formally).
Proposition

Let $\phi$ be a formula with propositional atoms $p_1, p_2, \ldots, p_n$. Let $l$ be a line in $\phi$'s truth table. For all $1 \leq i \leq n$, let $\hat{p}_i$ be $p_i$ if $p_i$ is T in $l$; otherwise $\hat{p}_i$ is $\neg p_i$. Then

1. $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi$ is valid if the entry for $\phi$ at $l$ is T;
2. $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi$ is valid if the entry for $\phi$ at $l$ is F.

Proof.

We prove by induction on the height of the parse tree of $\phi$.

- $\phi$ is a propositional atom $p$. Then $p \vdash p$ or $\neg p \vdash \neg p$ have one-line proof.
- $\phi$ is $\neg \phi_1$.
  - If $\phi$ is T at $l$. Then $\phi_1$ is F. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi_1 (\equiv \phi)$.
  - If $\phi$ is F at $l$. Then $\phi_1$ is T. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$. Using $\neg \neg i$, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \neg \phi_1 (\equiv \neg \phi)$. 
Completeness Theorem for Propositional Logic (Step 2)

Proof (cont’d).

- \( \phi \) is \( \phi_1 \implies \phi_2 \).
  - If \( \phi \) is F at \( l \), then \( \phi_1 \) is T and \( \phi_2 \) is F at \( l \). By IH, \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \) and \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg \phi_2 \). Consider
    
    $\begin{array}{cccc}
    1 & \phi_1 & \implies & \phi_2 & \text{assumption} \\
    \vdots \\
    i & \phi_1 & \text{IH} \\
    i + 1 & \phi_2 & \implies & e \ i \ , \ 1 \\
    \vdots \\
    j & \neg \phi_2 & \text{IH} \\
    j + 1 & \bot & \neg e \ i+1 \ , \ j \\
    j + 2 & \neg (\phi_1 \implies \phi_2) & \neg i \ 1-(j+1) \\
    \end{array}$
Proof (cont’d).

- \( \phi \) is \( \phi_1 \implies \phi_2 \).

  - If \( \phi \) is T at \( l \), we have three subcases. Consider the case where \( \phi_1 \) and \( \phi_2 \) are F at \( l \). Then

    \[
    \begin{array}{cccccc}
    1 & \phi_1 & \text{assumption} & \vdots & & \\
    i & \neg \phi_1 & \text{IH} & i + 1 & \bot & \neg \text{ e } 1, i \\
    i + 2 & \phi_2 & \bot & \text{ e } (i+1) \\
    i + 3 & \phi_1 \implies \phi_2 & \implies & i \ 1-(i+2)
    \end{array}
    \]

    The other two subcases are simple exercises.
Proof (cont’d).

**φ is φ₁ ∧ φ₂.**

- If φ is T at l, then φ₁ and φ₂ are T at l. By IH, we have  
  \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \) and \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_2 \). Using \( \land i \), we have \( \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \land \phi_2 \).

- If φ is F at l, there are three subcases. Consider the subcase where φ₁ and φ₂ are F at l. Then

\[
\begin{align*}
1 & \quad \phi_1 \land \phi_2 \quad \text{assumption} \\
2 & \quad \phi_1 \quad \land e_1 \ 1 \\
\vdots & \\
i & \quad \neg \phi_1 \quad \text{IH} \\
i + 1 & \quad \bot \quad \neg e_2, i \\
i + 2 & \quad \neg(\phi_1 \land \phi_2) \quad \neg i \ 1-(i+1)
\end{align*}
\]

The other two subcases are simple exercises.
Completeness Theorem for Propositional Logic (Step 2)

**Proof.**

- $\phi$ is $\phi_1 \lor \phi_2$.
  - If $\phi$ is F at $l$, then $\phi_1$ and $\phi_2$ are F at $l$. Then
    
    1. $\phi_1 \lor \phi_2$ \hspace{1cm} assumption
    2. $\phi_1$ \hspace{1cm} assumption
    $\vdots$
    i. $\neg \phi_1$ \hspace{1cm} IH
    i + 1. $\bot$ \hspace{1cm} $\neg$ e 2, i
    i + 2. $\phi_2$ \hspace{1cm} assumption
    $\vdots$
    j. $\neg \phi_2$ \hspace{1cm} IH
    j + 1. $\bot$ \hspace{1cm} $\neg$ e i+2, j
    j + 2. $\bot$ \hspace{1cm} $\lor$ e 2-(i+1), (i+2)-(j+1)
    j + 3. $\neg (\phi_1 \lor \phi_2)$ \hspace{1cm} $\neg$ i 1-(j+2)
  - If $\phi$ is T at $l$, there are three subcases. All of them are simple exercises.
Completeness Theorem for Propositional Logic (Step 2)

**Theorem**

If $\phi$ is a tautology, then $\phi$ is a theorem.

**Proof.**

Let $\phi$ have propositional atoms $p_1, p_2, \ldots, p_n$. Since $\phi$ is a tautology, each line in $\phi$’s truth table is T. By the above proposition, we have the following $2^n$ proofs for $\phi$:

\[
\neg p_1, \neg p_2, \ldots, \neg p_n \vdash \phi \\
\neg p_1, p_2, \ldots, \neg p_n \vdash \phi \\
p_1, p_2, \ldots, \neg p_n \vdash \phi \\
\vdots \\
p_1, p_2, \ldots, p_n \vdash \phi
\]

We apply the rule LEM and the $\lor$ rule to obtain a proof for $\vdash \phi$. (See the following example.)
Example

Observe that \( \vdash p \implies (q \implies p) \). Prove \( \vdash p \implies (q \implies p) \).

Proof.

1. \( p \lor \neg p \)  
   LEM
2. \( p \)  
   assumption
3. \( q \lor \neg q \)  
   LEM
4. \( q \)  
   assumption
   ...
5. \( p \implies (q \implies p) \)  
   \( p, q \vdash p \implies (q \implies p) \)
6. \( \neg q \)  
   assumption
   ...
7. \( p \implies (q \implies p) \)  
   \( p, \neg q \vdash p \implies (q \implies p) \)
8. \( p \implies (q \implies p) \)  
   \( \lor e 3, 4-i, (i+1)-j \)
9. \( \neg p \)  
   assumption
10. \( q \lor \neg q \)  
   LEM
11. \( q \)  
   assumption
   ...
12. \( p \implies (q \implies p) \)  
   \( \neg p, q \vdash p \implies (q \implies p) \)
13. \( \neg q \)  
   assumption
   ...
14. \( p \implies (q \implies p) \)  
   \( \neg p, \neg q \vdash p \implies (q \implies p) \)
15. \( \lor e (j+3), (j+4)-k, (k+1)-l \)
16. \( p \implies (q \implies p) \)  
   \( \lor e 1, 2-(j+1), (j+2)-(l+1) \)
Completeness Theorem for Propositional Logic (Step 3)

Lemma

If \( \phi_1 \implies (\phi_2 \implies (\cdots (\phi_n \implies \psi))) \) is a theorem, then \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) is valid.

Proof.

Consider

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( \phi_1 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( \phi_2 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>\vdots</td>
</tr>
<tr>
<td>n</td>
<td>n</td>
<td>( \phi_n )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>\vdots</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>( \phi_1 \implies (\phi_2 \implies (\cdots (\phi_n \implies \psi))) )</td>
</tr>
<tr>
<td>i + 1</td>
<td>i</td>
<td>( \phi_2 \implies (\cdots (\phi_n \implies \psi)) )</td>
</tr>
<tr>
<td>i + 2</td>
<td>i</td>
<td>( \phi_3 \implies (\cdots (\phi_n \implies \psi)) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>\vdots</td>
</tr>
<tr>
<td>i + n - 1</td>
<td>i</td>
<td>( \phi_n \implies \psi )</td>
</tr>
<tr>
<td>i + n</td>
<td>i</td>
<td>( \psi )</td>
</tr>
</tbody>
</table>